On calculation of exponential growth rates

by

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Abstract. Let $h : M \to M$ be a pseudo-Anosov homeomorphism on an orientable surface with boundary. For the induced homomorphism $h_* : \pi_1(M, *) \to \pi_1(M, *)$, we will simplify the formula on calculation of exponential growth rate. If we choose $\alpha \in \pi_1(M, *)$ which is not represented by a boundary parallel closed path, the simple limit $\lim_{n \to \infty} (1/n) \log |h_*^m(\alpha)|$ gives the exponential growth rate of $h_*$. 

1. Introduction. In this article we will give a formula on calculation of exponential growth rates. Especially we are interested in exponential growth rates for homomorphisms on the fundamental group of a compact oriented surface $M$ induced by homeomorphisms $h : M \to M$. Choosing a generating system of $\pi_1(M, *)$, we denote $|\alpha|$ the word length of $\alpha \in \pi_1(M, *)$ with respect to this generating system. Recall that for a homomorphism $\phi$ of the group $\pi_1(M, *)$, the exponential growth rate $\text{EGR}(\pi_1(M, *), \phi)$ is given by the formula

$$\text{EGR}(\pi_1(M, *), \phi) = \sup \limsup_{m \to \infty} \frac{1}{m} \log |\phi^m(\alpha)|.$$ 

Note that the definition of the exponential growth rate is independent of the choice of generating systems of the group $\pi_1(M, *)$. 

In the case when a homeomorphism $h$ is pseudo-Anosov and $M$ has non-empty boundary, this formula is slightly simplified for suitable choice of an element $\alpha \in \pi_1(M, *)$.

Theorem. Assume that $h$ is a pseudo-Anosov homeomorphism on a compact oriented surface $M$ with non-empty boundary. If an element $\alpha \in \pi_1(M, *)$ is not represented by a boundary parallel closed path, then

$$\text{EGR}(\pi_1(M, *), h_*) = \lim_{m \to \infty} \frac{1}{m} \log |h_*^m(\alpha)|.$$  

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Note that even if \( h \) does not preserve the base point, the exponential growth rate is well defined, because we have an isomorphism \( \pi_1(M, h^n(*) \to \pi_1(M, *) \) which is uniquely determined up to inner automorphism of \( \pi_1(M, *) \).

In §2 for a pseudo-Anosov homeomorphism \( h \) we will construct a graph so called a train track \([3]\) from the unstable and stable foliations \([1]\) associated to \( h \). In §3 using the train track we will give a lemma to describe the topology of the image of a closed path under an iteration of \( h \), and then prove the theorem.

2. Train track. As the proof of 4.1. Theorem \([2]\), remove from \( M \) a neighborhood of the singularity of the unstable foliation associated to \( h \) and denote by \( M_1 \) the obtained holed surface. We make a quotient space of \( M_1 \) by collapsing each connected component of the intersection of \( M_1 \) with the stable leaves to a single point. Then we obtain a train track \( G_1 \), and furthermore \( h : M \to M \) and the projection \( \pi : M_1 \to G_1 \) determine a homotopy equivalence \( f_1 : G_1 \to G_1 \). Note that \( G_1 \) may be viewed as to be embedded in \( M \). Let \( k \) be the number of singular points of the unstable foliation which are in the interior of \( M \), and let \( F_k \) be the free subgroup of \( \pi_1(G_1, *) \) which is defined by the boundary circles of \( M_1 \) corresponding to the inner singularity. Then if we regard \( \pi_1(M, *) \) as a subgroup of \( \pi_1(G_1, *) \), it is obvious that \( \pi_1(G_1, *) = F_k \ast \pi_1(M, *) \). Let us choose a minimal generating system of \( \pi_1(G_1, *), \alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+k} \), such that \( \pi_1(M, *) = \langle \alpha_1, \ldots, \alpha_n \rangle \) and \( F_k = \langle \alpha_{n+1}, \ldots, \alpha_{n+k} \rangle \), and let \( p : \pi_1(G_1, *) \to \pi_1(M, *) \) denote the projection.

Recall that the neighborhood of the singularity chosen in the previous paragraph is the disjoint union of polygons (where we mean by a polygon the closed region bounded by it) with edges transverse to the stable leaves and vertices on the unstable separatrices. Furthermore for each single inner singular point we choose a distinct single polygon, and for all singular points on each single boundary component of \( M \) we choose a single polygon with boundary in \( \text{Int} M \) and with hole which is bounded by the boundary component itself. If we choose polygons corresponding to the inner singularity as to be sufficiently small relative to the neighborhood of \( \partial M \) with respect to the sum of the length of boundary edges measured by the transvers measure, then collapsing each subgraph of \( G_1 \) corresponding to the inner singularity, we obtain a graph \( G \) homotopy equivalent to \( M \). Let us denote this projection by \( q : G_1 \to G \). Then clearly \( q_* = p_* \) and \( f_1 : G_1 \to G_1 \) projects down to a homotopy equivalence \( f : G \to G \). Furthermore we may assume that any vertex of \( G_1 \) has three prongs.

3. Proof of Theorem. Using the train track constructed in the previous section we will show the following lemma.
Lemma. Under the same condition as Theorem, there exists a positive integer $m$ such that for the generating system chosen in the previous section the reduced word of $h_i^m(\alpha)$ includes $\alpha_j$ or its inverse $\bar{\alpha}_j$ for all $j$.

**Proof.** Let $a$ be the closed path in $G_1$ which determines $\alpha$. We will construct a closed path $\tilde{a}$ in $M$ which is homotopic to $a$ and which is the concatenation of segments lying in the stable, unstable separatrices and boundary circles, and with one endpoint the singularity. More precisely in the following paragraph we will choose $\tilde{a}$ as the concatenation $s_0 \cdot u_1 \cdot s_1 \cdot u_2 \cdot s_2 \cdots u_l \cdot s_l$, where each $s_i$ is in a stable separatrix, or in the union of a boundary circle and a stable separatrix emanating from it, perhaps $s_0$ is trivial, *i.e.* a vertex, and each $u_i$ is in an unstable separatrix, and the initial points of $u_j$ and the terminal points of $s_j$ are singular points.

By the assumption on $G_1$, we can detect which singular point a vertex $v$ of $G_1$ corresponds to, and more precisely which vertex of removed polygon it corresponds to. Assume that $a$ passes an edge $\epsilon$ with end points vertices $v_1$ and $v_2$ in the direction from $v_1$ to $v_2$, and assume that $v_j$ correspond to distinct singular points $x_j$ for $j = 1, 2$. By construction $\pi^{-1}(\text{Int } \epsilon)$ is a foliated rectangle, and we can uniquely determine an unstable separatrix $\zeta$ at $x_1$ which goes into $\pi^{-1}(\text{Int } \epsilon)$ and a stable separatrix $\xi$ at $x_2$ which intersects with $\zeta$ at $y$ in $\pi^{-1}(\pi(P_2))$ where $P_2$ denotes the boundary of the removed polygon for $x_2$. Then for paths $u_j$ and $s_j$, with suitable index $j$, we choose the paths in $\zeta$ and $\xi$ bounded by $x_1$ and $y$, and $y$ and $x_2$ respectively. If $x_1 = x_2$, we do not need to choose $u_j$ and $s_j$ for the edge $\epsilon$, and if $x_1$ and $x_2$ lie in the same boundary component, we also do not need to choose $u_j$ and $s_j$, but in this case we replace the last chosen $s_{j-1}$ by the concatenation of it with the arc $\eta$ in the boundary bounded by $x_1$ and $x_2$ if $j \geq 2$, and if $j = 1$, we choose $\eta$ as $s_0$. Note that by the assumption on $a$, there exists a non-trivial, *i.e.* not reduced to a vertex, $u_j$.

For each vertex $v$ of $G_1$ let us choose an edge path $\delta_v$ in $G_1$ connecting it to the base point $\ast$. Then for any path $\gamma$ in $G_1$ with endpoints $v_0$ and $v_1$ vertices, concatenating the reverse path $\delta_{v_0}$ to the one chosen for $v_0, \gamma$, and the path $\delta_{v_1}$ chosen for $v_1$ in this order, we obtain a closed path. Let us denote by $\langle \gamma \rangle$ the element of $\pi_1(G_1, \ast)$ determined by this closed path.

Assume that $u_i \cdot s_i$ has the initial point $x_i^0$ and the terminal point $x_i^1$. For $\epsilon = 0$ and 1 let $B_i^\epsilon$ be the subgraphs of $G_1$ corresponding to the boundary components of $M_i$ with respect to $x_i^\epsilon$. Pushing out $u_i$ from the polygons chosen for the singularity as sliding along stable leaves and keeping the terminal point in the sector bounded by two adjacent unstable separatrices, we obtain a path $u'_i$ and then this projects down to a path $\tilde{u}_i$ in $G_1$. For each $i = 1, 2, \cdots, l$ the terminal point $x_i^1$ of $\tilde{u}_i$ and the initial point $x_{i+1}^0$ of $\tilde{u}_{i+1}$ are in $B_i^1 = B_{i+1}^0$, and let us choose a path $\tilde{t}_i$ in $B_i^1$ which connects $x_i^1$ with $x_{i+1}^0$, where $l + 1$ is viewed as 1. Then the concatenation $\tilde{b} = \tilde{u}_1 \cdot \tilde{t}_1 \cdot \tilde{u}_2 \cdot \tilde{t}_2 \cdots \tilde{u}_l \cdot \tilde{t}_l$ is a closed path in $G_1$, and
it defines the element $\alpha$ again even though the choice of $\tilde{t}_i$ is not unique. Replacing $\tilde{a}$, if necessary, we may assume that $\tilde{b}$ has no back track, and more precisely we may assume that in a neighborhood of the singular point $z_0^{i+1} u_i$ is pushed to a separatrix $\chi_i$ along stable leaves, then the separatrix $\chi_{i+1}$ in which $u_{i+1}$ lies is distinct from $\chi_i$.

Set $u_i(m) = h^m(u_i)$ and $s_i(m) = h^m(s_i)$. For paths $u_i(m)$ we perform the same construction as to obtain $\tilde{u}_i$ from $u_i$, and then we obtain the paths $\tilde{u}_i(m)$. Since $h$ maps the singularity into itself, $\tilde{u}_i(m)$ is well determined. We choose paths $\tilde{t}_i(m)$ in the same way as to choose $\tilde{t}_i$. Set $\bar{b}(m) = \tilde{u}_1(m) \cdot \tilde{t}_1(m) \cdot \tilde{u}_2(m) \cdot \tilde{t}_2(m) \cdots \tilde{u}_i(m) \cdot \tilde{t}_i(m)$. By construction $q(\bar{b}(m)) = f^m(q(\bar{b}))$ and $q(\tilde{u}_i(m)) = f^m(q(\tilde{u}_i))$. Replace $\tilde{u}_i(m)$ by the inner most edge path $\tilde{u}_i(m)$. Then by construction we have $q(\bar{b}(m)) = q(\tilde{u}_1(m)) \cdot q(\tilde{u}_2(m)) \cdots q(\tilde{u}_i(m))$, and $h^m(\alpha) = q_*(\langle \tilde{u}_1(m) \rangle \cdot \langle \tilde{u}_2(m) \rangle \cdots \langle \tilde{u}_i(m) \rangle)$ up to conjugacy.

Now we will assert that a reduced word to represent $h^m(\alpha)$ is produced by formally making a product $\langle \tilde{u}_1(m) \rangle \cdot \langle \tilde{u}_2(m) \rangle \cdots \langle \tilde{u}_i(m) \rangle$ and removing letters $\alpha_{n+1}, \alpha_{n+2}, \cdots, \alpha_{n+k}$ from it. Then this completes the proof, because each unstable separatrix is dense in $M$, and thus for a large $m$ $\tilde{u}_i(m)$ laps each edges sufficiently large times. Therefore the above assertion implies that the reduced word to represent $h^m(\alpha)$ includes any $\alpha_j$ or its inverse for $j = 1, 2, \cdots, n$.

We denote by $\langle \langle \tilde{u}_i(m) \rangle \rangle$ the word obtained from $\langle \tilde{u}_i(m) \rangle$ by removing letters $\alpha_{n+1}, \alpha_{n+2}, \cdots, \alpha_{n+k}$. Let $\gamma_i$ and $\gamma_{i+1}$ be the last letter and the first letter of $\langle \langle \tilde{u}_i(m) \rangle \rangle$ and $\langle \langle \tilde{u}_{i+1}(m) \rangle \rangle$. Since $h$ maps a singular point with $k$ unstable separatrices to a singular point of the same type, by construction we have that $\gamma_i \neq \gamma_{i+1}$, and thus cancellation may be done only in each word $\langle \langle \tilde{u}_i(m) \rangle \rangle$. Suppose by contradiction that the word $\langle \langle \tilde{u}_i(m) \rangle \rangle$ includes letters to be cancelled. Then it follows that $\tilde{u}_i(m)$ has a back track, but this is impossible because $\tilde{u}_i(m)$ is an immersed curve in $G_1$. This completes the proof.

We have done the all preparations to prove the theorem. To complete the proof, only a little bit argument is needed.

**Proof of Theorem.** As shown in the proof of Lemma, for a sufficiently large $\tilde{m}$ $\langle \tilde{u}_i(m) \rangle$ has a word $\alpha_j$ or $\bar{\alpha}_j$ for any $j, 1 \leq j \leq n$, and when we remove $\alpha_j, n+1 \leq j$, any other cancellation does not occur. Therefore we have

$$\limsup_{m \to \infty} \frac{1}{m} \log |h^m(\alpha)| \geq \limsup_{m \to \infty} \frac{1}{m} \log |h^m(\alpha_j)|$$

Furthermore by the above argument $|h^m(\alpha_j)|$ increase monotonically, and thus in the above inequality we may replace the limit supremums by simple limits:

$$\lim_{m \to \infty} \frac{1}{m} \log |h^m(\alpha)| \geq \lim_{m \to \infty} \frac{1}{m} \log |h^m(\alpha_j)|$$

Since the contrary inequality is obvious, we complete the proof of Theorem.
References


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