A REMARK ON THE NON-SCARRING OF $\triangle u_j = \lambda_j u_{j \cdot}$ (Microlocal Analysis of the Schrödinger Equation and Related Topics)

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Citation
数理解析研究所講究録 2000, 1176: 18-25

Issue Date
2000-11

URL
http://hdl.handle.net/2433/64496

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
A REMARK ON THE NON-SCARING OF $-\triangle u_j = \lambda_j u_j$.

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ABSTRACT. Scar is the singular support of the mass distribution related to Laplace eigenfunctions. We claim that if the singular support exists, then the Hausdorff dimension is at least 1. For example there exists a subsequence of eigenfunctions such that this singular support is a closed geodesic curve.

§1. Introduction and results.

Let $(M, g)$ be a compact Riemannian manifold without boundary, and let $(\lambda_j, u_j)$ be eigenvalues and normalized eigenfunction of $-\Delta$. So $\{u_j\}$ forms a complete orthonormal base in $L^2(M)$.

In this paper, our main concern is the property of the probability measure $d\nu_j = |u_j(x)|^2 dvol_M$, when $j \to \infty$.

The typical example are as follows.

In the asymptotic theory of high-frequency eigenfunctions, if the phase flow on the cosphere bundle $S^* M$ is ergodic, then the "almost all" of the probability measure are asymptotically uniformly distributed. So there exists a subsequence satisfying $d\nu_{j_k} \to dvol_M (as \ k \to \infty).$ (See [1,2,3,4].)

On the other hand, another example illustrates different behavior of eigenfunctions. It is a subsequence of eigenfunctions concentrated near the stable closed geodesic line $\gamma$ on $M$. In this case, $d\nu_{j_k} \to \delta_{\gamma} dvol_M (as \ k \to \infty)$, where $\delta_{\gamma} dvol_M$ denotes a measure distributed uniformly along $\gamma$. (See[5].) We call $\nu_{j_k}$ scars to $\gamma$ in this case, too.
Moreover we generally conjecture a mixed type of concentration of eigenfunctions called "Scar". (See [6].) "Scar" is the singular part of mass distribution of eigenfunctions (rigorous definition is given below), and a few things are known about "Scar". (See [6].) In this paper, we claim that strong scarring on isolated points is impossible. (See §2.) Furthermore if the one-dimensional Hausdorff measure $H^1(S) = 0$, then strong scarring on $S$ is impossible. (See §3.) This is the best possible estimate for general compact manifolds. (See the above example.)

§2. Non-scarring on isolated points.

In this section we define "Scar", and we prove that the strong scarring on isolated points is impossible.

**Definition (Scar) (See [6].)** A subsequence $\nu_{j_k}$ is said to scar strongly to a closed subset $S \in M$ if $\nu_{j_k} \rightarrow \mu$ and $\text{supp} \mu_s \in S$, where $\mu = \mu_s + \mu_r$ is the Lebesgue decomposition of $\mu$ into singular parts and regular parts with respect to $\text{vol}_M$ (the volume form on $(M, g)$).

We have the next theorem.

**Theorem 1.** A subsequence $\nu_{j_k}$ scars strongly to a closed subset $S$ and let $x_0 \in S$ be an isolated point. Then $\nu_{j_k}$ scars to $S \setminus \{x_0\}$.

**Proof.** We assume that a subsequence $\nu_{j_k}$ scars strongly to a closed subset $S$ and let $x_0 \in S$ be an isolated point. (i.e. there exists an openset $M'$ such that $x_0 \in M'$ and $S \cap M' = \{x_0\}$.)

Let $\phi_\epsilon(x) \in C_0^\infty(M)$ be a smooth real valued function satisfying $\phi_\epsilon(x_0) = 1$ with a compact support $B_\epsilon(x_0) \equiv \{x; \text{dist}(x, x_0) < \epsilon\}$, where $\text{dist}(x, y)$ denotes the Riemannian distance on $M$. 
We consider the following estimate.

\[ \int_{M} \phi_{\epsilon}(x) d\nu_{j_{k}} = \int_{M} \phi_{\epsilon}(x) |u_{j_{k}}|^{2} d\text{vol}_{M} \]

\[ = \langle \phi_{\epsilon}(x) u_{j_{k}}, u_{j_{k}} \rangle_{L^{2}(M)} \]

\[ = \langle e^{-it\sqrt{\lambda_{j_{k}}}} u_{j_{k}}, \phi_{\epsilon}(x) e^{it\sqrt{\lambda_{j_{k}}}} u_{j_{k}} \rangle_{L^{2}(M)} \]

\[ = \langle e^{-it\sqrt{-\Delta}} u_{j_{k}}, \phi_{\epsilon}(x) e^{it\sqrt{-\Delta}} u_{j_{k}} \rangle_{L^{2}(M)} \]

where \( \langle , \rangle_{L^{2}(M)} \) denotes the scalar product in \( L^{2}(M) \).

The Egorov theorem states, that if \( \hat{A} \) is a pseudo-differential operator with principal symbol \( A(x, \xi) \in C_{0}^{\infty}(S^{*}M) \), then \( e^{-it\sqrt{\Delta}} \hat{A} e^{it\sqrt{\Delta}} \) is also a pseudo-differential operator, and its principal symbol is \( \exp(tX)^{*} A(x, \xi) \). Here \( \exp(tX) \) is a Hamiltonian phase flow in \( S^{*}M \) generated by the Hamiltonian function \( H = \sqrt{g(\xi, \xi)} \). We apply the Egorov theorem to \( \phi_{\epsilon}(x) \).

So we have

\[ \liminf_{k \rightarrow \infty} \inf_{\infty} \langle \exp(tX)^{*} \phi_{\epsilon}(x) u_{j_{k}}, u_{j_{k}} \rangle_{L^{2}(M)} = \liminf_{k \rightarrow \infty} \langle \exp(tX)^{*} \phi_{\epsilon}(x) u_{j_{k}}, u_{j_{k}} \rangle_{L^{2}(M)} \cdots (1) \]

where we consider that \( \phi_{\epsilon}(x) \) is a pseudo-differential operator with principal symbol \( \pi^{*} \phi_{\epsilon}(x) \). Here \( \pi : S^{*}M \rightarrow M \) is a projection operator on the cosphere bundle.

We assume that \( u_{j_{k}} \) scars to \( S \) and \( S \) contains isolated point \( x_{0} \).

(i.e. \( |u_{j_{k}}|^{2} d\text{vol} \rightarrow \text{const.} \delta_{x_{0}} + \cdots \))

Therefore

\[ \liminf_{k \rightarrow \infty} \int_{M} \phi_{\epsilon}(x) d\nu_{j_{k}} = \liminf_{k \rightarrow \infty} \int_{M} \phi_{\epsilon}(x) |u_{j_{k}}|^{2} d\text{vol}_{M} = \text{const} > 0 (\text{indep. of } \epsilon) \cdots (2) \]

By (1) and (2), we obtain

\[ \liminf_{k \rightarrow \infty} \langle \exp(tX)^{*} \phi_{\epsilon}(x) u_{j_{k}}, u_{j_{k}} \rangle_{L^{2}(M)} = \text{const} > 0 (\text{indep. of } t \text{ and } \epsilon) \cdots (3) \]

On the other hand, \( \exp(tX)^{*} \phi_{\epsilon}(x) \) is a smooth function with compact support \( B_{t, \epsilon} = \{(x, p) \in S^{*}M : \exp(tX) B_{\epsilon}(x_{0})\} \), and \( \text{vol}_{M}(\pi B_{t, \epsilon}) \rightarrow 0 \ (\epsilon \rightarrow 0) \). By the
assumption of this theorem, there exists $t > 0$ satisfying $\pi B_{t,\epsilon} \subset \text{supp}(\mu_{r})$ for all small $\epsilon > 0$. By applying Garding inequality, we have

$$
\limsup_{k \to \infty} \langle \exp(tX)^{\ast} \phi_{\epsilon}(x) u_{j_k}, u_{j_k} \rangle_{L^{2}(M)} \leq \limsup_{k \to \infty} \int_{B_{t,\epsilon}} |u_{j_k}|^{2}dvol_{S^{\ast}M} \\
\leq \limsup_{k \to \infty} \int_{\pi B_{t,\epsilon}} |u_{j_k}|^{2}dvol_{M} \\
= \mu(\pi B_{t,\epsilon}) \\
= \mu_{r}(\pi B_{t,\epsilon}) \\
\leq \exists \text{const} \ vol(\pi B_{t,\epsilon}) \to 0 \text{ as } \epsilon \to 0.
$$

which is a contradiction of (3), thus we have proved the theorem.

**Corollary.** Let a subsequence $\nu_{j_k}$ scars to $\bigcup_{i=1}^{n}\{x_i\}$. Then $\nu_{j_k}$ scars to $\emptyset$. Thus $\nu_{j_k}$ converges to some regular measure weakly.

§3. Non-Scarring on Cantor-like sets.

Next we proof that if the closed set $S$ satisfies one-dimensional Hausdorff measure $H^{1}(S) = 0$, then strong scarring on $S$ is impossible. This proof is the same method as the above theorem.

**Definition.** Let $S \subset R^{n}$ be a set, $0 \leq s < \infty, 0 < \delta \leq \infty$. Define

$$H^{s}_{\delta}(S) \equiv \inf \left\{ \sum_{j=1}^{\infty} \alpha(s)(\frac{\text{diam}(C_j)}{2})^{s} |S \subset \bigcup C_j, \text{diam}(C_j) \leq \delta \right\}
$$

Here $\Gamma(s) \equiv \int_{0}^{\infty} e^{-x}x^{s-1}dx,(0 < s < \infty)$ is the usual gamma function, $\alpha(s) = \frac{\pi^{s/2}\Gamma(s)}{\Gamma(s/2+1)}$, $\{C_j\}$ is a collection of closed balls, and $\text{diam}(C_j)$ means the diameter of $C_j$.

**Definition**($s$-dimensional Hausdorff measure)(See[7].). For $S$ and $s$ as above, define

$$H^{s}(S) \equiv \lim_{\delta \to 0} H^{s}_{\delta}(S) = \sup_{\delta > 0} H^{s}_{\delta}(S)
$$

We call $H^{s}$ $s$-dimensional Hausdorff measure on $R^{n}$.
Lemma. Let \( f : B^n \rightarrow \mathbb{R} \) be Lipschitz, \( S \subset B^n, 0 \leq s < \infty \). Then
\[
H^s(f(S)) \leq (\text{Lip}(f))^s H^s(S),
\]
where \( B^n \) is a \( n \)-dimensional closed ball in \( \mathbb{R}^n \), \( \text{Lip}(f) \) means the Lipschitz constant of \( f \).

Proof. Fix \( \delta > 0 \) and choose sets \( \{C_j\} \subset B^n \) such that \( \text{diam}(C_j) \leq \delta, S \subset \bigcup_{i=1}^{\infty} C_i \).

Then \( \text{diam}(f(C_i)) \leq \text{Lip}(f) \text{diam}(C_i) \leq \text{Lip}(f) \delta \) and \( f(S) \subset \bigcup_{i=1}^{\infty} f(C_i) \). Thus
\[
H^s_{\text{Lip}(f)\delta}f(S) \leq \sum_{i=1}^{\infty} \alpha(s) \left( \frac{\text{diam}(f(C_i))}{2} \right)^s 
\leq (\text{Lip}(f))^s \sum_{i=1}^{\infty} \alpha(s) \left( \frac{\text{diam}(C_i)}{2} \right)^s.
\]
Taking infima over all such sets \( \{C_i\} \), we find
\[
H^s_{\text{Lip}(f)\delta}f(S) \leq (\text{Lip}(f))^s H^s_{\delta}(S).
\]

Send \( \delta \rightarrow 0 \) to finish the proof.

Key lemma. Let \( S \subset \mathbb{R}^n \) be a closed set satisfying \( H^1(S) = 0 \). Then for all \( x_0 \in S, \epsilon > 0 \), there exists an annulus
\[
A_\delta(x_0, \epsilon') \equiv \{ x \in \mathbb{R}^n | 0 < \epsilon' \leq \text{dist}|x - x_0| \leq \epsilon' + \delta \}
\]
such that \( S \cap A_\delta(x_0, \epsilon') = \emptyset \) and \( \text{diam}(A_\delta(x_0, \epsilon')) \leq \epsilon \).

Proof. \( S \) is a closed set. So if the statement is not true, we may assume there exists \( x_0 \in S, \epsilon > 0 \) such that \( A_0(x_0, \epsilon') \cap S \neq \emptyset \) for all \( 0 \leq \epsilon' \leq \epsilon \).

Let \( f : B_\epsilon(x_0) \ni (r, \theta) \rightarrow \mathbb{R} \ni r \) be a radial function, where \( B_\epsilon(x_0) \) is a closed ball with radius \( \epsilon \). Therefore \( f \) is a Lipschitz continuous. We apply the above lemma for \( f \). So we have
\[
\epsilon = H^1([0, \epsilon]) = H^1(f(B_\epsilon(x_0))) \leq (\text{Lip}(f))^1 H^1((B_\epsilon(x_0))) \leq (\text{Lip}(f))^1 H^1(S).
\]
\( \text{Lip}(f) = 1 \), thus \( H^1(S) \geq \epsilon > 0 \). This is a contradiction.

By the following corollary, we may assume the uniform estimate for \( \delta > 0 \) (the width of the annulus).
Corollary. Let $S \subset \mathbb{R}^n$ be a compact set. For all $\epsilon > 0$, there exists $\delta > 0$ such that

$$A_\delta(x', \epsilon'(x')) \equiv \{x \in \mathbb{R}^n | \epsilon'(x') \leq \text{dist}|x - x'| \leq \epsilon'(x') + \delta < \epsilon \} \cap S = \emptyset \text{ for all } x' \in S,$$

where $\epsilon'(x') > 0$ depends on $x'$.

proof. $S$ is a compact set. The usual covering statement means the uniformity of $\delta$.

Remark. On compact Riemannian manifolds, we can easily show the same lemma. So we use the lemma on compact manifolds.

Using the above corollary, we obtain the following main theorem.

Theorem 2 (Non-scarring). A subsequence $\nu_{j_k}$ scars strongly to a closed subset $S$ and $H^1(S) = 0$. Then $\nu_{j_k}$ scars to $\emptyset$.

Remark. This theorem states if the strong scarring on closed sets happens, the Hausdorff dimension is larger than 1. So strong scarring on Cantor-like sets is impossible.

Remark. $H^1(S) = 0$ is the best possible estimate. For example, there exists the strong scarring on the stable closed line $\gamma$. (See [5].) This means if $H^1(S) > 0$, strong scarring on $S$ is possible.

Proof. This proof is the same method as Theorem 1.

We assume that a subsequence $\nu_{j_k}$ scars strongly to a closed subset $S$ and $H^1(S) = 0$.

By the definition of the one-dimensional Hausdorff measure, for all small $L > 0$, $\epsilon > 0$ there exists a finite cover $S \subset \bigcup_{l=1}^n B_l$ such that $\text{diam}(B_l) < \epsilon$ and $\sum_{l=1}^n \text{diam}(B_l) < L$.

We fix $\delta > 0$ as the above corollary and we assume $0 < \epsilon < \delta$.

Let $\phi^l(x) \in C^\infty_0(M)$ be a partition of unity satisfying $\sum_{l=1}^n \phi^l(x) = 1$ on $S$ with a compact support $\text{support}(\phi^l(x)) \subset B_l$. 

We consider the following estimate.

\[
\nu(S) \leq \lim_{k \to \infty} \inf \int_{M} \sum_{l=1}^{n} \phi^{l}(x) d\nu_{j_{k}} \\
= \lim_{k \to \infty} \sum_{l=1}^{n} \int_{M} \phi^{l}(x) |u_{j_{k}}|^{2} d\text{vol}_{M} \\
= \lim_{k \to \infty} \sum_{l=1}^{n} \langle \phi^{l}(x) u_{j_{k}}, u_{j_{k}} \rangle_{L^{2}(M)} \\
= \lim_{k \to \infty} \sum_{l=1}^{n} \langle e^{-it\sqrt{\lambda_{j_{k}}}} u_{j_{k}}, \phi^{l}(x) e^{it\sqrt{\lambda_{j_{k}}}} u_{j_{k}} \rangle_{L^{2}(M)} \\
= \lim_{k \to \infty} \sum_{l=1}^{n} \langle e^{-it\sqrt{-\Delta}} u_{j_{k}}, \phi^{l}(x) e^{it\sqrt{-\Delta}} u_{j_{k}} \rangle_{L^{2}(M)}
\]

We apply the Egorov theorem for \( \phi^{l}(x) \). So we have

\[
\lim_{k \to \infty} \inf \langle e^{-it\sqrt{\Delta}} u_{j_{k}}, \phi^{l}(x) e^{it\sqrt{\Delta}} u_{j_{k}} \rangle_{L^{2}(M)} = \lim_{k \to \infty} \inf \langle \exp(tX)^{*} \phi^{l}(x) u_{j_{k}}, u_{j_{k}} \rangle_{L^{2}(M)}
\]

By the above lemma, we can choose \( t_{l} > 0 \) (uniform bounded) satisfying \( A_{\delta}(x_{1}, t_{l}) \cap S = \emptyset \). Therefore, by applying Garding inequality, we have

\[
\lim_{k \to \infty} \inf \sum_{l=1}^{n} \exp(t_{l}X)^{*} \phi^{l}(x) u_{j_{k}}, u_{j_{k}} \rangle_{L^{2}(M)} \leq \lim_{k \to \infty} \inf \sum_{l=1}^{n} \int_{A_{\delta}(x_{l}, t_{l})} |u_{j_{k}}|^{2} d\text{vol}_{S^{*}M} \\
= \mu(\bigcup_{l=1}^{n} (A_{\delta}(x_{l}, t_{l}))) \\
\leq \exists C \text{vol}_{M}(\bigcup_{l=1}^{n} (A_{\delta}(x_{l}, t_{l}))) \\
\leq \exists C \sum_{l=1}^{n} \text{diam}(B_{l}) \\
\leq \exists C'L,
\]

where \( C' > 0 \) is independent of \( L \). For all \( L > 0 \), we obtain \( \mu(S) < C'L \). Thus we obtain \( \mu_{s}(S) \leq \mu(S) = 0 \).
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