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FINITE SUMS OF NILPOTENT ELEMENTS IN PROPERLY INFINITE $C^*$-ALGEBRAS

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ABSTRACT. We prove that $A$ is the linear span of elements $x \in A$ with $x^2 = 0$ if $A$ is stable or properly infinite. Moreover, we prove the same statement for any closed two-sided ideal $I$ of such $C^*$-algebras.

1 Introduction

Recall that a $C^*$-algebra $A$ is called stable if $A$ is isomorphic to $A \otimes K$, and a unital $C^*$-algebra $A$ is called properly infinite if there exist projections $e, f \in A$ such that $e \sim f \sim 1$ and $ef = 0$, where $A \otimes K$ is the tensor product of $A$ and the $C^*$-algebra $K$ of compact operators on a separable infinite dimensional Hilbert space, and $e \sim f$ means that there exists a partial isometry $x \in A$ such that $e = x^*x, f = xx^*$. We prove that $A$ is the linear span of elements $x \in A$ with $x^2 = 0$ (or in particular the linear span of nilpotent elements of $A$) if $A$ is stable or properly infinite. Moreover, we prove the same statement for any closed two-sided ideal $I$ of such $C^*$-algebras. Denoting by $[A, A]$ the linear span of commutators $[a, b] = ab - ba$, with $a, b \in A$, T.Fack proved in [2] that $[A, A] = A$ if $A$ is stable or properly infinite. We also show the same statement for any closed two-sided ideal $I$ of such $C^*$-algebras.

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2 Main Results

For each $C^*$-algebra $A$, we denote by $N(A)$ the linear span of elements $x \in A$ with $x^2 = 0$. We have the following result;

Theorem 1. Let $A$ be a properly infinite unital $C^*$-algebra. Then $I = N(I)$ for any closed two-sided ideal $I$ of $A$.

When $\{A_k\}_{k=1}^\infty$ is a sequence of $C^*$-algebras, we denote by $\bigoplus_{k=1}^\infty A_k$ the direct sum $C^*$-algebra $\{\bigoplus_{k=1}^\infty a_k : a_k \in A_k, \lim_{k \to \infty} \|a_k\| = 0\}$. We also denote by $M_n$ the $n \times n$ matrix algebra, and by $I_n$ the unit of $M_n$. We begin with the following lemma;
Lemma 2 Let $B$ be a $C^*$-algebra, and suppose that $A = B \otimes (\oplus_{i=1}^{\infty}M_{3\ell})$. Define $E_0^{\ell} \in M_{3\ell}$ for $\ell \in \mathbb{N}$ by

$$E_0^{\ell} = \frac{1}{\ell} \begin{pmatrix} I_{\ell} & 0 & 0 \\ 0 & -\frac{1}{2}I_{\ell} & 0 \\ 0 & 0 & -\frac{1}{2}I_{\ell} \end{pmatrix}.$$  

Then $x \otimes (\oplus_{i=1}^{\infty}E_0^{\ell}) \in A$ is a element in $N(A)$ for any $x \in B$.

**Proof.** Define $E_m^{\ell} \in M_{3\ell}$ for $\ell \in \mathbb{N}, m=1,2,3,4$ by

$$E_1^{\ell} = \frac{1}{\ell} \begin{pmatrix} I_{\ell} & I_{\ell} & I_{\ell} \\ -\frac{1}{2}I_{\ell} & -\frac{1}{2}I_{\ell} & -\frac{1}{2}I_{\ell} \\ -\frac{1}{2}I_{\ell} & -\frac{1}{2}I_{\ell} & -\frac{1}{2}I_{\ell} \end{pmatrix},$$

$$E_2^{\ell} = \frac{1}{\ell} \begin{pmatrix} 0 & -I_{\ell} & -I_{\ell} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_3^{\ell} = \frac{1}{\ell} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_4^{\ell} = \frac{1}{\ell} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Then $(E_m^{\ell})^2 = 0, E_0^{\ell} = \sum_{m=1}^{4} E_m^{\ell}$ and $\lim_{\ell \to \infty} ||E_m^{\ell}|| = 0$ for each $m$. Thus $x \otimes (\oplus_{i=1}^{\infty}E_m^{\ell}) \in A, (x \otimes (\oplus_{i=1}^{\infty}E_m^{\ell}))^2 = 0$ for each $m$ and

$$x \otimes (\oplus_{i=1}^{\infty}E_0^{\ell}) = x \otimes (\oplus_{i=1}^{\infty} \sum_{m=1}^{4} E_m^{\ell})$$

$$= \sum_{m=1}^{4} x \otimes (\oplus_{i=1}^{\infty}E_m^{\ell}) \in N(A).$$

Note that $E_0^{\ell}$ equals $\frac{1}{2}(\sum_{i=1}^{\ell} e_{i,i}^{\ell} - \frac{1}{2} \sum_{i=\ell+1}^{3\ell} e_{i,i}^{\ell})$ by denoting the matrix units of $M_{3\ell}$ by $\{e_{i,j}^{\ell}\}$.

Lemma 3 Let $B$ be a $C^*$-algebra, and suppose that $A = B \otimes \mathbb{K}$. Denote by $\{e_{i,j}\}$ the matrix units of $\mathbb{K}$. Then $x \otimes e_{1,1} \in N(A)$ for each $x \in B$.

**Proof.** Define a sequence $(\lambda_i)_{i=1}^{\infty}$ by

$$\lambda_i = \begin{cases} \frac{1}{4^k-1} & (4^{k-1} \leq i \leq 2 \cdot 4^{k-1} - 1) \\ \frac{1}{4^k-1} \cdot (-\frac{1}{2}) & (2 \cdot 4^{k-1} \leq i \leq 4 \cdot 4^{k-1} - 1) \end{cases},$$

where $k = \lfloor \log_4 i \rfloor$.
for each $i \in \mathbb{N}$ (i.e. $(\lambda_i)_{i=1}^{\infty} = (1, -\frac{1}{2}, -\frac{1}{2}, \cdots, (-\frac{1}{2})^{k-1}, \cdots, (-\frac{1}{2})^{k-1}, \cdots)$). Then
\[
e_{1,1} = \sum_{i=1}^{\infty} \lambda_i e_{i,i} - \sum_{i=2}^{\infty} \lambda_i e_{i,i}
\]
\[
= \sum_{k=1}^{\infty} \sum_{i=4^{k-1}}^{4^{k}-1} \lambda_i e_{i,i} + \sum_{k=1}^{\infty} \sum_{i=2\cdot 4^{k-1}}^{2\cdot 4^{k}-1} (-\lambda_i) e_{i,i}
\]
\[
= \sum_{k=1}^{\infty} \frac{1}{4^{k-1}} \left\{ \sum_{i=4^{k-1}}^{2\cdot 4^{k}-1} e_{i,i} - \frac{1}{2} \sum_{i=2\cdot 4^{k-1}}^{4\cdot 4^{k}-1} e_{i,i} \right\}
+ \sum_{k=1}^{\infty} \frac{1}{2\cdot 4^{k-1}} \left\{ \sum_{i=2\cdot 4^{k-1}}^{2\cdot 2\cdot 4^{k-1}-1} e_{i,i} - \frac{1}{2} \sum_{i=2\cdot 2\cdot 4^{k-1}}^{4\cdot 2\cdot 4^{k-1}-1} e_{i,i} \right\}.
\]

For each $\ell \in \mathbb{N}$, define $*$-monomorphisms $\iota^\ell : M_{3\ell} \to \mathbb{K}$ by
\[
\iota^\ell(e_{i,j}^f) = e_{l+i-1,l+j-1}, \quad 1 \leq i,j \leq 3\ell.
\]
Then
\[
\iota^\ell(E_0^f) = \frac{1}{\ell} \left\{ \sum_{i=1}^{2\ell-1} e_{i,i} - \frac{1}{2} \sum_{i=2\ell}^{4\ell-1} e_{i,i} \right\}
\]
and $\text{Ran}(\iota^{4^{k-1}}) \perp \text{Ran}(\iota^{4^{k'-1}}), \text{Ran}(\iota^{2\cdot 4^{k-1}}) \perp \text{Ran}(\iota^{2\cdot 4^{k'-1}})$ for each $k, k' \in \mathbb{N}, k \neq k'$, where $\text{Ran}(\iota^\ell)$ is the range of $\iota^\ell$ and $\perp$ means the orthogonality relation. Thus the maps
\[
i_1 = \text{id}_B \otimes (\oplus_{k=1}^{\infty} \iota^{4^{k-1}}) : B \otimes (\oplus_{k=1}^{\infty} M_{3\cdot 4^{k-1}}) \to A
\]
\[
i_2 = \text{id}_B \otimes (\oplus_{k=1}^{\infty} \iota^{2\cdot 4^{k-1}}) : B \otimes (\oplus_{k=1}^{\infty} M_{3\cdot 2\cdot 4^{k-1}}) \to A
\]
are well-defined homomorphisms and injective, where $\text{id}_B$ is the identity map on $B$. Since $i_1(N(B \otimes (\oplus_{k=1}^{\infty} M_{3\cdot 4^{k-1}}))), i_2(N(B \otimes (\oplus_{k=1}^{\infty} M_{3\cdot 2\cdot 4^{k-1}}))) \subseteq N(A)$, it follows by lemma 2 that
\[
x \otimes e_{1,1} = i_1(x \otimes (\oplus_{k=1}^{\infty} E_0^{4^{k-1}})) - i_2(x \otimes (\oplus_{k=1}^{\infty} E_0^{2\cdot 4^{k-1}})) \in N(A)
\]
for each $x \in B$. $\blacksquare$

**Proof of Theorem 1.** Let $e, f \in A$ be projectors such that $e \sim 1 \sim f$, $ef = 0$, and $u, v \in A$ be isometries such that $u^*u = v^*v = 1, uu^* = e$ and $vv^* = f$. For each $x \in I$, since $x = exe + ex(1-e) + (1-e)xe + (1-e)x(1-e)$ and $(1-e)xe, ex(1-e) \in N(I)$, we only have to prove that $exe, (1-e)x(1-e) \in N(I)$. Set $e_i, f_i \in A$ for each $i \in \mathbb{N}$ by
\[
e_i = \begin{cases}
     v^iue & (i \geq 2) \\
     e & (i = 1),
\end{cases}
\]
\[
f_i = \begin{cases}
     u^i(1-e) & (i \geq 2) \\
     1-e & (i = 1).
\end{cases}
\]
Note that $e_i$'s are partial isometries with mutually orthogonal range projection and with the same initial projection $e$, and that $f_j$'s are partial isometries with mutually orthogonal range projections and with the same initial projection $1 - e$. Define *-isomorphisms $\varphi : eIe \otimes \mathbb{K} \to I, \psi : (1 - e)I(1 - e) \otimes \mathbb{K} \to I$ by

\[
\begin{align*}
\varphi(a \otimes e_{i,j}) &= e_i a e_j^*, & a &\in eIe, & i, j &\in \mathbb{N}, \\
\psi(b \otimes e_{i,j}) &= f_i b f_j^*, & b &\in (1 - e)I(1 - e), & i, j &\in \mathbb{N}.
\end{align*}
\]

Then $\varphi(a \otimes e_{1,1}) = a$ for each $a \in eIe$ and $\psi(b \otimes e_{1,1}) = b$ for each $b \in (1 - e)I(1 - e)$. Thus by lemma 3,

\[exe = \varphi(exe \otimes e_{1,1}) \in \varphi(N(eIe \otimes \mathbb{K})) \subseteq N(I),\]

\[(1 - e)x(1 - e) = \psi((1 - e)x(1 - e) \otimes e_{1,1}) \in \psi(N((1 - e)I(1 - e) \otimes \mathbb{K})) \subseteq N(I).\]

\[\blacksquare\]

**Corollary 4.** Let $B$ be a $C^*$-algebra such that the multiplier algebra $M(B)$ is properly infinite, and $C$ be a $C^*$-algebra. If $A = B \otimes C$ (for instance, if $A$ is a stable algebra, a tensor product with a Cuntz-algebra $\mathcal{O}_n$) then $I = N(I)$ for any closed two-sided ideal $I$ of $A$, where the tensor product can be taken with respect to any $C^*$-norm.

**Proof.** The multiplier algebra $M(A)$ of $A$ is properly infinite and $A$ is a closed two-sided ideal of $M(A)$. Thus $I$ is a closed two-sided ideal of $M(A)$ and $I = N(I)$ by theorem 1. \[\blacksquare\]

Recall that a unital $C^*$-algebra $A$ is called infinite if there exists a projection $e \in A$ such $e \neq 1, e \sim 1$.

**Corollary 5.** If $A$ is a simple unital infinite $C^*$-algebra then $A = N(A)$

**Proof.** By [1], $A$ is properly infinite. Thus $A = N(A)$ by Theorem 1. \[\blacksquare\]

If we do not assume that $A$ is simple in corollary 5 then the conclusion does not follow in general. For instance, the Toeplitz algebra $\mathcal{T}$ is a unital infinite $C^*$-algebra with a closed two-sided ideal $\mathbb{K}$, and the quotient $C^*$-algebra $\mathcal{T}/\mathbb{K}$ is isomorphic to $C(\mathbb{S})$, where $C(\mathbb{S})$ is the $C^*$-algebra of complex continuous functions on $\mathbb{S}$ and $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$. Then $N(\mathcal{T}) = \mathbb{K} 
eq \mathcal{T}$. For $N(\mathbb{K}) = \mathbb{K}$ by corollary 4, and $N(\mathcal{T}/\mathbb{K}) = N(C(\mathbb{S})) = \{0\}$. Thus $N(\mathcal{T}) \subseteq Ker(\pi) = \mathbb{K} = N(\mathbb{K}) \subseteq N(\mathcal{T})$ since $\pi(N(\mathcal{T})) \subseteq N(\mathcal{T}/\mathbb{K}) = \{0\}$, where $\pi$ is the quotient map from $\mathcal{T}$ onto $\mathcal{T}/\mathbb{K}$.

Finally we consider the relation between $[A, A]$ and $N(A)$.

**Proposition 6.** For any $C^*$-algebra $A$, $N(A) \subseteq [A, A]$. 
Proof. For each $x \in A$ with $x^2 = 0$, set $x = u|x|$, where $|x| = (x^*x)^{\frac{1}{2}}$ and $u$ in the double dual $A^{**}$ of $A$ is the partial isometry of the polar-decomposition of $x$. Then since $u|x|^\frac{1}{2} \in A$,

$$x = [u|x|^\frac{1}{2}, |x|^\frac{1}{2}] \in [A, A].$$

\[\blacksquare\]


Proof. By theorem 1 and proposition 6, $I = N(I) \subseteq [I, I] \subseteq I$.\[\blacksquare\]

Corollary 8. Let $B$ be a $C^*$-algebra such that the multiplier algebra $M(B)$ is properly infinite, and $C$ be a $C^*$-algebra. If $A = B \otimes C$ then $I = [I, I]$ for any closed two-sided ideal $I$ of $A$, where the tensor product can be taken with respect to any $C^*$-norm.

Proof. The multiplier algebra $M(A)$ is properly infinite and $A$ is a closed two-sided ideal of $M(A)$. Thus $I$ is a closed two-sided ideal of $M(A)$ and $I = [I, I]$ by corollary 7.\[\blacksquare\]

References


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