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Kyoto University
Fat solenoidal attractors

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Abstract
We study dynamical systems generated by skew products
\[ T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \quad T(x, y) = (\ell x, \lambda y + f(x)) \]
where \( \ell \geq 2, 1/\ell < \lambda < 1 \) and \( f \) is a \( C^2 \) function on \( S^1 \). We show that the SBR measure for \( T \) is absolutely continuous for almost every \( f \).

1 Introduction
In this paper, we study a class of dynamical systems that stably admit an absolutely continuous ergodic measure (acem) with a negative Lyapunov exponent. It is well-known that expanding dynamical systems generally admit acem's whose Lyapunov exponents are all positive. The aim of this paper is to study another kind of acem's which is produced by a quite different mechanism: overlap and sliding in short.

We can find a typical example of such acem's in a paper of Alexander and Yorke[1], where the so-called generalized baker's transformation is considered:

\[ B : [-1, 1] \times [-1, 1] \cap, \quad B(x, y) = \begin{cases} 
(2x - 1, \beta y + (1 - \beta)) & x \geq 0 \\
(2x + 1, \beta y - (1 - \beta)) & x < 0.
\end{cases} \]

When \( \beta = 1/2 \), this map \( B \) is nothing but the ordinary baker's transformation. Alexander and Yorke studied the case \( 1/2 < \beta \leq 1 \). In such case, the images of left and right halves of the domain, i.e., \( B([-1, 0] \times [-1, 1]) \) and \( B([0, 1] \times [-1, 1]) \) overlap with some sliding. This makes the dynamical nature of the map \( B \) more complicated and interesting. They observed that the map \( B \) admits an acem if and only if the number \( \beta \) satisfies a delicate numerical condition: absolute continuity of the corresponding infinitely convoluted Bernoulli measure. As they noted, there are infinitely many numbers in \( (1/2, 1] \) (e.g. \( (\sqrt{5} - 1)/2 \)) for which \( B \) admits no acem's, according to a result of Erdős[2]. On the other hand, \( B \) admits an acem for Lebesgue almost every \( \beta \) in \( (1/2, 1] \) according to a more recent result of Solomyak[3].
In this paper, we consider a class of dynamical systems generated by maps

\[ T: S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \quad T(x, y) = (\ell x, \lambda y + f(x)) \quad (1) \]

where \( \ell \geq 2 \) is an integer, \( 0 < \lambda < 1 \) is a real number, and \( f \) is a \( C^2 \) function on \( S^1 = \mathbb{R}/\mathbb{Z} \). We may regard this class of maps as a conceptual generalization of the generalized baker’s transformations \( B \) in the sense that the translation in vertical direction depends smoothly on \( x \).

The map \( T \) is a skew product on the expanding map \( \tau : x \mapsto \ell x \) and it is uniformly contracting in the fiber direction. So \( T \) is an Anosov endomorphism.

The ergodic property of \( T \) is rather simple: there exists an ergodic probability measure \( \mu \) on \( S^1 \times \mathbb{R} \), for which Lebesgue almost every point \( x \in S^1 \times \mathbb{R} \) is generic, that is,

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)} = \mu \quad \text{weakly}. \]

We will call this measure \( \mu \) the SBR measure for \( T \).

The question is smoothness of the SBR measure \( \mu \) with respect to the Lebesgue measure on \( S^1 \times \mathbb{R} \). In the case \( \lambda \ell < 1 \), the SBR measure is totally singular because \( T \) contracts area. The case \( \lambda \ell > 1 \), which corresponds to the case \( \beta > 1/2 \) for the generalized baker’s transformations, is more interesting. We will focus on this case. First we give two examples in opposite directions.

**Example 1** Let \( \ell = 2, 0.5 < \lambda \leq 0.51 \) and \( f(x) = \sin 2\pi x \). Then the SBR measure \( \mu \) for \( T \) is absolutely continuous with respect to the Lebesgue measure of \( S^1 \times \mathbb{R} \). (See figure 1.)

**Example 2** If \( f(x) = \varphi(\tau(x)) - \lambda \varphi(x) \) for some measurable function \( \varphi \) on \( S^1 \), the SBR measure for \( T \) is supported on the graph of \( \varphi \) and totally singular.

We claim that the SBR measure is absolutely continuous for almost every \( T \) and, moreover, that the absolute continuity is robust. Fix an integer \( \ell \geq 2 \). Let \( D \subset (0, 1) \times C^2(S^1, \mathbb{R}) \) be the set of combinations \( (\lambda, f) \) for which the SBR measure is absolutely continuous w.r.t. the Lebesgue measure on \( S^1 \times \mathbb{R} \). We consider the interior \( D^o \) of \( D \) with respect to the topology that is defined as the product of the canonical topology on \( (0, 1) \) and \( C^2 \)-topology on \( C^2(S^1, \mathbb{R}) \). The main result of this paper is the following.

**Theorem 1** Let \( \ell^{-1} < \lambda < 1 \). There exists a finite collection of \( C^\infty \) functions \( \varphi_i : S^1 \to \mathbb{R}, \ i = 1, 2, \cdots, m \), such that, for any \( C^2 \) function \( g \in C^2(S^1, \mathbb{R}) \), the subset of \( \mathbb{R}^m \),

\[ \left\{ (t_1, t_2, \cdots, t_m) \in \mathbb{R}^m \mid \left( \lambda, g(x) + \sum_{i=1}^{m} t_i \varphi_i(x) \right) \notin D^o \right\}, \]

is a null set with respect to the Lebesgue measure on \( \mathbb{R}^m \).
As simple consequences, we obtain

**Corollary 2** \( \mathcal{D} \) contains an open and dense subset of \( (1/\ell, 1) \times C^2(S^1, \mathbb{R}) \).

**Corollary 3** For \( \ell^{-1} < \lambda < 1 \) and \( 2 \leq r \leq \infty \), the set of functions

\[
\mathcal{D}_\lambda^r = \{ f \in C^r(S^1, \mathbb{R}) \mid (\lambda, f) \in \mathcal{D}^o \}
\]

is an open and dense subset of \( C^r(S^1, \mathbb{R}) \).

Moreover, the claim of theorem 1 implies that the subset \( \mathcal{D}_\lambda^r \) above occupies almost everywhere in \( C^r(S^1, \mathbb{R}) \). In fact, if \( C^r(S^1, \mathbb{R}) \) were a finite dimensional Euclidean space, the claim would imply that the subset \( \mathcal{D}_\lambda^r \) had full measure with respect to the 'Lebesgue measure' on \( C^r(S^1, \mathbb{R}) \). See [5] and [6] for discussions about measure-theoretical conditions that imply "almost everywhere" for subsets in infinite dimensional spaces.

The proof of theorem 1 is based on an idea that transversality of the unstable manifolds leads to absolute continuity of the SBR measure. We took this idea from a paper of Solomyak and Peres[4] where the authors gave a simplified proof of the above mentioned result of Solomyak.

One can download the full paper at

**http://www.math.sci.hokudai.ac.jp/~tsujii/index.html**
References


