Title	An invariant for projectively Anosov diffeomorphisms on the two-dimensional torus (New developments in dynamical systems)
Author(s)	Asaoka, Masayuki
Citation	数理解析研究所講究録 (2000), 1179: 94-98
Issue Date	2000-12
URL	http://hdl.handle.net/2433/64533
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1179 巻 2000 年 94-98

An invariant for projectively Anosov diffeomorphisms on the two-dimensional torus

浅岡 正幸 (Masayuki Asaoka) *
徳島大学総合科学部
asaoka@ias.tokushima-u.ac.jp

2000.9.21

Let f be a diffeomorphism on a closed smooth manifold M and Λ a compact invariant set of f. We say a splitting $E_1 \oplus E_2$ of $TM|_\Lambda$ is a dominated splitting when E_1 and E_2 are Df invariant and there exists a positive number C and $\lambda < 1$ such that

$$\|Df^n|_{E_1(z)}\| \cdot \|Df^{-n}|_{E_2(f^n(z))}\| < C\lambda^n$$

for any point z in Λ and any positive integer n. We also say a diffeomorphism f is projectively Anosov when there exists a dominated splitting $E_1 \oplus E_2$ of TM. One of the example of projectively Anosov diffemorphisms is an Anosov diffeomorphism (*i.e* Diffeomorphisms which possess a hyperbolic splitting on the whole manifold. See [2] for example). Another example of projectively Anosov diffeomorphisms is the DA diffeomorphism given by Smale [5]. More examples are given in the next section. We remark that projectively Anosov diffeomorphisms are not structurally stable in general. An example is seen in the deformation from an Anosov diffeomorphism to the DA diffeomorphism.

One of the motivation for studying two-dimensional projectively Anosov diffeomorphisms is that they are the discrete analogue of three-dimensional projectively Anosov flows. It is known that any orientable close three-dimensional manifolds possess projectively Anosov flows. Moreover, projectively Anosov flows on three-dimensional manifolds are related to contact structures of the manifold (See [1], [3]).

Let $\mathrm{Diff}^r(M)$ be the set of C^r diffeomorphisms on M with C^r topology and $\mathrm{PA}^r(M)$ the subset consisting of C^r projectively Anosov diffeomorphisms. We denote the subset $\mathrm{PA}^r_+(M)$ of $\mathrm{PA}^r(M)$ by the projectively Anosov diffeomorphisms f such that the dominated splitting $E_1 \oplus E_2$ satisfies that E_1 and E_2 are orientable and Df preserves the orientations of E_1 and E_2. It can be shown that $\mathrm{PA}^r(M)$ is open in $\mathrm{Diff}^r(M)$ and $\mathrm{PA}^r_+(M)$ is open and closed in $\mathrm{PA}^r(M)$. We define an equivalence relation between projectively Anosov diffeomorphisms.

Definition 1 . Let f_1 and f_2 be projectively Anosov diffeomorphisms on a manifold M. We say f_1 and f_2 is C^r **PA-isotopic** when there exists a continuous map $H : [0,1] \to \mathrm{PA}^r(M)$ such that $H(0) = f_1$, $H(1) = f_2$. We call H a C^r **PA-isotopy** between f_1 and f_2.

*Partially supported by Grant-in Aid for Encouragement of Young Scientists

A projectively Anosov diffeomorphism f is called **regular** if all periodic points of f are hyperbolic. By Kupka-Smale's theorem, regular projectively Anosov diffeomorphisms are generic in $\mathrm{PA}^r_+(M)$.

Our aim in this paper is to construct an PA-isotopy invariant of C^2 regular projectively Anosov diffeomorphisms on $\mathbf{T}^2$. To construct it, we have to define of the space of circles and the natural action of the diffeomorphism on it. Let f be in $\mathrm{PA}^r_+(\mathbf{T}^2)$ and $E_1 \oplus E_2$ the dominated splitting associated to f. We denote the set consisting of C^1 immersions of S^1 to $\mathbf{T}^2$ such that $Dc(TS^1) \subset E_2$ by $\mathrm{Imm}_{E_2}(S^1, \mathbf{T}^2)$. Since a C^1 diffeomorphism s of S^1 naturally acts on $\mathrm{Imm}_{E_2}(S^1, \mathbf{T}^2)$ by $c \mapsto c \circ s^{-1}$, we define the space $\mathrm{Circ}(E_2)$ of circles tangent to E_2 by $\mathrm{Circ}(E_2) = \mathrm{Imm}_{E_2}(S^1, \mathbf{T}^2)/\mathrm{Diff}^1(S^1)$. The space $\mathrm{Circ}(E_2)$ has the natural topology induced from that of $\mathrm{Imm}_{E_2}(S^1, \mathbf{T}^2)$.

For a continuous loop $\gamma : S^1 \to \mathbf{T}^2$, let $\mathrm{Circ}(E_2, \gamma)$ be the set consisting of $[c] \in \mathrm{Circ}(E_2)$ such that c is homotopic to γ. If $f \circ \gamma$ is homotopic to γ, we can define the action $\Gamma_{F,\gamma}$ of f on $\mathrm{Circ}(E_2, \gamma)$ by $\Gamma_{f,\gamma}[c] = [f \circ c]$.

We say a continous loop γ on $\mathbf{T}^2$ is simple when γ is not homotpic to $(t \mapsto \gamma'(nt))$ for any integer $n \geq 2$ and any continuous loop γ'. If $f \in \mathrm{PA}^2_+$ is regular and a continuous loop γ on $\mathbf{T}^2$ satisfies $f \circ \gamma \simeq \gamma$, we define the index $\mathrm{ind}[c]$ of periodic point $[c]$ of $\Gamma_{f,\gamma}$ by

$$\mathrm{ind}[c] := \#\{t \in S^1 \mid c(t) \text{ is a repelling periodic point of } f\}.$$

Main results are the followings:

Main Theorem A. *Let $f \in \mathrm{PA}^2_+(\mathbf{T}^2)$ be regular and $\gamma : S^1 \to \mathbf{T}^2$ a simple continuous loop satisfying that $f \circ \gamma \simeq \gamma$. Then $\mathrm{Circ}_\gamma(E_2)$ is a CW complex with the cells $\{(e_{[c]}, \lambda_{[c]})\}$ indexed by $\mathrm{Per}(\Gamma_{f,\gamma})$ such that $(e_{[c]}, \lambda_{[c]})$ is k-cell if $\mathrm{ind}[c] = k$.*

Moreover, if $0 < \#\mathrm{Per}(\Gamma_{f,\gamma}) < \infty$ in addition, then $\mathrm{Circ}(E_2, \gamma)$ is compact.

Corollary. *If f and γ satisfy the assumption of the above, then the following inequality holds:*

$$\#\{[c] \in \mathrm{Per}(\Gamma_{f,\gamma}) \mid \mathrm{ind}[c] = k\} \geq \mathrm{rank} H_k(\mathrm{Circ}(E_2, \gamma)).$$

In fact, for a periodic point $[c]$ of $\Gamma_{f,\gamma}$ with $\mathrm{ind}[c] = k$ and the period l, the unstable set

$$W^u([c]; \Gamma_{f,\gamma}) = \left\{ [c'] \in \mathrm{Circ}_\gamma(E_2) \,\middle|\, \lim_{n \to -\infty} (\Gamma_{f,\gamma})^{-ln}([c']) = [c] \right\}$$

is a manifold diffeomorphic to k-dimensional open disk and the characteristic map $e_{[c]}$ is given by the map of which image is $W^u([c]; \Gamma_{f,\gamma})$.

Main Theorem B. *Let $f \in \mathrm{PA}^2_+(\mathbf{T}^2)$ be regular and $\gamma : S^1 \to \mathbf{T}^2$ a simple continuous loop satisfying that $f \circ \gamma \simeq \gamma$. Assume $0 < \#\mathrm{Per}(\Gamma_{f,\gamma}) < \infty$ Then, for any regular $g \in \mathrm{PA}^2_+(\mathbf{T}^2)$ PA-isotopic to f, we have that $0 < \#\mathrm{Per}(\Gamma_{g,\gamma}) < \infty$ and*

$$\mathrm{Circ}(E_2^g, \gamma) \simeq \mathrm{Circ}(E_2^f, \gamma),$$

where '$\simeq$' means homotopy equivalence.

We give some examples of projectively Anosov diffeomorphisms on 2-dimensional torus and calculate $\mathrm{Circ}(E_2, f)$ for them.

DA-diffeomorphisms One of the most popular examples are DA-diffeomorphisms. We construct them by deformation of Anosov automorphisms on 2-dimensional torus around a fixed point. See [5]. If we choose the deformation correctly, the DA-diffeomorphism can be a projectively Anosov diffeomorphism. For a DA-diffeomorphism f, a loop γ such that $f \circ \gamma$ is free homotopic to γ is null-homotopic. By Poincaré-Bendixon's theorem, $\mathrm{Fix}(\Gamma_{f,\gamma}) = \emptyset$. Therefore, $\mathrm{Circ}(E_2, \gamma) \simeq \emptyset$ if it is defined.

Algebraic type projectively Anosov diffeomorphisms The example in [1, p.28] is a projectively Anosov diffeomorphism homotopic to identity. Especially, it is not PA-isotopic to any DA-diffeomorphisms.

First, we identify the 2-dimensional torus and $\Re^2 \setminus \{(0,0)\}/(x,y) \sim (2x, 2y)$. Define a diffeomorphism f_α on the torus by $f_\alpha(x,y) = (\alpha, \frac{1}{\alpha} y)$ for a real number $\alpha > 1$. See Figure 1. It is easy to check that f_α is well-defined and is a

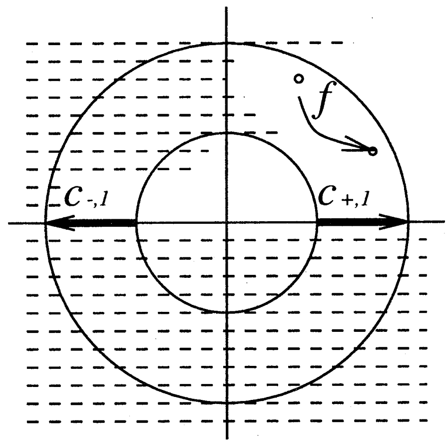

Figure 1: An Algebraic type projectively Anosov diffeomorphism

projectively Anosov diffeomorphism (The subspaces parallel to x-axis and y-axis form the dominated splitting associated to f_α). The circles in $\mathrm{Circ}(E_2)$ periodic under the action of f_α are given by $c_{\pm,n} : t \mapsto \pm 2^{nt}$ for each natural number n. The images of $c_{\pm,n}$ are normally attracting invariant manifolds and $c_{+,n}$ and $c_{-,n}$ are homotopic.

Assume that α is irrational. Then, the diffeomorphism f_α is regular and the index $\mathrm{ind}[c_{\pm,n}] = 0$ since any image of $c_{\pm,n}$ does not contains repelling periodic points. Theorem A implies that the $\mathrm{Circ}(E_2, \gamma)$ is homotopy equivalent to the 2-point set $\{[c_{-,n}], [c_{+,n}]\}$. if γ is homotopic to $c_{+,n}$ for some n, or equivalent to empty set otherwise. By Theorem B, we conclude that any regular C^2 projectively Anosov diffeomorphism g PA isotopic to f_0 has at least two normally attracting invariant circles homotopic to $c_{+,1}$.

Non-algebraic type projectivelt Anosov diffeomorphisms There exists another example homotopic to identity. First, we take a one-dimensional diffeomorphism $h(t) = t + \frac{1}{2} sin(2\pi t)$. Define the diffeomorphism g_0 on $\mathbf{T}^2 = \Re^2/\mathbf{Z}^2$ by $g(x,y) = (h(x), h(y))$.

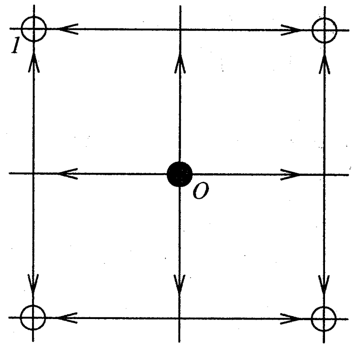

Figure 2: the diffeomorphism g_0

A projectively Anosov diffeomorphism g is given by an appropriate perturbation of g_0 around the repelling and attracting fixed points $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$.

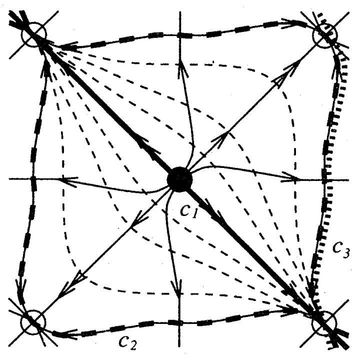

Figure 3: the projectively Anosov diffeomorphism g

Let $\gamma_1(t) = (t, 1-t)$ be a continuous loop on $\mathbf{T}^2$. Then, $\mathrm{Per}(\Gamma_{g,\gamma_1})$ consists of two elements $[c_1], [c_2]$ such that the image of c_1 contains the repelling fixed point $(0,0)$ and that of $[c_2]$ does not contain no repelling periodic point. See Figure 3. These facts imply that $\mathrm{ind}[c_1] = 1$ and $\mathrm{ind}[c_2] = 0$. Therefore, we conclude that $\mathrm{Circ}(E_2, \gamma)$ is homotopy equivalent to 1-dimensional circle and that any regular C^2 projectively Anosov diffeomorphisms PA-isotopic to g have at least one repelling fixed orbit.

Next, let $\gamma_1(t) = (0,t)$ be another loop on $\mathbf{T}^2$. We see that there exists the unique element $[c_3]$ of $\mathrm{Per}(\Gamma_{g,\gamma_2})$ and that the image of $[c_3]$ contains no repelling periodic points. Therefore, we conclude that $\mathrm{Circ}(E_2, \gamma_2)$ is homotopy equivalent to the one point set $\{[c_3]\}$. By Theorem B, g is not PA-isotopic to f_α in the previous subsection.

References

[1] Y. Eliashberg and W. Thurston, *Confoliations*, Amer. Math. Soc., 1998.

[2] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems, Encyclopedia of Math. and its Appl.* **54**, Cambridge Univ. Press, 1995.

[3] Y. Mitsumatsu Anosov flows and non-stein symplectic manifolds, *Ann. Inst. Fourier*, **45** (1995), 503-540.

[4] E. Pujals and M. Sambarino, Homoclinic tangencies and hyperbolicity for surfaces diffeomorphisms : a conjecture of Palis, preprint.

[5] S. Smale, Differentiable dynamical systems, *Bull.Amer.Math.Soc.* **73** (1967), 747 - 817.