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A REMARK ON THE NON-SCARRING OF $-\triangle u_j = \lambda_j u_j$.

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ABSTRACT. Scar is the singular support of the mass distribution related to Laplace eigenfunctions. We claim that if the singular support exists, then the Haussdorff dimension is at least 1. For example there exists a subseqence of eigenfunctions such that this singular support is a closed geodesic curve.

§1.Introduction and results.

Let (M, g) be a compact Riemannian manifold without boundary, and let (λ_j, u_j) be eigenvalues and normalized eigenfunctioin of $-\triangle$. So $\{u_j\}$ forms a complete orthonormal base in $L^2(M)$.

In this paper, our main concern is the property of the probability measure $d\nu_j = |u_j(x)|^2 dvol_M$, when $j \to \infty$.

The typical example are as follows.

In the asymptotic theory of high-frequency eigenfunctions, if the phase flow on the cosphere bundle S^*M is ergodic, then the "almost all" of the probability measure are asymptotically uniformly distributed. So there exists a subsequence satisfying $d\nu_{j_k} \to dvol_M$(as $k \to \infty$).(See [1,2,3,4].)

On the other hand, another example illustrates different behavior of eigenfunctions. It is a subsequence of eigenfunctions concentrated near the stable closed geodesic line γ on M. In this case, $d\nu_{j_k} \to \delta_\gamma dvol_M$(as $k \to \infty$), where $\delta_\gamma dvol_M$ denotes a measure distributed uniformly along γ.(See[5].) We call ν_{j_k} scars to γ in this case, too.

Moreover we generally conjecture a mixed type of concentration of eigenfunctions called "Scar".(See [6].) "Scar" is the singular part of mass distribution of eigenfunctions (rigorous definition is given below), and a few things are known about "Scar".(See [6].) In this paper, we claim that strong scarring on isolated points is impossible.(See §2.) Furthermore if the one-dimensional Hausdorff measure $H^1(S) = 0$, then strong scarring on S is impossible.(See §3.) This is the best possible estimate for general compact manifolds.(See the above example.)

§2. Non-scarring on isolated points.

In this section we define "Scar", and we proof that the strong scarring on isolated points is impossible.

Definition(Scar)(See [6].). A subsequence ν_{j_k} is said to scar strongly to a closed subset $S \in M$ if $\nu_{j_k} \rightarrow \mu$ and supp$\mu_s \in S$, where $\mu = \mu_s + \mu_r$ is the Lebesgue decomposition of μ into *singular parts* and *regular parts* with respect to vol_M(the volume form on (M, g)).

We have the next theorem.

Theorem1. *A subsequence ν_{j_k} scars strongly to a closed subset S and let $x_0 \in S$ be an isolated point. Then ν_{j_k} scars to $S \backslash \{x_0\}$.*

Proof. We assume that a subsequence ν_{j_k} scars strongly to a closed subset S and let $x_0 \in S$ be an isolated point. (i.e. there exists an openset M' such that $x_0 \in M'$ and $S \cap M' = \{x_0\}$.)

Let $\phi_\epsilon(x) \in C_0^\infty(M)$ be a smooth real valued function satisfying $\phi_\epsilon(x_0) = 1$ with a compact support $B_\epsilon(x_0) \equiv \{x; dist(x, x_0) < \epsilon\}$, where $dist(x, y)$ denotes the Riemannian distance on M.

We consider the following estimate.

$$
\begin{aligned}
\int_M \phi_\epsilon(x) d\nu_{j_k} =& \int_M \phi_\epsilon(x) |u_{j_k}|^2 dvol_M \\
=& \langle \phi_\epsilon(x) u_{j_k}, u_{j_k} \rangle_{L^2(M)} \\
=& \langle e^{-it\sqrt{\lambda_{j_k}}} u_{j_k}, \phi_\epsilon(x) e^{it\sqrt{\lambda_{j_k}}} u_{j_k} \rangle_{L^2(M)} \\
=& \langle e^{-it\sqrt{-\triangle}} u_{j_k}, \phi_\epsilon(x) e^{it\sqrt{-\triangle}} u_{j_k} \rangle_{L^2(M)}
\end{aligned}
$$

where $\langle , \rangle_{L^2(M)}$ denotes the scalar product in $L^2(M)$.

The Egorov theorem states, that if $\hat{A}$ is a pseudo-differential operator with principal symbol $A(x,\xi) \in C_0^\infty(S^*M)$, then $e^{-it\sqrt{\triangle}} \hat{A} e^{it\sqrt{\triangle}}$ is also a pseudo-differential operator, and its principal symbol is $\exp(tX)^* A(x,\xi)$. Here $\exp(tX)$ is a Hamiltonian phase flow in S^*M generated by the Hamiltonian function $H = \sqrt{g(\xi,\xi)}$. We applies the Egorov theorem to $\phi_\epsilon(x)$.

So we have

$$
\liminf_{k\to\infty} \langle e^{-it\sqrt{\triangle}} u_{j_k}, \phi_\epsilon(x) e^{it\sqrt{\triangle}} u_{j_k} \rangle_{L^2(M)} = \liminf_{k\to\infty} \langle exp(tX)^* \phi_\epsilon(x) u_{j_k}, u_{j_k} \rangle_{L^2(M)} \cdots (1)
$$

where we consider that $\phi_\epsilon(x)$ is a pseudo-differntial opeataor with prinipal symbol $\pi^*\phi_\epsilon(x)$. Here $\pi : S^*M \to M$ is a projection operator on the cosphere bundle.

We assume that u_{j_k} scars to S and S contains isolated point x_0.

(i.e. $|u_{j_k}|^2 dvol \to const.\delta_{x_0} + \cdots$)

Therefore

$$
\liminf_{k\to\infty} \int_M \phi_\epsilon(x) d\nu_{j_k} = \liminf_{k\to\infty} \int_M \phi_\epsilon(x) |u_{j_k}|^2 dvol_M = const > 0 (\text{indep. of } \epsilon). \cdots (2).
$$

By (1) and (2), we obtain

$$
\liminf_{k\to\infty} \langle exp(tX)^* \phi_\epsilon(x) u_{j_k}, u_{j_k} \rangle_{L^2(M)} = const > 0 (\text{indep. of } t \text{ and } \epsilon). \cdots (3).
$$

On the other hand, $exp(tX)^*\phi_{\epsilon(x)}$ is a smooth function with compact support $B_{t,\epsilon} = \{(x,p) \in S^*M : \exp(tX) B_\epsilon(x_0)\}$, and $vol_M(\pi B_{t,\epsilon}) \to 0 \quad (\epsilon \to 0)$. By the assumption of this theorem, there exists $t > 0$ satisfying $\pi B_{t,\epsilon} \subset \mathrm{supp}(\mu_r)$ for all small $\epsilon > 0$. By applying Garding inequality, we have

$$\begin{aligned}\limsup_{k\to\infty}\langle exp(tX)^*\phi_\epsilon(x)u_{j_k}, u_{j_k}\rangle_{L^2(M)} &\le \limsup_{k\to\infty}\int_{B_{t,\epsilon}}|u_{j_k}|^2 dvol_{S^*M}\\ &\le \limsup_{k\to\infty}\int_{\pi B_{t,\epsilon}}|u_{j_k}|^2 dvol_M\\ &= \mu(\pi B_{t,\epsilon})\\ &= \mu_r(\pi B_{t,\epsilon})\\ &\le \exists\mathrm{const}\ vol(\pi B_{t,\epsilon}) \to 0 \quad \text{as } \epsilon \to 0.\end{aligned}$$

which is a contradiction of (3), thus we have proved the theorem.

Corollary. *Let a subsequence ν_{j_k} scars to $\bigcup_{i=1}^n\{x_i\}$. Then ν_{j_k} scars to $\emptyset$. Thus ν_{j_k} converges to some regular measure weakly.*

§3. Non-Scarring on Cantor-like sets.

Next we proof that if the closed set S satisfies one-dimensional Hausdorff measure $H^1(S) = 0$, then strong scarring on S is impossible. This proof is the same method as the above theorem.

Definition. Let $S \subset \boldsymbol{R}^n$ be a set, $0 \le s < \infty, 0 < \delta \le \infty$. Define

$$H^s_\delta(S) \equiv \inf\{\sum_{j=1}^{\infty}\alpha(s)(\frac{diam(C_j)}{2})^s | S \subset \bigcup C_j, diam(C_j) \le \delta\}$$

Here $\Gamma(s) \equiv \int_0^\infty e^{-x}x^{s-1}dx, (0 < s < \infty)$ is the usual gamma function, $\alpha(s) = \frac{\pi^{s/2}\Gamma(s)}{\Gamma(s/2+1)}$, $\{C_j\}$ is a collection of closed balls, and $diam(C_j)$ means the diameter of C_j.

Definition(s-dimensional Hausdorff measure)(See[7].). For S and s as above, define

$$H^s(S) \equiv \lim_{\delta\to 0} H^s_\delta(S) = \sup_{\delta>0} H^s_\delta(S)$$

We call H^s s-dimensional Hausdorff measure on $\boldsymbol{R}^n$

Lemma. *Let $f : \boldsymbol{B}^n \to \boldsymbol{R}$ be Lipschitz, $S \subset \boldsymbol{B}^n, 0 \leq s < \infty$. Then*

$$H^s(f(S)) \leq (Lip(f))^s H^s(S),$$

where B^n is a n-dimensional closed ball in $\boldsymbol{R}^n$, $Lip(f)$ means the Lipschitz constant of f.

Proof. Fix $\delta > 0$ and choose sets $\{C_j\} \subset \boldsymbol{B}^n$ such that $diam(C_j) \leq \delta, S \subset \bigcup_{i=1}^{\infty} C_i$. Then $diam(f(C_i)) \leq Lip(f) diam(C_i) \leq Lip(f)\delta$ and $f(S) \subset \bigcup_{i=1}^{\infty} f(C_i)$.Thus

$$\begin{aligned} H^s_{Lip(f)\delta} f(S) \leq & \sum_{i=1}^{\infty} \alpha(s) (\frac{diam f(C_i)}{2})^s \\ \leq & (Lip(f))^s \sum_{i=1}^{\infty} \alpha(s) (\frac{diam f(C_i)}{2})^s. \end{aligned}$$

Taking infima over all such sets$\{C_i\}$, we find

$$H^s_{Lip(f)\delta} f(S) \leq (Lip(f))^s H^s_\delta(S).$$

Send $\delta \to 0$ to finish the proof.

Key lemma. *Let $S \subset \boldsymbol{R}^n$ be a closed set satisfying $H^1(S) = 0$. Then for all $x_0 \in S, \epsilon > 0$, there exists an annulus*

$$A_\delta(x_0, \epsilon') \equiv \{x \in \boldsymbol{R}^n | 0 < \epsilon' \leq dist|x - x_0| \leq \epsilon' + \delta\}$$

such that $S \cap A_\delta(x_0, \epsilon') = \emptyset$ and $diam(A_\delta(x_0, \epsilon')) \leq \epsilon$.

proof. S is a closed set. So if the statement is not true, we may assume there exists $x_0 \in S, \epsilon > 0$ such that $A_0(x_0, \epsilon') \cap S \neq \emptyset$ for all $0 \leq \epsilon' \leq \epsilon$.

Let $f : B_\epsilon(x_0) \ni (r, \theta) \to \boldsymbol{R} \ni r$ be a radial function, where $B_\epsilon(x_0)$ is a closed ball with radius ϵ. Therefore f is a Lipschitz continuous. We apply the above lemma for f. So we have

$$\epsilon = H^1([0, \epsilon]) = H^1(f(B_\epsilon(x_0))) \leq (Lip(f))^1 H^1((B_\epsilon(x_0))) \leq (Lip(f))^1 H^1(S).$$

$Lip(f) = 1$, thus $H^1(S) \geq \epsilon > 0$.This is a contradiction.

By the following corollary, we may assume the uniform estimate for $\delta > 0$(the width of the annulus).

Corollary. *Let $S \subset \boldsymbol{R}^n$ be a compact set. For all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$A_\delta(x', \epsilon'(x')) \equiv \{x \in \boldsymbol{R}^n | \epsilon'(x') \leq dist|x - x'| \leq \epsilon'(x') + \delta < \epsilon\} \cap S = \emptyset \; for \; all \; x' \in S,$$

where $\epsilon'(x') > 0$ depends on x'.

proof. S is a compact set. The usual covering statement means the uniformity of δ.

Remark. On compact Riemannian manifolds, we can easily show the same lemma. So we use the lemma on compact manifolds.

Using the above corollary, we obtain the following main theorem.

Theorem2(Non-scarring). *A subsequence ν_{j_k} scars strongly to a closed subset S and $H^1(S) = 0$. Then ν_{j_k} scars to $\emptyset$.*

Remark. This theorem states if the strong scarring on closed sets happens, the Hausdorff dimension is larger than 1.So strong scarring on Cantor-like sets is impossible.

Remark. $H^1(S) = 0$ is the best possible estimate.For example, there exists the strong scarring on the stable closed lineγ.(See[5].) This means if $H^1(S) > 0$, strong scarring on S is possible.

Proof. This proof is the same method as Theorem 1.

We assume that a subsequence ν_{j_k} scars strongly to a closed subset S and $H^1(S) = 0$.

By the definition of the one-dimensional Haussdorff measure, for all small $L > 0$, $\epsilon > 0$ there exists a finite cover $S \subset \bigcup_{l=1}^n B_l$ such that $diam(B_l) < \epsilon$ and $\sum_{l=1}^n diam(B_l) < L$.

We fix $\delta > 0$ as the above corollary and we assume $0 < \epsilon < \delta$.

Let $\phi^l(x) \in C_0^\infty(M)$ be a partition of unity satisfying $\sum\limits_{l=1}^n \phi^l(x) = 1$ on S with a compact support $support(\phi^l(x)) \subset B_l$.

We consider the following estimate.

$$\begin{aligned}
\nu(S) &\leq \liminf_{k\to\infty} \int_M \sum_{l=1}^{n} \phi^l(x) d\nu_{j_k} \\
&= \liminf_{k\to\infty} \sum_{l=1}^{n} \int_M \phi^l(x) |u_{j_k}|^2 dvol_M \\
&= \liminf_{k\to\infty} \sum_{l=1}^{n} \langle \phi^l(x) u_{j_k}, u_{j_k} \rangle_{L^2(M)} \\
&= \liminf_{k\to\infty} \sum_{l=1}^{n} \langle e^{-it\sqrt{\lambda_{j_k}}} u_{j_k}, \phi^l(x) e^{it\sqrt{\lambda_{j_k}}} u_{j_k} \rangle_{L^2(M)} \\
&= \liminf_{k\to\infty} \sum_{l=1}^{n} \langle e^{-it\sqrt{-\triangle}} u_{j_k}, \phi^l(x) e^{it\sqrt{-\triangle}} u_{j_k} \rangle_{L^2(M)}
\end{aligned}$$

We applies the Egorov theorem for $\phi^l(x)$.So we have

$$\liminf_{k\to\infty} \langle e^{-it\sqrt{\triangle}} u_{j_k}, \phi^l(x) e^{it\sqrt{\triangle}} u_{j_k} \rangle_{L^2(M)} = \liminf_{k\to\infty} \langle exp(tX)^* \phi^l(x) u_{j_k}, u_{j_k} \rangle_{L^2(M)}$$

By the above lemma, we can choose $t_l > 0$(uniform bounded) satisfying $A_\delta(x_l, t_l) \cap S = \emptyset$. Thererfore by applying Garding inequality, we have

$$\begin{aligned}
\liminf_{k\to\infty} \langle \sum_{l=1}^{n} \exp(t_l X)^* \phi^l(x) u_{j_k}, u_{j_k} \rangle_{L^2(M)} &\leq \liminf_{k\to\infty} \sum_{l=1}^{n} \int_{A_\delta(x_l,t_l)} |u_{j_k}|^2 dvol_{S^*M} \\
&= \mu(\bigcup_{l=1}^{n} (A_\delta(x_l, t_l))) \\
&\leq \exists C vol_M(\bigcup_{l=1}^{n} (A_\delta(x_l, t_l))) \\
&\leq \exists C' \sum_{l=1}^{n} diam(B_l) \\
&\leq \exists C' L,
\end{aligned}$$

where $C' > 0$ is independent of L. For all $L > 0$, we obtain $\mu(S) < C'L$. Thus we obtain $\mu_s(S) \leq \mu(S) = 0$.

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