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Kyoto University
On smoothing properties of Nonlinear
Schrödinger Equations

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1 Introduction and Main Results

In this paper, we estimate smoothing properties of local solutions to nonlinear Schrödinger equations with initial data which belong to Sobolev spaces $H^s$, $s \geq 0$.

We consider the following equation;

\begin{equation}
    i\partial_t u = -\Delta u + F(u),
\end{equation}
where $u$ is a complex-valued function of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$, $\Delta$ is the Laplacian on $\mathbb{R}^n$, and $F(u) = F \circ u$ is a local nonlinear operator given by a complex-valued function $F$ on $\mathbb{C}$.

Here we consider the following assumptions on the nonlinear term $F$ in Kato[8];

**Assumption (F1)** $F \in C^{\{s\}}(\mathbb{C}; \mathbb{C})$, with $F(0) = 0$, where \( \{s\} = [s] + 1 \) if $s \not\in \mathbb{Z}$ and \( \{s\} = s \) if $s \in \mathbb{Z}$, and \( \{0\} = 1 \).

**Assumption (F2)** If $s > n/2$, no assumption.

If $s \leq n/2$ and if $F$ is a polynomial in $\zeta = \xi + i\eta$ and $\bar{\zeta}$, then degree of $F$ is equal to $k \leq 1 + 4/(n - 2s)$.

If $s \leq n/2$ and if $F$ is not a polynomial,

$$|D^i F(\zeta)| = \max_{0 \leq j \leq i} \{|\partial^j \zeta \partial_{\zeta}^{i-j} F|\} \leq M_i |\zeta|^{k-i}, \quad \text{for} \quad |\zeta| \geq 1,$$

for $i = 0, 1, \ldots, \{s\}$, where $\partial_\zeta = (\partial_\xi - i\partial_\eta)/2$, $\partial_{\bar{\zeta}} = (\partial_\xi + i\partial_\eta)/2$,
and $k$ is a finite number such that

$$\{s\} \leq k \leq 1 + \frac{4}{n-2s}.$$ 

We denote $\partial_x = (\partial_1, \ldots, \partial_n)$, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\partial_x^\alpha = (\partial_1^{\alpha_1}, \ldots, \partial_n^{\alpha_n})$ and we often denote $\partial_x$ by $\partial$, and use $\partial$ as $\partial^\alpha$ if $|\alpha| = 1$.

The Cauchy problems of equation (1) with above assumptions were studied by many authors. In Kato[6], Tsutsumi[17], they discussed about local or global wellposedness in the case that the initial data belongs to $L^2$, in Ginibre and Velo[3], Kato[5][6], they discussed about local wellposedness in the case that the initial data belongs to $H^1$, and in Ginibre and Velo[4], Kato[5][6], about the existence of global solution. Sjölin’s result[15] that we noted below was based on the existence of local solutions discussed in Kato[5]. In Kato[6], he discussed about local wellposedness in the case that the initial data belongs to $H^2$, and in
Kato[5][6], Tsutsumi[18], about the existence of global solution.

We try to estimate regularity of local solutions to equation (1) with \( u(0) = u_0 \in H^s \), for \( s \geq 0 \), which were obtained by Kato[8].

In Sjölin[14], he obtained the following inequality; For some \( C > 0 \), depending on \( \phi \in C_0^\infty(\mathbb{R}^{n+1}) \),

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^n} |\phi(t,x)(1-\Delta)^{1/4} e^{it\Delta} f|^2 \, dx \, dt \leq C \|f\|^2_{L^2}, \quad \forall f \in L^2. \tag{2}
\]

This inequality manifests that the free Schrödinger propagator \( e^{it\Delta} \) has the smoothing effect which can improve the properties of differentiability locally in time and space. Independently, Vega[19] was obtained the same as the local smoothing property for the free Schrödinger equation, Constantin and Saut[2] were obtained the same as the one for general dispersive equations in homogeneous or inhomogeneous cases.

Later the similar property for \( e^{-itH} \), where \( H = -\Delta + V \) is a self-adjoint operator and \( V = V(x) \) or \( V(t,x) \) are various scalar
potentials, was studied by many authors (Ben-Artzi[1], Ruiz and Vega[12], etc.). In particular, Kato and Yajima[9] obtained the inequality replacing $\phi$ in (2) by $(1 + |x|^2)^{-\frac{1}{4} - \varepsilon}$, $\varepsilon > 0$, and Yajima[21] obtained the similar estimate for the propagators of Schrödinger equations with time dependent magnetic and scalar potentials which may increase at infinity $|x| \to \infty$.

In Sjögren and Sjölin[13], they obtained the extension of (2) in the following form; They defined

$$\mathcal{A} \equiv \{ \varphi \in C^\infty(\mathbb{R}^n) \mid \text{There exists } \varepsilon > 0 \text{ such that } |\partial^\alpha \varphi(x)| \leq C_\alpha (1 + |x|)^{-\frac{1}{2} - \varepsilon}, \forall \alpha \},$$

where introduced mixed Sobolev spaces $H^{r,\rho} = H^{r,\rho}(\mathbb{R} \times \mathbb{R}^n) = (G_r \otimes G_\rho) * L^2(\mathbb{R}^{n+1})$, where $G_r$ and $G_\rho$ are Bessel kernels in $\mathbb{R}$ and $\mathbb{R}^n$, respectively. If $r \geq 0$, $\rho \geq 0$, then, for each $\varphi \in \mathcal{A}$, $\psi \in C_0^\infty(\mathbb{R})$,

$$\|\psi \varphi e^{-it\overline{H}} u\|_{H^{r,\rho}} \leq C_{\psi \varphi} \|u\|_{H^{mr+\rho-\frac{1}{2}(m-1)}},$$

(3)
for some $C_{\varphi, \psi} > 0$. Here $H = -P + V$, where $P$ is a elliptic operators with constant coefficient which degree is $m \geq 2$ and $V = V(x)$ is a real-valued function in $C^\infty$ with $D^\alpha V$ bounded for every $\alpha$.

In Sjölin[15][16] he adapted these estimates for equation (1), nonlinear Schödinger equation, with $H^1$ or $H^2$-initial data. The author extended in [10] $p$'s range when the initial data belong to $H^2$.

Kato[8] proved that assume (F1) and (F2), if $s \geq n/2$, or if $s < n/2$ and $k < 1 + \max\{4, 2s + 2\}/(n - 2s)$, and $k < 2/(1 - 2s)$ if $k = 1$, then there exists a number $T > 0$ and a unique solution $u \in C(I; H^s)$ of equation (1) with $u(0) = u_0 \in H^s$, $s > 0$, where $I = [0, T]$. Moreover $u \in \mathcal{Y}_s(I) \equiv \{u \in \mathcal{X}(I)|\Lambda^\sigma u \in \mathcal{X}(I), 0 \leq \sigma \leq s\}$, where

$$\mathcal{X}(I) \equiv (\cap L^s(I; L^q)) \cap C(I; L^2),$$  \hspace{1cm} (4)
where $\cap$ is the intersection in $(q, s)$ satisfying $1/q + 2/ns = 1/2$ and $1/2 - 1/n < 1/q \leq 1/2$ and $\Lambda^\sigma u = (-\Delta)^{\sigma/2}u$. In this paper we shall adapt Sjölin’s estimates to the solution of equation (1) with $H^s$-initial values. We obtain the following Theorem.

**Theorem 1.** Assume (F1) and (F2). If $s > 1$, then the unique solution $u \in C(I; H^s)$ of the equation (1) with $u(0) = u_0 \in H^s$ satisfies $\varphi u \in L^2(I; H^{s+1/2})$ for each $\varphi \in \mathcal{A}$. If $0 \leq s \leq 1$, then, in $1 \leq n \leq 2s + 4$, $\varphi u \in L^2(I; H^{s+1/2})$ for each $\varphi \in \mathcal{A}$. When $n \geq 2s + 5$, under the additional condition $p < 1 + 2/(n - 2s - 2)$ $\varphi u \in L^2(I; H^{s+1/2})$ for each $\varphi \in \mathcal{A}$.

**Remark 1** Applying this theorem to Theorem 1 in Yajima[21], we can obtain the regularity estimate of solutions to nonlinear Shrödinger equations with magnetic fields in the case that the initial data belongs to $H^s$. The author[11] obtained the results in the case of $H^1$. 

We use the standard notations. We abbreviate $L^p(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n)$ to $L^p$ and $H^k$, respectively. We denote usual $L^p$-norm by $\| \cdot \|_p$. For $I = [0, T]$, put $L^{p,r} = L^r(I; L^p)$, where $1 < p < \infty, 1 \leq r \leq \infty$, with its norm denoted by

$$
\| f \|_{p,r} \equiv \left( \int_I \| f(t) \|_p^r \, dt \right)^{\frac{1}{r}}.
$$

We denote various constants by $C, M, \ldots$. They may differ from line to line.

2 Proof of Theorem

We introduce the following linear operators.

$$(\Gamma \phi)(t) = U(t)\phi = e^{it\Delta} \phi, \quad t \in I, \quad (5)$$

$$(Gf)(t) = \int_0^t U(t - \tau)f(\tau)\, d\tau, \quad t \in I. \quad (6)$$

The following lemmas is well known (see, for example, Kato[5][6][7], Yajima[20])
Lemma 2.1 $\Gamma$ is a bounded operator from $L^2$ to $L^{p_1,r_1}$, where $(p_1, r_1)$ satisfies $1/p_1 + 2/nr_1 = 1/2$ and $1/2 - 1/n < 1/p_1 \leq 1/2$. The bound is independent of $T$, and is uniform for any $(p_1, r_1)$. Here $L^{2,\infty}$ may be replaced by $C(I;L^2)$.

Lemma 2.2 $G$ is a bounded operator from $L^{p_2,r_2}$ to $L^{p_1,r_1}$, where $(p_1, r_1)$ and $(p_2, r_2)$ satisfy $1/p_1 + 2/nr_1 = 1/2$ and $1/2 + 1/n < 1/p_1 \leq 1/2$, $1/p_2 + 2/nr_2 = 1/2 + 2/n$ and $1/2 \leq 1/p_2 < 1/2 + 1/n$, respectively. The bound is independent of $T$, and is uniform for any $(p_1, r_1)$ and $(p_2, r_2)$. Here $L^{2,\infty}$ may be replaced by $C(I;L^2)$.

For the nonlinear term $F$, the following lemma is well known.

Lemma 2.3 (Kato[8]) Assume (F1) and (F2) with $s \leq n/2$. If $F$ is a polynomial of degree $k$, it is obviously the sum of homogeneous polynomials of degree 1 to $k$. Otherwise, $F$ can
be written in the form

\[ F = F_1 + F_2 + \cdots + F_{\{s\}-1} + F_{\{s\}} + F_k, \]

where the \( F_j \) for \( j = 1, 2, \ldots, \{s\} - 1 \) are homogeneous polynomials of degree \( j \), while \( F_{\{s\}} \) and \( F_k \) are quasi-homogeneous of order \( \{s\} \) and of degree \( \{s\} \) and \( k \), respectively. If \( k = \{s\} \), \( F_{\{s\}} \) is redundant and should be omitted.

We say that a function \( F : \mathbb{C} \rightarrow \mathbb{C} \) is quasi-homogeneous of degree \( k \) and order \( m \), if the following estimate hold;

\[ |D^i F(\zeta)| \leq M|\zeta|^{k-i}, \text{ for } 0 \leq i \leq m, \ \zeta \in \mathbb{C}. \]

**Lemma 2.4** (Kato[8]) Let \( F \in C^j(\mathbb{C};\mathbb{C}), j \in \mathbb{Z} \). Assume that there is \( k \geq j \) such that

\[ |D^i F(\zeta)| \leq M|\zeta|^{k-i}, \quad i = 1, 2, \ldots, j. \]

If \( 0 \leq \sigma \leq j \), then

\[ \|\Lambda^\sigma F(\phi)\|_r \leq c\|\phi\|_q^{k-1}\|\Lambda^\sigma \phi\|_p, 1/r = 1/p + (k - 1)/q, \ p, q, r \in (1, \infty), \]
where $c$ depends on $\sigma$, $p$, $q$ and $r$. Moreover,

$$
\|\Lambda^\sigma F(u)\|_{r_1, r_2} \leq c \|u\|^{k-1}_{q_1, q_2} \|\Lambda^\sigma u\|_{p_1, p_2}, \text{ for } 1/r_j = 1/p_j + (k - 1)/q_j, \ j = 1, 2,
$$

where $c$ depends on $\sigma$, $p_j$, $q_j$ and $r_j$. And if $F(\zeta)$ is a polynomial in $\zeta$ and $\overline{\zeta}$ of degree $k \geq 1$, then the above inequalities are true for any $\sigma \geq 0$.

**Lemma 2.5** Assume (F1) and (F2). Suppose $u \in \tilde{Y}_{s}(I_0)$, where $I_0 = [0, T_0]$ with $T_0 < \infty$, and $\tilde{X}(I_0)$ is defined in (6) with $I_0$ instead of $I$. If $s > 1$, then $F(u) \in L^1(I_0; H^s)$. If $0 \leq s \leq 1$, and $1 \leq n \leq 2s + 4$, then $F(u) \in L^1(I_0; H^s)$. If $0 \leq s \leq 1$, and $n \geq 2s + 5$, then $F(u) \in L^1(I_0; H^s)$ with the additional assumption $k < 1 + 2/(n - 2s - 2)$.

**Proof.** We may estimate each $F_j(u)$, $j = 1, 2, \ldots, \{s\}, k$, in Lemma 2.3. It suffices to find $(p_1, r_1)$, $(p_2, r_2)$ which satisfy

$$
\frac{j - 1}{p_1} + \frac{1}{p_2} = \frac{1}{2}, \quad \text{ (7)}
$$
\[
\frac{j-1}{r_1} + \frac{1}{r_2} < 1, \quad (8)
\]
\[
\frac{1}{2} - \frac{s+1}{n} < \frac{1}{p_1} \leq \frac{1}{2}, \quad (9)
\]
\[
\frac{1}{2} - \frac{1}{n} < \frac{1}{p_2} \leq \frac{1}{2}, \quad (10)
\]

and
\[
\frac{1}{r_1} = \max\{0, \frac{n}{2} (\frac{1}{2} - \frac{s}{n} - \frac{1}{p_1})/2\}, \quad (11)
\]
\[
\frac{1}{r_2} = \frac{n}{2} (\frac{1}{2} - \frac{1}{p_2}), \quad (12)
\]

Indeed, by (10), (12) and the definition of $\bar{\mathcal{X}}(I_0)$, we have $\bar{\mathcal{X}}(I_0) \subset L^{p_3,r_2}$, and, set $1/p_3 = 1/p_1 + s/n$, then, from (9) and (11), it follows $\bar{\mathcal{X}}(I_0) \subset L^{p_3,r_1}$. Hence, by Sobolev’s embedding theorem, we have $\bar{\mathcal{Y}}_s(I_0) \subset L^{p_1,r_1}$. And, if (7) and (8) hold, then we have, by Hölder inequality,
\[
\|\Lambda^\sigma u F_j(u)\|_{2,1} \leq MT^{1-1/r_3} \|u\|_{p_1,r_1}^{j-1} \|\Lambda^\sigma u\|_{p_2,r_2}, \quad (13)
\]
where $(p-1)/r_1 + 1/r_2 = 1/r_3 < 1$ and for $j = 1, 2, \ldots, \{s\}$, $k$ and $0 \leq \sigma \leq s$. Hence the right hand side of (13) converge,
respectively.

Actually, there are \((p_1, r_1)\) and \((p_2, r_2)\) satisfying the equalities and inequalities from (7) to (12). Solving (1) and (6), we obtain that \(1/p_1 = 2/nr_2(j-1)\). By \(1/2-(s+1) < 1/p_1 \leq 1/2\), we have \(1/2-(s+1) < 2/nr_2(j-1) \leq 1/2\), namely \((n-2s-2)(j-1)/4 < 1/r_2 \leq n(j-1)/4\). By \(1 \leq j \leq k\), \((n-2s-2)(k-1)/4 < 1/r_2 \leq n(k-1)/4\). (10) and (12) implies that \(2 < r_2 \leq \infty\). Since we assume the condition \(p < 1+2/(n-2s-2)\), there is a desirable \(r_2\).

Proof of Theorem 1. Recall that \(u = \Gamma u_0 - iGF(u)\) is the unique solution of equation (1). Let \(\varphi \in A\). Then \(\varphi u = \varphi \Gamma u_0 - i\varphi GF(u)\). Since \(\|\varphi \Gamma u_0\|_{L^2(I; H^{s+1/2})} \leq C\|u_0\|_{H^s}\) by (3) with \(H = -\Delta\), \(r = 0\), \(\rho = s + 1/2\), it suffices to estimate \(\|\varphi GF(u)\|_{L^2(I; H^{s+1/2})}\). But it is easily seen that Theorem 3.1 in Canstantin and Saut [2] holds for \(\varphi \in A\) instead of \(\chi \in C(\mathbb{R}^{n+1})\).
We note $\varphi(\check{x}_j) \in L^2(\mathbb{R}_{x_j})$, where $\check{x}_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$, for $j = 1, 2, \ldots, n$. Thus it suffices to prove $F(u) \in L^1(I; H^s)$, $s \geq 0$. By Lemma 2.5, we can prove Theorem.

References


