数理解析研究所講究録
1176 巻 2000 年 39-55

On smoothing properties of Nonlinear Schrödinger Equations

Yoshihisa Nakamura

Graduate School of Science and Technology

Kunamoto University

1 Introduction and Main Results

In this paper, we estimate smoothing properties of local solutions to nonlinear Schrödinger equations with initial data which belong to Sobolev spaces H^s, $s \geq 0$.

We consider the following equation;

$$i\partial_t u = -\Delta u + F(u), \tag{1}$$

where u is a complex-valued function of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$, Δ is the Laplacian on $\mathbb{R}^n$, and $F(u) = F \circ u$ is a local nonlinear operator given by a complex-valued function F on $\mathbb{C}$.

Here we consider the following assumptions on the nonlinear term F in Kato[8];

Assumption (F1) $F \in C^{\{s\}}(\mathbb{C}; \mathbb{C})$, with $F(0) = 0$, where $\{s\} = [s] + 1$ if $s \notin \mathbb{Z}$ and $\{s\} = s$ if $s \in \mathbb{Z}$, and $\{0\} = 1$.

Assumption (F2) If $s > n/2$, no assumption.

If $s \leq n/2$ and if F is a polynomial in $\zeta = \xi + i\eta$ and $\bar{\zeta}$, then degree of F is equal to $k \leq 1 + 4/(n - 2s)$.

If $s \leq n/2$ and if F is not a polynomial,

$$|D^i F(\zeta)| \equiv \max_{0 \leq j \leq i} \{|\partial_\zeta^j \partial_{\bar{\zeta}}^{i-j} F|\} \leq M_i |\zeta|^{k-i}, \quad \text{for} \quad |\zeta| \geq 1,$$

for $i = 0, 1, \ldots, \{s\}$, where $\partial_\zeta = (\partial_\xi - i\partial_\eta)/2$, $\partial_{\bar{\zeta}} = (\partial_\xi + i\partial_\eta)/2$, and k is a finite number such that

$$\{s\} \leq k \leq 1 + \frac{4}{n-2s}.$$

We denote $\partial_x = (\partial_1, \ldots, \partial_n)$, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\partial_x^\alpha = (\partial_1^{\alpha_n}, \ldots, \partial_n^{\alpha_n})$ and we often denote ∂_x by ∂, and use ∂ as ∂^α if $|\alpha| = 1$.

The Cauchy problems of equation (1) with above assumptions were studied by many authors. In Kato[6], Tsutsumi[17], they discussed about local or global wellposedness in the case that the initial data belongs to L^2, in Ginibre and Velo[3], Kato[5][6], they discussed about local wellposedness in the case that the initial data belongs to H^1, and in Ginibre and Velo[4], Kato[5][6], about the existence of global solution. Sjölin's result[15] that we noted below was based on the existence of local solutions discussed in Kato[5]. In Kato[6], he discussed about local wellposedness in the case that the initial data belongs to H^2, and in

Kato[5][6], Tsutsumi[18], about the existence of global solution. We try to estimate regularity of local solutions to equation (1) with $u(0) = u_0 \in H^s$, for $s \geq 0$, which were obtained by Kato[8].

In Sjölin[14], he obtained the following inequality; For some $C > 0$, depending on $\phi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |\phi(t,x)(1-\Delta)^{\frac{1}{4}} e^{it\Delta} f|^2 dx dt \leq C \|f\|_{L^2}^2, \quad \forall f \in L^2. \quad (2)$$

This inequality manifests that the free Schrödinger propagator $e^{it\Delta}$ has the smoothing effect which can improve the properties of differentiability locally in time and space. Independently, Vega[19] was obtained the same as the local smoothing property for the free Schrödinger equation, Constantin and Saut[2] were obtained the same as the one for general dipersive equations in homogeneous or inhomogeneous cases.

Later the similar property for e^{-itH}, where $H = -\Delta + V$ is a self-adjoint operator and $V = V(x)$ or $V(t,x)$ are various scalar potentials, was studied by many authors(Ben-Artzi[1], Ruiz and Vega[12], etc.). In particular, Kato and Yajima[9] obtained the inequality replacing ϕ in (2) by $(1+|x|^2)^{-\frac{1}{4}-\varepsilon}$, $\varepsilon > 0$, and Yajima[21] obtained the similar estimate for the propagators of Schrödinger equations with time dependent magnetic and scalar potentials which may increase at infinity $|x| \to \infty$.

In Sjögren and Sjölin[13], they obtained the extension of (2) in the following form; They defined

$$\mathcal{A} \equiv \{\varphi \in C^\infty(\mathbb{R}^n) \quad | \quad \text{There exists } \varepsilon > 0 \text{ such that}$$
$$|\partial^\alpha \varphi(x)| \le C_\alpha (1+|x|)^{-\frac{1}{2}-\varepsilon},\ \forall \alpha\},$$

where introduced mixed Sobolev spaces $H^{r,\rho} = H^{r,\rho}(\mathbb{R}\times\mathbb{R}^n) = (G_r \bigotimes G_\rho) * L^2(\mathbb{R}^{n+1})$, where G_r and G_ρ are Bessel kernels in $\mathbb{R}$ and $\mathbb{R}^n$, respectively. If $r \ge 0$, $\rho \ge 0$, then, for each $\varphi \in \mathcal{A}$, $\psi \in C_0^\infty(\mathbb{R})$,

$$||\psi\varphi e^{-it\overline{H}} u||_{H^{r,\rho}} \le C_{\psi\varphi} ||u||_{H^{mr+\rho-\frac{1}{2}(m-1)}}, \tag{3}$$

for some $C_{\varphi,\psi} > 0$. Here $\overline{H} = -P + V$, where P is a elliptic operators with constant coefficient which degree is $m \geq 2$ and $V = V(x)$ is a real-valued function in C^∞ with $D^\alpha V$ bounded for every α.

In Sjölin[15][16] he adapted these estimates for equation (1), nonlinear Schödinger equation, with H^1 or H^2-initial data. The author extended in [10] p's range when the initial data belong to H^2.

Kato[8] proved that assume (F1) and (F2), if $s \geq n/2$, or if $s < n/2$ and $k < 1+\max\{4, 2s+2\}/(n-2s)$, and $k < 2/(1-2s)$ if $k = 1$, then there exists a number $T > 0$ and a unique solution $u \in C(I; H^s)$ of equation (1) with $u(0) = u_0 \in H^s$, $s > 0$, where $I = [0, T]$. Moreover $u \in \bar{\mathcal{Y}}_s(I) \equiv \{u \in \bar{\mathcal{X}}(I) | \Lambda^\sigma u \in \bar{\mathcal{X}}(I), 0 \leq \sigma \leq s\}$, where

$$\bar{\mathcal{X}}(I) \equiv (\bigcap L^s(I; L^q)) \cap C(I; L^2), \tag{4}$$

where $\bigcap$ is the intersection in (q, s) satisfying $1/q + 2/ns = 1/2$ and $1/2 - 1/n < 1/q \leq 1/2$ and $\Lambda^\sigma u = (-\Delta)^{\sigma/2} u$. In this paper we shall adapt Sjölin's estimates to the solution of equation (1) with H^s-initial values. We obtain the following Theorem.

Theorem 1. Assume (F1) and (F2). If $s > 1$, then the unique solution $u \in C(I; H^s)$ of the equation (1) with $u(0) = u_0 \in H^s$ satisfies $\varphi u \in L^2(I; H^{s+1/2})$ for each $\varphi \in \mathcal{A}$. If $0 \leq s \leq 1$, then, in $1 \leq n \leq 2s + 4$, $\varphi u \in L^2(I; H^{s+1/2})$ for each $\varphi \in \mathcal{A}$. When $n \geq 2s + 5$, under the additional condition $p < 1 + 2/(n - 2s - 2)$ $\varphi u \in L^2(I; H^{s+1/2})$ for each $\varphi \in \mathcal{A}$.

Remark 1 Applying this theorem to Theorem 1 in Yajima[21], we can obtain the regularity estimate of solutions to nonlinear Shrödinger equations with magnetic fields in the case that the initial data belongs to H^s. The author[11] obtained the results in the case of H^1.

We use the standard notations. We abbreviate $L^p(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n)$ to L^p and H^k, respectively. We denote usual L^p-norm by $\| \quad \|_p$. For $I = [0, T]$, put $L^{p,r} = L^r(I; L^p)$, where $1 < p < \infty$,$1 \leq r \leq \infty$, with its norm denoted by

$$\|f\|_{p,r} \equiv (\int_I \|f(t)\|_p^r dt)^{\frac{1}{r}}.$$

We denote various constants by C, M, etc. They may differ from line to line.

2 Proof of Theorem

We introduce the following linear operators.

$$(\Gamma\phi)(t) = U(t)\phi = e^{it\Delta}\phi, \qquad t \in I, \tag{5}$$

$$(Gf)(t) = \int_0^t U(t-\tau)f(\tau)d\tau, \qquad t \in I. \tag{6}$$

The following lemmas is well known.(see,for example, Kato[5][6][7], Yajima[20])

Lemma 2.1 Γ is a bounded operator from L^2 to L^{p_1,r_1}, where (p_1, r_1) satisfies $1/p_1 + 2/nr_1 = 1/2$ and $1/2 - 1/n < 1/p_1 \leq 1/2$. The bound is independent of T, and is uniform for any (p_1, r_1). Here $L^{2,\infty}$ may be replaced by $C(I; L^2)$.

Lemma 2.2 G is a bounded operator from L^{p_2,r_2} to L^{p_1,r_1}, where (p_1, r_1) and (p_2, r_2) satisfy $1/p_1 + 2/nr_1 = 1/2$ and $1/2 + 1/n < 1/p_1 \leq 1/2$, $1/p_2 + 2/nr_2 = 1/2 + 2/n$ and $1/2 \leq 1/p_2 < 1/2 + 1/n$, respectively. The bound is independent of T, and is uniform for any (p_1, r_1) and (p_2, r_2). Here $L^{2,\infty}$ may be replaced by $C(I; L^2)$.

For the nonlinear term F, the following lemma is well known.

Lemma 2.3 (Kato[8]) Assume (F1) and (F2) with $s \leq n/2$. If F is a polynomial of degree k, it is obviously the sum of homogeneous polynomials of degree 1 to k. Otherwsie, F can be written in the form

$$F = F_1 + F_2 + \cdots + F_{\{s\}-1} + F_{\{s\}} + F_k,$$

where the F_j for $j = 1, 2, \ldots, \{s\} - 1$ are homogeneouse polynomials of degree j, while $F_{\{}s\}$ and F_k are quasi-homogeneouse of order $\{s\}$ and of degree $\{s\}$ and k, respectively. If $k = \{s\}$, $F_{\{s\}}$ is redundunt and should be omitted.

We say that a function $F : \mathbb{C} \to \mathbb{C}$ is quasi-homogeneouse of degree k and order m, if the following estimate hold;

$$|D^i F(\zeta)| \leq M|\zeta|^{k-i}, \text{ for } 0 \leq i \leq m,\ \zeta \in \mathbb{C}.$$

Lemma 2.4 (Kato[8]) Let $F \in C^j(\mathbb{C}; \mathbb{C})$, $j \in \mathbb{Z}$. Assume that there is $k \geq j$ such that

$$|D^i F(\zeta)| \leq M|\zeta|^{k-i}, \quad i = 1, 2, \ldots, j.$$

If $0 \leq \sigma \leq j$, then

$$\|\Lambda^\sigma F(\phi)\|_r \leq c\|\phi\|_q^{k-1}\|\Lambda^\sigma \phi\|_p, 1/r = 1/p + (k-1)/q,\ p, q, r \in (1, \infty),$$

where c depends on σ, p, q and r. Moreover,

$$\|\Lambda^\sigma F(u)\|_{r_1,r_2} \leq c\|u\|_{q_1,q_2}^{k-1}\|\Lambda^\sigma u\|_{p_1,p_2}, \text{for}, 1/r_j = 1/p_j + (k-1)/q_j,\ j = 1,2,$$

where c depends on σs, p_j, q_j and r_j. And if $F(\zeta)$ is a polynomial in ζ and $\bar{\zeta}$ of degree $k \geq 1$, then the above inequalities are true for any $\sigma \geq 0$.

Lemma 2.5 Assume (F1) and (F2). Suppose $u \in \bar{\mathcal{Y}}_s(I_0)$, where $I_0 = [0, T_0]$ with $T_0 < \infty$, and $\bar{\mathcal{X}}(I_0)$ is defined in (6) with I_0 instead of I. If $s > 1$, then $F(u) \in L^1(I_0; H^s)$. If $0 \leq s \leq 1$, and $1 \leq n \leq 2s + 4$, then $F(u) \in L^1(I_0; H^s)$. If $0 \leq s \leq 1$, and $n \geq 2s + 5$, then $F(u) \in L^1(I_0; H^s)$ with the additional assumption $k < 1 + 2/(n - 2s - 2)$.

Proof. We may estimate each $F_j(u)$, $j = 1, 2, \ldots, \{s\}, k$, in Lemma 2.3. It suffices to find (p_1, r_1), (p_2, r_2) which satisfy

$$\frac{j-1}{p_1} + \frac{1}{p_2} = \frac{1}{2}, \tag{7}$$

$$
\begin{aligned}
\frac{j-1}{r_1}+\frac{1}{r_2} &< 1, && (8)\\
\frac{1}{2}-\frac{s+1}{n} < \frac{1}{p_1} &\le \frac{1}{2}, && (9)\\
\frac{1}{2}-\frac{1}{n} < \frac{1}{p_2} &\le \frac{1}{2}, && (10)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{r_1} &= \max\{0, \frac{n}{2}(\frac{1}{2}-\frac{s}{n}-\frac{1}{p_1})/2\}, && (11)\\
\frac{1}{r_2} &= \frac{n}{2}(\frac{1}{2}-\frac{1}{p_2}), && (12)
\end{aligned}
$$

Indeed, by (10), (12) and the definition of $\bar{\mathcal{X}}(I_0)$, we have $\bar{\mathcal{X}}(I_0) \subset L^{p_2,r_2}$, and, set $1/p_3 = 1/p_1 + s/n$, then, from (9) and (11), it follows $\bar{\mathcal{X}}(I_0) \subset L^{p_3,r_1}$. Hence, by Sobolev's embedding theorem, we have $\bar{\mathcal{Y}}_s(I_0) \subset L^{p_1,r_1}$. And, if (7) and (8) hold, then we have, by Hölder inequality,

$$
||\Lambda^\sigma u F_j(u)||_{2,1} \ \le\ MT^{1-1/r3}||u||^{j-1}_{p_1,r_1}||\Lambda^\sigma u||_{p_2,r_2}, \qquad (13)
$$

where $(p-1)/r_1 + 1/r_2 = 1/r_3 < 1$ and for $j = 1, 2, \ldots, \{s\}, k$ and $0 \le \sigma \le s$. Hence the right hand side of (13) converge, respectively.

Actually, there are (p_1, r_1) and (p_2, r_2) satisfying the equalities and inequalities from (7) to (12). Solving () and (). we obtain that $1/p_1 = 2/nr_2(j-1)$. By $1/2-(s+1) < 1/p_1 \leq 1/2$, we have $1/2-(s+1) < 2/nr_2(j-1) \leq 1/2$, namely $(n-2s-2)(j-1)/4 < 1/r_2 \leq n(j-1)/4$. By $1 \leq j \leq k$, $(n-2s-2)(k-1)/4 < 1/r_2 \leq n(k-1)/4$. (10) and (12) implies that $2 < r_2 \leq \infty$. Since we assume the condition $p < 1+2/(n-2s-2)$, there is a desirable r_2.

Proof of Theorem 1. Recall that $u = \Gamma u_0 - iGF(u)$ is the unique solution of equation (1). Let $\varphi \in \mathcal{A}$. Then $\varphi u = \varphi\Gamma u_0 - i\varphi GF(u)$. Since $\|\varphi\Gamma u_0\|_{L^2(I;H^{s+1/2})} \leq C\|u_0\|_{H^s}$ by (3) with $\overline{H} = -\Delta$, $r = 0$, $\rho = s+1/2$, it suffices to estimate $\|\varphi GF(u)\|_{L^2(I;H^{s+1/2})}$. But it is easily seen that Theorem 3.1 in Canstantin and Saut[2] holds for $\varphi \in \mathcal{A}$ instead of $\chi \in C(\mathbb{R}^{n+1})$.

We note $\varphi(\check{x}_j) \in L^2(\mathbb{R}_{x_j})$, where $\check{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$, for $j = 1, 2, \dots, n$. Thus it suffices to prove $F(u) \in L^1(I; H^s)$, $s \geq 0$. By Lemma 2.5, we can prove Theorem.

References

[1] M. Ben-Artzi, Global estimates for Schrödinger equation, J. Funct. Anal. 107(1992)

[2] P. Constantin and J. C. Saut, Local smoothing properties of dispersive equations, J. Amer. Math. Soc. 1 (1988)

[3] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations, I. The Cauchy problem, General case, J. Funct. Anal. 32 (1979)

[4] J. Ginibre and G. Velo, The global Cauchy problem for the non linear Schrödinger equation revisited, Ann.Inst.Henri Poincaré, analyse non linéaire 2 (1985)

[5] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. Henri Poincaré, physique theorique 46 (1987)

[6] T. Kato, Nonlinear Schrödinger equations,in Schrödinger operators, Lecture Notes in Physics 345, Springer-Verlag, Berlin-New York, 1989

[7] T. Kato, An $L^{q,r}$-Theory for Nonlinear Schrödinger equations,in Spectral and Scattering Theory and Applications, Advanced Studies in Pure Math. 23, 1994

[8] T. Kato, On nonlinear Schrödinger equations II. H^s-solutions and unconditional well-posedness, J. d'Analyse Math. 67 (1995)

[9] T. Kato and K. Yajima, Some examples of smooth operators and the associated smoothing effect, Rev. Math. Phys., 1 (1989)

[10] Y. Nakamura, Regularity of solutions to nonlinear Schrödinger equations with H^2 initial data, Yokohama Math. J., 47 (1999)

[11] Y. Nakamura, Local solvability and smoothing effects of nonlinear Schrödinger equations with magnetic fields, preprint (1999)

[12] A. Ruiz and L. Vega, On local regularity of Schrödinger equations, Internat. Math. Res. Notices (1993)

[13] P. Sjögren and P. Sjölin, Local regularity of solutions to time-dependent Schrödinger equations with smooth potentials, Ann. Acad. Sci. Fenn. Ser. A. I. Math. 16 (1991)

[14] P. Sjölin, Regularity of solutions to the Schrödinger equations, Duke Math. J. 55 (1987)

[15] P. Sjölin, Regularity of solutions to nonlinear Schrödinger equations, Ark. mat. 28 (1990)

[16] P. Sjölin, Regularity of solutions to nonlinear equations of Schrödinger type, Tohoku Math. J. 45 (1993)

[17] Y. Tsutsumi, L^2-Solution for nonlinear Schrödinger equations and nonlinear groups, Funkcial. Ekvac. 30 (1987)

[18] Y. Tsutsumi, Global strong solutions for nonlinear Schrödinger equations, Nonlinear Anal. TMA 11 (1987)

[19] L. Vega, The Schrödinger equation : pointwise convergence to the initial data, Proc. Amer. Math Soc. 102 (1988)

[20] K. Yajima, Existence of solutions for Schrödinger evolution equations, Comm.Math.Phys. 110 (1987)

[21] K. Yajima, Schrödinger evolution equations with magnetic fields, J. d'Analyse Math. 56 (1991)