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Kyoto University
GLOBAL CONTINUATION BEYOND SINGULARITY ON THE BOUNDARY

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1. INTRODUCTION

We consider problems of the form

\[ \begin{align*}
    u_t &= u_{xx}, & 0 < x < 1, & 0 < t < T, \\
    u_x(0, t) &= 0, & 0 < t < T, \\
    u_x(1, t) &= f(u(1, t)), & 0 < t < T, \\
    u(x, 0) &= u_0(x) > 0, & 0 \leq x \leq 1,
\end{align*} \]

where \( f(u) = -u^{-p}, \ p > 0, \) or \( f(u) = u^p, \ p > 1. \) We shall call them Problem (Q) and Problem (B), respectively. We discuss them separately.

1.1. **Problem (Q) \( (f(u) = -u^{-p}) \)**. This problem was studied before by Fila & Levine(1993) where it was shown that that every solution quenches in a finite time \( T = T(u_0) \) in the sense that \( u > 0 \) in \([0, 1] \times [0, T)\) and \( u(1, t) \to 0 \) as \( t \to T \). The behavior of \( u \) near \((1, T)\) for \( t \leq T \) was also studied.

The question whether it is possible to continue the solution beyond \( t = T \) (in some suitable sense) was raised by Levine(1993). Since \( u(\cdot, T) \in C([0, 1]) \) and \( u(1, T) = 0, \) an obvious possibility of continuing the solution is to extend it for \( t > T \) by \( \tilde{u} \) which solves

\[ \begin{align*}
    \tilde{u}_t &= \tilde{u}_{xx}, & 0 < x < 1, & t > T, \\
    \tilde{u}_x(0, t) &= 0, & t > T, \\
    \tilde{u}(1, t) &= 0, & t > T, \\
    \tilde{u}(x, T) &= u(x, T), & 0 \leq x \leq 1.
\end{align*} \]

We show that this continuation is natural since it can be obtained as a limit of a sequence of solutions of regularized problems. More precisely, if \( \epsilon > 0 \) and \( f_\epsilon \in C^1([0, \infty)) \) is such that \( f_\epsilon(0) = 0 \) and

\[ \begin{align*}
    f_\epsilon(s) &= -s^{-p} & \text{for } s \geq \epsilon, \\
    f(s) &\leq f_{\epsilon_1}(s) \leq f_{\epsilon_2}(s) & \text{for } s > 0 \text{ and } \epsilon_1 < \epsilon_2,
\end{align*} \]
then the solutions of \((Q_\epsilon)\):

\[
\begin{align*}
\left\{ 
& u_t^\epsilon = u_{xx}^\epsilon, & 0 < x < 1, & 0 < t < \infty, \\
& u_x^\epsilon(0, t) = 0, & 0 < t < \infty, \\
& u_x^\epsilon(1, t) = f_\epsilon(u^\epsilon(1, t)), & 0 < t < \infty, \\
& u^\epsilon(x, 0) = u_0(x), & 0 \leq x \leq 1,
\end{align*}
\]

converge to the extension of \(u\) by \(\tilde{u}\).

The fact that solutions of Problem \((Q)\) can be continued beyond \(t = T\) for all \(p > 0\) is in contrast with the situation when quenching occurs in the interior. Namely, for the problem

\[
\begin{align*}
& u_t = u_{xx} - u^{-p}, & 0 < x < 1, & 0 < t < T, \\
& u_x(0, t) = 0, & 0 < t < T, \\
& u(1, t) = 1, & 0 < t < T, \\
& u(x, 0) = u_0(x), & 0 \leq x \leq 1,
\end{align*}
\]
solutions can be continued beyond quenching if and only if \(0 < p < 1\) (cf. Phillips(1987), Galaktionov & Vazquez(1995)).

Let us also mention here that a similar phenomenon when the continuation beyond gradient blow-up does not satisfy the original boundary condition was observed by Fila & Lieberman(1994).

1.2. Problem \((B)\) \((f(u) = u^p)\). The study of blow-up of solutions of the heat equation with a nonlinear boundary condition was initiated by Levine & Payne(1974) and it has attracted considerable attention (see a survey paper of Fila & Filo(1996)). It was shown by Fila(1989) that every solution of Problem \((B)\) blows up in a finite time \(T = T(u_0)\) and it is also known (cf. López Gómez, Márquez, & Wolanski(1991)) that the only blow-up point is \(x = 1\).

(By a blow-up point we mean a point \(a \in [0, 1]\) such that there are \(\{x_n\} \subset [0, 1]\) and \(t_n \to T\) such that \(x_n \to a\) and \(u(x_n, t_n) \to \infty\) as \(n \to \infty\).)

We show that for Problem \((B)\) blow-up is always complete in the following sense. If

\[
f^n(s) = \min\{s^p, n^p\}, \quad s \geq 0, \quad n \in \mathbb{N}, \tag{1.1}
\]

and \(u^n\) is the solution of \((B^n)\):

\[
\begin{align*}
& u_t^n = u_{xx}^n, & 0 < x < 1, & 0 < t < \infty, \\
& u_x^n(0, t) = 0, & 0 < t < \infty, \\
& u_x^n(1, t) = f^n(u^n(1, t)), & 0 < t < \infty, \\
& u^n(x, 0) = u_0(x), & 0 \leq x \leq 1,
\end{align*}
\]

then \(u^n(x, t) \to \infty\) for \((x, t) \in [0, 1] \times (T, \infty)\).

For results on complete blow-up for the problem when the nonlinearity occurs in the equation we refer to the papers of Baras & Cohen(1987), Lacey & Tzanetis(1988), Galaktionov & Vazquez(1995, 1997), Martel(1998), etc.

Our method is different and it is restricted to one space dimension since we are using an intersection-comparison (or zero number(cf. [14])) argument.
2. INCOMPLETE QUENCHING

In this section we show that if \( u(x, t) \) is the solution of the problem

\[
\begin{aligned}
    u_t &= u_{xx}, & & 0 < x < 1, \quad 0 < t < T, \\
    u_x(0, t) &= 0, & & 0 < t < T, \\
    u_x(1, t) &= -u^{-p}(1, t), & & 0 < t < T, \\
    u(x, 0) &= u_0(x) > 0, & & 0 \leq x \leq 1,
\end{aligned}
\]

(Q)

where \( p > 0 \) and \( T \) is the quenching time for \( u \) then there is a natural continuation of \( u \) beyond \( T \). We shall assume that \( u_0 \in C^1([0,1]) \) and the compatibility conditions

\[ u_0'(0) = 0, \quad u_0'(1) = -u_0^{-p}(1) \]

are satisfied.

Assume that \( 0 < \epsilon < u_0(1) \). Then there exists a unique global (in time) solution \( u^\epsilon \) of (Q) such that \( u^\epsilon \in C^{2,1}([0,1] \times [0, \tau]) \) for any \( \tau > 0 \) and

(i) \( u^\epsilon > 0 \) for \( (x, t) \in [0,1] \times [0, \infty) \),

(ii) \( u^{\epsilon_1} \leq u^{\epsilon_2} \) for \( 0 < \epsilon_1 < \epsilon_2 \) and \( (x, t) \in [0,1] \times [0, \infty) \),

(iii) \( u^\epsilon \geq u \) for \( (x, t) \in [0,1] \times [0, T) \).

Also, by the maximum principle, it is clear that

\[ u^\epsilon \leq K \equiv \max_{0 \leq x \leq 1} u_0(x) \]

for all \( \epsilon > 0 \).

Now, let

\[ v(x, t) = \lim_{\epsilon \to 0} u^\epsilon(x, t), \quad (x, t) \in [0,1] \times [0, \infty). \]

(2.1)

Then \( v \) is well-defined and \( 0 \leq v \leq K \) in \( [0,1] \times [0, \infty) \). It follows from the regularity theory for parabolic equations that \( v \) satisfies the heat equation in \( (0,1) \times (0, \infty) \). By the maximum principle, \( v > 0 \) in \( (0,1) \times (0, \infty) \). Also, it is clear that \( v_x(0, t) = 0 \) for \( t > 0 \). Furthermore, if \( t \in (0, T) \), then

\[ v_x(1, t) = -v^{-p}(1, t). \]

It follows that \( v \) is a solution of (Q). By uniqueness, \( v = u \) in \( [0,1] \times [0, T) \). For the boundary condition for \( v \) on \( \{x = 1, \ t > T\} \), it can be shown that \( v(1, t) = 0 \) for \( t \geq T \).

We summarize the above results as follows:

**Theorem 2.1[15]**. The function \( v \) defined by (2.1) satisfies

\[
\begin{aligned}
    v_t &= v_{xx}, & & 0 < x < 1, \quad t > 0, \\
    v_x(0, t) &= 0, & & t > 0, \\
    v_x(1, t) &= -v^{-p}(1, t), & & 0 < t < T, \\
    v(1, t) &= 0, & & t \geq T, \\
    v(x, 0) &= u_0(x), & & 0 \leq x \leq 1.
\end{aligned}
\]

It coincides with the solution \( u \) of Problem (Q) for \( t \leq T \).
3. Complete Blow-up

Consider the problem

\[
\begin{align*}
  u_t &= u_{xx}, & 0 < x < 1, & 0 < t < T, \\
  u_x(0, t) &= 0, & 0 < t < T, \\
  u_x(1, t) &= u^p(1, t), & 0 < t < T, \\
  u(x, 0) &= u_0(x) > 0, & 0 \leq x \leq 1,
\end{align*}
\]

(B)

where \( p > 1 \), and \( T \) is the blow-up time for \( u \). We assume further that \( u_0'(0) = 0 \) and \( u_0'(1) = u_0^p(1) \).

Let \( K = \max_{0 \leq x \leq 1} u_0(x) \). For any \( n > K \), \( n \in \mathbb{N} \), we define \( f^n \) as in (1.1). Note that \( f^n \) is Lipschitz and \( u_0'(1) = f^n(u_0(1)) \) if \( n > K \). Hence, the solution of (B\(^n\)) is \( C^1 \) up to the boundary. We show that there exists a unique global (in time) solution \( u^n \) of (B\(^n\)) such that

(i) \( u^n > 0 \) for \( (x, t) \in [0, 1] \times [0, \infty) \),
(ii) \( u^n \leq u^{n+1} \) for \( (x, t) \in [0, 1] \times [0, \infty) \),
(iii) \( u^n \leq u \) for \( (x, t) \in [0, 1] \times [0, T) \).

Define

\[
  v(x, t) = \lim_{n \to \infty} u^n(x, t), \quad 0 \leq x \leq 1, \quad t \geq 0.
\]

(3.1)

Similarly, one can show that \( v_x(1, t) = v^p(1, t) \) for \( t \in (0, T) \). Then it is clear that \( v(x, t) = u(x, t) \) for \( 0 < t < T \). Note that \( v(1, T) = \infty \). Furthermore, there holds \( v(1, t) = \infty \) for \( t \geq T \).

This proves the following:

**Theorem 3.1** [15]. The function \( v \) defined in (3.1) coincides with the solution \( u \) of Problem (B) for \( t \leq T \) and \( v(x, t) = \infty \) for \( (x, t) \in [0, 1] \times (T, \infty) \).

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**References**


