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<th>FINITE SUMS OF NILPOTENT ELEMENTS IN PROPERLY INFINITE $C^*$-ALGEBRAS (Free products in operator algebras and related topics)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2000), 1177: 21-25</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2000-11</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64507">http://hdl.handle.net/2433/64507</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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FINITE SUMS OF NILPOTENT ELEMENTS IN PROPERLY INFINITE $C^*$-ALGEBRAS

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ABSTRACT. We prove that $A$ is the linear span of elements $x \in A$ with $x^2 = 0$ if $A$ is stable or properly infinite. Moreover, we prove the same statement for any closed two-sided ideal $I$ of such $C^*$-algebras.

1 Introduction

Recall that a $C^*$-algebra $A$ is called stable if $A$ is isomorphic to $A \otimes \mathbb{K}$, and a unital $C^*$-algebra $A$ is called properly infinite if there exist projections $e, f \in A$ such that $e \sim f \sim 1$ and $ef = 0$, where $A \otimes \mathbb{K}$ is the tensor product of $A$ and the $C^*$-algebra $\mathbb{K}$ of compact operators on a separable infinite dimensional Hilbert space, and $e \sim f$ means that there exists a partial isometry $x \in A$ such that $e = x^*x, f = xx^*$. We prove that $A$ is the linear span of elements $x \in A$ with $x^2 = 0$ (or in particular the linear span of nilpotent elements of $A$) if $A$ is stable or properly infinite. Moreover, we prove the same statement for any closed two-sided ideal $I$ of such $C^*$-algebras. Denoting by $[A, A]$ the linear span of commutators $[a, b] = ab - ba$, with $a, b \in A$, T. Fack proved in [2] that $[A, A] = A$ if $A$ is stable or properly infinite. We also show the same statement for any closed two-sided ideal $I$ of such $C^*$-algebras.

The author would like to thank Prof. A. Kishimoto for some helpful comments.

2 Main Results

For each $C^*$-algebra $A$, we denote by $N(A)$ the linear span of elements $x \in A$ with $x^2 = 0$. We have the following result;

Theorem 1. Let $A$ be a properly infinite unital $C^*$-algebra. Then $I = N(I)$ for any closed two-sided ideal $I$ of $A$.

When $\{A_k\}_{k=1}^\infty$ is a sequence of $C^*$-algebras, we denote by $\bigoplus_{k=1}^\infty A_k$ the direct sum $C^*$-algebra $\{\bigoplus_{k=1}^\infty a_k : a_k \in A_k, \lim_{k \to \infty} ||a_k|| = 0\}$. We also denote by $M_n$ the $n \times n$ matrix algebra, and by $I_n$ the unit of $M_n$. We begin with the following lemma;
Lemma 2  Let $B$ be a $C^*$-algebra, and suppose that $A = B \otimes (\oplus_{\ell=1}^\infty M_{3\ell})$. Define $E_0^\ell \in M_{3\ell}$ for $\ell \in \mathbb{N}$ by

$$E_0^\ell = \frac{1}{\ell} \begin{pmatrix} I_\ell & 0 & 0 \\ 0 & -\frac{1}{\ell} I_\ell & 0 \\ 0 & 0 & -\frac{1}{2} I_\ell \end{pmatrix}.$$ 

Then $x \otimes (\oplus_{\ell=1}^\infty E_0^\ell) \in A$ is a element in $N(A)$ for any $x \in B$.

Proof. Define $E_m^\ell \in M_{3\ell}$ for $\ell \in \mathbb{N}, m = 1, 2, 3, 4$ by

$$E_1^\ell = \frac{1}{\ell} \begin{pmatrix} I_\ell & I_\ell & I_\ell \\ -\frac{1}{\ell} I_\ell & -\frac{1}{\ell} I_\ell & -\frac{1}{\ell} I_\ell \\ -\frac{1}{\ell} I_\ell & -\frac{1}{\ell} I_\ell & -\frac{1}{\ell} I_\ell \end{pmatrix},$$

$$E_2^\ell = \frac{1}{\ell} \begin{pmatrix} 0 & -I_\ell & -I_\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_3^\ell = \frac{1}{\ell} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_4^\ell = \frac{1}{\ell} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Then $(E_m^\ell)^2 = 0$, $E_0^\ell = \sum_{m=1}^4 E_m^\ell$ and $\lim_{\ell \to \infty} ||E_m^\ell|| = 0$ for each $m$. Thus $x \otimes (\oplus_{\ell=1}^\infty E_m^\ell) \in A$, $(x \otimes (\oplus_{\ell=1}^\infty E_m^\ell))^2 = 0$ for each $m$ and

$$x \otimes (\oplus_{\ell=1}^\infty E_0^\ell) = x \otimes (\oplus_{\ell=1}^\infty \sum_{m=1}^4 E_m^\ell)$$

$$= \sum_{m=1}^4 x \otimes (\oplus_{\ell=1}^\infty E_m^\ell) \in N(A).$$ 

\[\square\]

Note that $E_0^\ell$ equals $\frac{1}{\ell} \{\sum_{i=1}^\ell e_{i,i}^\ell - \frac{1}{2} \sum_{i=\ell+1}^{3\ell} e_{i,i}^\ell\}$ by denoting the matrix units of $M_{3\ell}$ by $\{e_{i,j}\}$.

Lemma 3  Let $B$ be a $C^*$-algebra, and suppose that $A = B \otimes \mathbb{K}$. Denote by $\{e_{i,j}\}$ the matrix units of $\mathbb{K}$. Then $x \otimes e_{1,1} \in N(A)$ for each $x \in B$.

Proof. Define a sequence $(\lambda_i)_{i=1}^\infty$ by

$$\lambda_i = \begin{cases} \frac{1}{4^{k-1}} & \text{if } 4^{k-1} \leq i \leq 2 \cdot 4^{k-1} - 1 \\ \frac{1}{4^{k-1}} \cdot (-\frac{1}{2}) & \text{if } 2 \cdot 4^{k-1} \leq i \leq 4 \cdot 4^{k-1} - 1 \end{cases}$$
for each $i \in \mathbb{N}$ (i.e. $(\lambda_i)_{i=1}^{\infty} = (1, -\frac{1}{2}, -\frac{1}{2}, \cdots, (-\frac{1}{2})^{k-1}, \cdots, (-\frac{1}{2})^{k-1}, \cdots)$). Then

$$e_{1,1} = \sum_{i=1}^{\infty} \lambda_i e_{i,i} - \sum_{i=2}^{\infty} \lambda_i e_{i,i}$$

$$= \sum_{k=1}^{\infty} \frac{1}{4^{k-1}} \left\{ \sum_{i=4^{k-1}}^{2\cdot 4^{k-1}-1} e_{i,i} - \frac{1}{2} \sum_{i=2\cdot 4^{k-1}}^{4\cdot 4^{k-1}-1} e_{i,i} \right\}$$

For each $\ell \in \mathbb{N}$, define $*-$monomorphisms $\iota^\ell : M_{3\ell} \to \mathbb{K}$ by

$$\iota^\ell(e_{i,j}^{f}) = e_{\ell+i-1,\ell+j-1}$$

1 \leq i,j \leq 3\ell.$$

Then

$$\iota^\ell(E_0^{f}) = \frac{1}{\ell} \left\{ \sum_{i=\ell}^{2\ell-1} e_{i,i} - \frac{1}{2} \sum_{i=2\ell}^{4\ell-1} e_{i,i} \right\}$$

and $\text{Ran}(\iota^{4^k}) \perp \text{Ran}(\iota^{2\cdot 4^k}) \perp \text{Ran}(\iota^{4^{k'}})$ for each $k, k' \in \mathbb{N}, k \neq k'$, where $\text{Ran}(\iota^\ell)$ is the range of $\iota^\ell$ and $\perp$ means the orthogonality relation. Thus the maps

$$\iota_1 = id_B \otimes (\bigoplus_{k=1}^{\infty} \iota^{4^k}) : B \otimes (\bigoplus_{k=1}^{\infty} M_{3\cdot 4^{k-1}}) \to A$$

$$\iota_2 = id_B \otimes (\bigoplus_{k=1}^{\infty} \iota^{2\cdot 4^k}) : B \otimes (\bigoplus_{k=1}^{\infty} M_{3\cdot 2\cdot 4^{k-1}}) \to A$$

are well-defined homomorphisms and injective, where $id_B$ is the identity map on $B$. Since $\iota_1(N(B \otimes (\bigoplus_{k=1}^{\infty} M_{3\cdot 4^{k-1}}))), \iota_2(N(B \otimes (\bigoplus_{k=1}^{\infty} M_{3\cdot 2\cdot 4^{k-1}}))) \subseteq N(A)$, it follows by lemma 2 that

$$x \otimes e_{1,1} = \iota_1(x \otimes (\bigoplus_{k=1}^{\infty} E_0^{4^k})) - \iota_2(x \otimes (\bigoplus_{k=1}^{\infty} E_0^{2\cdot 4^k})) \in N(A)$$

for each $x \in B.$

**Proof of Theorem 1.** Let $e, f \in A$ be projectons such that $e \sim 1 \sim f$, $ef = 0$, and $u, v \in A$ be isometries such that $u^*u = v^*v = 1$, $uu^* = e$ and $vv^* = f$. For each $x \in I$, since $x = exe + ex(1-e) + (1-e)xe + (1-e)x(1-e)$ and $(1-e)xe, ex(1-e) \in N(I)$, we only have to prove that $exe, (1-e)x(1-e) \in N(I)$. Set $e_i, f_i \in A$ for each $i \in \mathbb{N}$ by

$$e_i = \begin{cases} v^{i-1}ue & (i \geq 2) \\ e & (i = 1) \end{cases}$$

$$f_i = \begin{cases} u^{i-1}v(1-e) & (i \geq 2) \\ 1-e & (i = 1) \end{cases}$$

Then

$$x \otimes e_{1,1} = \sum_{k=1}^{\infty} \lambda_k e_{4^{k-1},4^{k-1}} - \sum_{k=1}^{\infty} \lambda_k e_{4^{k-1}+1,4^{k-1}+1}$$

$$= \sum_{k=1}^{\infty} \frac{1}{4^{k-1}} \left\{ \sum_{i=4^{k-1}}^{2\cdot 4^{k-1}-1} e_{i,i} - \frac{1}{2} \sum_{i=2\cdot 4^{k-1}}^{4\cdot 4^{k-1}-1} e_{i,i} \right\}$$

and

$$\iota_1 = id_B \otimes (\bigoplus_{k=1}^{\infty} \iota^{4^k}) : B \otimes (\bigoplus_{k=1}^{\infty} M_{3\cdot 4^{k-1}}) \to A$$

$$\iota_2 = id_B \otimes (\bigoplus_{k=1}^{\infty} \iota^{2\cdot 4^k}) : B \otimes (\bigoplus_{k=1}^{\infty} M_{3\cdot 2\cdot 4^{k-1}}) \to A$$

are well-defined homomorphisms and injective, where $id_B$ is the identity map on $B$. Since $\iota_1(N(B \otimes (\bigoplus_{k=1}^{\infty} M_{3\cdot 4^{k-1}}))), \iota_2(N(B \otimes (\bigoplus_{k=1}^{\infty} M_{3\cdot 2\cdot 4^{k-1}}))) \subseteq N(A)$, it follows by lemma 2 that

$$x \otimes e_{1,1} = \iota_1(x \otimes (\bigoplus_{k=1}^{\infty} E_0^{4^k})) - \iota_2(x \otimes (\bigoplus_{k=1}^{\infty} E_0^{2\cdot 4^k})) \in N(A)$$

for each $x \in B.$
Note that $e_i$'s are partial isometries with mutually orthogonal range projection and with the same initial projection $e$, and that $f_j$'s are partial isometries with mutually orthogonal range projections and with the same initial projection $1 - e$. Define $\ast$-isomorphisms $\varphi: eIe \otimes \mathbb{K} \to I, \psi: (1 - e)I(1 - e) \otimes \mathbb{K} \to I$ by

$$\varphi(a \otimes e_{i,j}) = e_i a e_j^*, \quad a \in eIe, \quad i, j \in \mathbb{N},$$
$$\psi(b \otimes e_{i,j}) = f_i b f_j^*, \quad b \in (1 - e)I(1 - e), \quad i, j \in \mathbb{N}.$$  

Then $\varphi(a \otimes e_{1,1}) = a$ for each $a \in eIe$ and $\psi(b \otimes e_{1,1}) = b$ for each $b \in (1 - e)I(1 - e)$. Thus by lemma 3,

$$exe = \varphi(exe \otimes e_{1,1}) \in \varphi(N(eIe \otimes \mathbb{K})) \subseteq N(I),$$

$$(1 - e)x(1 - e) = \psi((1 - e)x(1 - e) \otimes e_{1,1}) \in \psi(N((1 - e)I(1 - e) \otimes \mathbb{K})) \subseteq N(I).$$

\[\blacksquare\]

**Corollary 4.** Let $B$ be a $C^\ast$-algebra such that the multiplier algebra $M(B)$ is properly infinite, and $C$ be a $C^\ast$-algebra. If $A = B \otimes C$ (for instance, if $A$ is a stable algebra, a tensor product with a Cuntz-algebra $O_n$) then $I = N(I)$ for any closed two-sided ideal $I$ of $A$, where the tensor product can be taken with respect to any $C^\ast$-norm.

**Proof.** The multiplier algebra $M(A)$ of $A$ is properly infinite and $A$ is a closed two-sided ideal of $M(A)$. Thus $I$ is a closed two-sided ideal of $M(A)$ and $I = N(I)$ by theorem 1.\[\blacksquare\]

Recall that a unital $C^\ast$-algebra $A$ is called infinite if there exists a projection $e \in A$ such $e \neq 1, e \sim 1$.

**Corollary 5.** If $A$ is a simple unital infinite $C^\ast$-algebra then $A = N(A)$

**Proof.** By [1], $A$ is properly infinite. Thus $A = N(A)$ by Theorem 1.\[\blacksquare\]

If we do not assume that $A$ is simple in corollary 5 then the conclusion does not follow in general. For instance, the Toeplitz algebra $\mathfrak{T}$ is a unital infinite $C^\ast$-algebra with a closed two-sided ideal $\mathbb{K}$, and the quotient $C^\ast$-algebra $\mathfrak{T}/\mathbb{K}$ is isomorphic to $C(S)$, where $C(S)$ is the $C^\ast$-algebra of complex continuous functions on $S$ and $S = \{ z \in \mathbb{C} : |z| = 1\}$. Then $N(\mathfrak{T}) = \mathbb{K} \neq \mathfrak{T}$. For $N(\mathbb{K}) = \mathbb{K}$ by corollary 4, and $N(\mathfrak{T}/\mathbb{K}) = N(C(S)) = \{0\}$. Thus $N(\mathfrak{T}) \subseteq \ker(\pi) = \mathbb{K} = N(\mathbb{K}) \subseteq N(\mathfrak{T})$ since $\pi(N(\mathfrak{T})) \subseteq N(\mathfrak{T}/\mathbb{K}) = \{0\}$, where $\pi$ is the quotient map from $\mathfrak{T}$ onto $\mathfrak{T}/\mathbb{K}$.

Finally we consider the relation between $[A, A]$ and $N(A)$.

**Proposition 6.** For any $C^\ast$-algebra $A$, $N(A) \subseteq [A, A]$.
Proof. For each $x \in A$ with $x^2 = 0$, set $x = u|x|$, where $|x| = (x^*x)^{\frac{1}{2}}$ and $u$ in the double dual $A^{**}$ of $A$ is the partial isometry of the polar-decomposition of $x$. Then since $u|x|^\frac{1}{2} \in A$,

$$x = [u|x|^\frac{1}{2}, |x|^\frac{1}{2}] \in [A, A].$$

\[\blacksquare\]


Proof. By theorem 1 and proposition 6, $I = N(I) \subseteq [I, I] \subseteq I$.$\blacksquare$

Corollary 8. Let $B$ be a $C^*$-algebra such that the multiplier algebra $M(B)$ is properly infinite, and $C$ be a $C^*$-algebra. If $A = B \otimes C$ then $I = [I, I]$ for any closed two-sided ideal $I$ of $A$, where the tensor product can be taken with respect to any $C^*$-norm.

Proof. The multiplier algebra $M(A)$ is properly infinite and $A$ is a closed two-sided ideal of $M(A)$. Thus $I$ is a closed two-sided ideal of $M(A)$ and $I = [I, I]$ by corollary 7.$\blacksquare$

References


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