Title	Cantor family of superstable manifolds of a double root in the dynamics of Newton's method (New developments in dynamical systems)
Author(s)	Yamagishi, Yoshikazu
Citation	数理解析研究所講究録 (2000), 1179: 60-68
Issue Date	2000-12
URL	http://hdl.handle.net/2433/64536
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1179 巻 2000 年 60-68

Cantor family of superstable manifolds of a double root in the dynamics of Newton's method *†

龍谷大学理工学部数理情報学科
山岸 義和 (Yamagishi Yoshikazu) ‡

September 22, 2000

Abstract

In the local dynamics of Newton's method, a generic double root of a holomorphic function of two variables has a Cantor family of holomorphic superstable manifolds.

1 Introduction

The aim of this paper is to give a geometric description on the local convergence of Newton's method toward a generic multiple root z_0, in the case of a holomorphic function of two variables.

Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a holomorphic function defined locally on a neighborhood of a point z_0. Newton's method of F is the mapping $NF(z) = z - (DF)_z^{-1}F(z)$ where $z = (x, y) \in \mathbf{C}^2$. If $L : \mathbf{C}^2 \to \mathbf{C}^2$ is a linear automorphism, then we have $N(L \circ F) = NF$ and $N(F \circ L) = L^{-1} \circ NF \circ L$. The point z_0 is called a multiple root of F if $F(z_0) = (0, 0)$ and $\det(DF)_{z_0} = 0$.

*2000 Mathematics Subject Classification. Primary 37D10; Secondary 65H10.

†Key Words and Phrases. Newton's method, multiple root, local convergence, indeterminate point, stable manifold.

‡yg@rins.st.ryukoku.ac.jp, http://dyn.math.ryukoku.ac.jp/~yg/

Suppose that $z_0 = (0,0)$ is a 'non-degenerate' multiple root, that is, F is written, after a linear coordinate change, by

$$F(z) = \left(x + a_2x^2 + a_1xy + a_0y^2 + O(\|z\|^3), y^2 - x^2 + O(\|z\|^3)\right) \tag{1}$$

where $\|z\| = \max(|x|, |y|)$ is the box norm. Suppose furthermore that

$$a_2 + a_0 \neq \pm a_1. \tag{2}$$

We are going to show the followings. There exists a neighrborhood $\mathbf{K} \ni z_0$ that is divided into three subsets

$$\mathbf{K} \setminus \{z_0\} = A \cup B \cup C \tag{3}$$

where

- A is called an attracting set. $NF(A) \subset A$. For each $z \in A$, we have $\|(NF)^{n+1}(z)\| \,/\, \|(NF)^n(z)\| \to 1/2$ as $n \to \infty$.
- B is called a bursting set. $B = \bigcup_{n=0}^{\infty} B_n$ where $B_0 = (NF|_{\mathbf{K}})^{-1}(\mathbf{C}^2 \backslash \mathbf{K})$, $B_n = (NF|_{\mathbf{K}})^{-n}(B_0)$. The image $(NF)^{n+1}(B_n)$ is unbounded. Each B_n consists of 2^n components.
- C is called a chaotic set, or a Cantor family of holomorphic superstable manifolds. There exist constants $0 < c_1 < c_2$ such that $c_1 |x|^2 \leq \|NF(z)\| \leq c_2 |x|^2$ for each $z \in C$.

Section 2 gives the decomposition (3). A keypoint is that the multiple root z_0 of F is an indeterminate point of NF. By choosing appropriate coordinates, we find a local blow-up transformation that is defined on a pair of polydiscs and is mapped to an unbounded region transversing themselves. Section 3 studies such a mapping, which we call a critical 'dango' (or 'barbecue') transformation.

By the C^r center manifold theorem (see [2]), we know that there exists a C^r invariant manifold of z_0 in the attracting set A, but its analyticity is not known. In section 4 we consider this problem in a general situation.

Acknowledgements. The author would like to express his gratitude to Prof. Kameyama Atsushi, Prof. Takeuchi Izumi, Prof. Kokubu Hiroshi, Prof. Ushiki Shigehiro, and Prof. Ito Toshikazu for their fruitful discussions.

2 A multiple root in Newton's method

Here we give the decomposition (3).

Newton's method of F as in (1) is written by

$$(X,Y) = NF(z) = \left(\frac{h_1(z)}{2y + h_0(z)}, \frac{y^2 - x^2 + h_2(z)}{2y + h_0(z)} \right) \tag{4}$$

where $h_0 = O(\|z\|^2)$, $h_1 = a_1x^3 + 2(a_2 + a_0)x^2y + a_1xy^2 + O(\|z\|^4)$, and $h_2 = O(\|z\|^3)$. Let c_{20} be the coefficient of x^2 in h_0.

Given $0 < \varepsilon < 1$, let $A_0 = \{|x| < \varepsilon\,|y|\}$, $B_0' = \{|y| < \varepsilon\,|x|\}$, and $C_0 = C_0^+ \cup C_0^- = \{|y - x| < \varepsilon\,|x|\} \cup \{|y + x| < \varepsilon\,|x|\}$. Given $\varepsilon' > 0$ and $0 < \delta < \varepsilon^2/4$, there exists a $0 < \rho \le 1/(3\varepsilon')$ such that (i) $|h_0| < \delta\,\|z\|$, $|h_1| < \delta\,\|z\|^2$, and $|h_2| < \delta\,\|z\|^2$ in $\mathbf{K} = \{(x,y) \in \mathbf{C}^2 \mid |x| < \delta\rho, |y| < \rho\}$, and (ii) $|y^2 + h_2| < \frac{1}{3}|x|^2$ and $|h_0 - c_{20}x^2| < \varepsilon'\,|x|^2$ in $B_0'' = \left\{(x,y) \in \mathbf{K} \mid |2y + c_{20}x^2| < \varepsilon'\,|x|^2\right\}$.

Lemma 1 $B_0'' \subset B_0 \subset B_0' \cap \mathbf{K}$.

(proof) If $(x,y) \in \mathbf{K} \setminus B_0'$ we have

$$|X| = \left|\frac{h_1}{2y + h_0}\right| < \frac{\delta\rho^2}{2\rho - \delta\rho} < \delta\rho,$$

$$|Y| = \left|\frac{y^2 - x^2 + h_2}{2y + h_0}\right| < \frac{\rho^2 + \delta^2\rho^2 + \delta\rho^2}{2\rho - \delta\rho} < \rho.$$

Thus $NF(x,y) \in \mathbf{K}$. For $(x,y) \in B_0''$, we have

$$|Y| > \frac{|x|^2 - \frac{1}{3}|x|^2}{\varepsilon'\,|x|^2 + \varepsilon'\,|x|^2} \ge \rho.$$

(qed)

Lemma 2 $NF(\mathbf{K} \setminus C_0) \subset A_0$.

(proof) If $(x,y) \notin C_0$, we have $|y^2 - x^2| > \frac{1}{2}\varepsilon\,\|z\|^2$ and

$$\left|\frac{X}{Y}\right| = \left|\frac{h_1}{y^2 - x^2 + h_2}\right| \le \frac{\delta\,\|z\|^2}{\frac{1}{2}\varepsilon\,\|z\|^2 - \delta\,\|z\|^2} < \varepsilon.$$

(qed)

By this Lemma, $B_n \subset C_0$ for $n \geq 1$. Define $C = \bigcap_{n=0}^{\infty} C_n$, $C_n = (NF|_{\mathbf{K}})^{-n}(C_0)$, and $A = N \setminus (B \cup C) = \bigcup_{n=0}^{\infty}(NF|_{\mathbf{K}})^{-n}(A_0)$.

In the following three subsections, we describe the sub-dynamics in A_0, B_0, and C_0.

2.1 Attracting set

Here we consider the dynamics in A_0. Let $(x, y) = \phi(u, v) = (uv, v)$, $(U, V) = (U_1, V_1)$ where $(U_n, V_n) = (\phi^{-1} \circ NF \circ \phi)^n(u, v)$. Both U and V are divisible by v, and $(U/v, V/v)|_{(u,v)=(0,0)} = (0, 1/2)$. Thus by the standard argument similar to Schröder's equation (see [1]), $\varphi = \varphi(u, v) = \lim_{n\to\infty} 2^n V_n = v + \cdots$ is uniformly convergent in a neighborhood of $(u, v) = (0, 0)$. Since φ/v is holomorphic around the origin $(u, v) = (0, 0)$, U is divisible by φ, and $\psi = U/\varphi$ is also holomorphic. By the new coordinates $(\xi, \eta) = (u, \varphi)$, we obtain the dynamics

$$(\xi, \eta) \mapsto (\eta\psi(\xi, \eta), \eta/2). \tag{5}$$

By the C^r center manifold theorem (see [2], Appendix III), we know that there exists a C^r differentiable function $\xi = \sigma(\eta) = \sigma(\mathrm{re}(\eta), \mathrm{im}(\eta))$ around the origin, whose graph is invariant under the dynamics (5). In section 4, we consider the problem whether this invariant manifold is holomorphic, in a general context.

2.2 Bursting set

Lemma 3 *The image $NF(B_0'') \subset NF(B_0)$ is unbounded.*

(proof) Given any $0 < \varepsilon'' < \varepsilon'$, take a point $z \in B_0''$ such that $|2y + c_{20}x^2| < \varepsilon'' |x|^2$ and $|h_0 - c_{20}x^2| < \varepsilon'' |x|^2$. Then we have

$$|Y| \geq \frac{|x|^2 - \frac{1}{3}|x|^2}{\varepsilon'' |x|^2 + \varepsilon'' |x|^2} = \frac{1}{3\varepsilon''}.$$

(qed)

As a description by coordinate geometry, let $(u, v) = (x, y/x^2)$ and $(\tilde{X}, \tilde{Z}) = (X/Y, 1/Y)$. Then $(u, v) = (0, v)$ is mapped to $(\tilde{X}, \tilde{Z}) = (0, -2v - c_{20})$. If $a_1 \neq 0$, this is a local diffeomorphism around each $(u, v) = (0, v)$.

2.3 Chaotic set

In (4), choose the coordinates $(u,v)=(x,y/x)$, $(U,V)=(X,X/Y)$. Let K_1 and K_2 be neighborhoods of $(u,v)=q_1=(0,1)$ and $q_2=(0,-1)$ respectively. Let K be a neighborhood of the line $u=0$. Around each q_i, the mapping $(u,v)\mapsto(\sqrt{U},\sqrt{U}V)$ is a local diffeomorphism with

$$\left.\frac{\partial(\sqrt{U},\sqrt{U}V)}{\partial(u,v)}\right|_{(u,v)=(0,\pm1)}=\begin{pmatrix}\sqrt{\pm\frac{1}{2}(a_2+a_0\pm a_1)} & 0\\ * & \sqrt{\pm 2(a_2+a_0\pm a_1)^{-1}}\end{pmatrix}$$

where $\sqrt{U}$ is any branch. Thus we can apply Theorem 4, given in Section 3, to the local dynamics $K_1\cup K_2\to K$ to obtain the Cantor family of holomorphic curves $\sigma:\Sigma(2)\to\mathbf{H}_1\cup\mathbf{H}_2$. By re-choosing δ sufficiently small if necessary, we obtain the chaotic set C as the graph $G(\sigma)$.

3 Cantor family of superstable manifolds

Here we give a prototype of a local dynamics that makes a Cantor family of holomorphic superstable manifolds. Let $i,j=1,2$ throughout this section.

Let $\pi(u,v)=(u,uv)$ and $\mathrm{sq}(u,v)=(u^2,v)$ be mappings of $\mathbf{C}^2$. Let K_0 be a neighborhood of the origin in $\mathbf{C}^2$, and let $K=\pi^{-1}(K_0)$. Consider two points $q_i=(0,\alpha_i)$ and their neighborhoods $K_i\ni q_i$. Let $g_i:K_0\to K_i$, $g_i(0,0)=q_i$, be a biholomorphic map with its linear part $S_i(u,v)=(a_iu+b_iv,\alpha_i+c_iu+d_iv)$. We consider the local dynamics

$$f:K_1\cup K_2\to K,\qquad \text{where } f|_{K_i}=\mathrm{sq}\circ\pi^{-1}\circ g_i^{-1}.$$

(Note that the dynamics of a mapping like $\pi^{-1}\circ g_i^{-1}:K_i\to K$ was studied in [3].)

Let $\mathbf{B}_0=\overline{\mathbf{D}}(0,\rho)\times\overline{\mathbf{D}}(0,r_0)\subset\overline{\mathbf{D}}(0,\sqrt{\rho})\times\overline{\mathbf{D}}(0,r_0)\subset K_0$ be closed polydiscs where $0<\rho<1$ and $\mathbf{B}_i=\overline{\mathbf{D}}(0,\rho)\times\overline{\mathbf{D}}(\alpha_i,r)\subset K_i$. Let $\mathbf{L}_i=\mathrm{Lip}_M(\overline{\mathbf{D}}(0,\rho),\overline{\mathbf{D}}(\alpha_i,r))$ be the set of Lipschitz functions of $\overline{\mathbf{D}}(0,\rho)$ to $\overline{\mathbf{D}}(\alpha_i,r)$ with Lipschitz constant $\leq M$, and its subset

$$\mathbf{H}_i=\left\{\tau_i\in\mathbf{L}_i\mid\tau_i|_{\mathbf{D}(0,\rho)}\text{ is holomorphic}\right\}.$$

Let $\Sigma(2)=\{1,2\}^{\mathbf{N}}\ni w=w_0w_1\cdots$ be a Cantor set.

Theorem 4 *Suppose that* $|a_i + b_i\alpha_j| \neq 0$, $i, j = 1, 2$. *There exist* $r, r_0, M > 0$, $0 < \rho < 1$, *and a unique embedding (homeomorphism onto its image)* $\sigma : \Sigma(2) \to \mathbf{H}_1 \cup \mathbf{H}_2$ *such that*

1. $\mathrm{graph}(\sigma(w)) \cap \mathrm{graph}(\sigma(w')) = \{q_{w_0}\}$ *for any* $w, w' \in \Sigma(2)$ *with* $w_0 = w'_0$.
2. σ *is invariant under* f: $\mathrm{graph}(\sigma(w)) = \mathbf{B}_{w_0} \cap f^{-1}(\mathrm{graph}(\sigma(s(w))))$ *for each* $w \in \Sigma(2)$.
3. *The graph* $G(\sigma) = \bigcup_{w\in\Sigma(2)} \mathrm{graph}(\sigma(w))$ *is the maximal local invariant set in* $\mathbf{B}_1 \cup \mathbf{B}_2$: $G(\sigma) = \bigcap_{n=0}^{\infty} f^{-n}(\mathbf{B}_1 \cup \mathbf{B}_2)$.
4. $G(\sigma)$ *is the local stable set of* $\{q_1, q_2\}$, *written by* $W^s_{\mathrm{loc}}(\{q_1, q_2\})$: $f^n(z) \to \{q_1, q_2\}$ *as* $n \to \infty$ *for each* $z \in G(\sigma) \setminus \{q_1, q_2\}$.
5. $G(\sigma)$ *is the local 'superstable' set of* $\{q_1, q_2\}$: *there exist constants* $0 < c_1 < c_2$ *such that* $c_1 |x|^2 \leq |p_1 f(z)| \leq c_2 |x|^2$ *for each* $z = (x, y) \in G(\sigma) \setminus \{q_1, q_2\}$.

The remainder of this section is a proof of this theorem.

Let $b = \max(|b_1|, |b_2|, |d_1|, |d_2|)$. Given $r > 0$ and $M > \left|\frac{c_i + d_i\alpha_j}{a_i + b_i\alpha_j}\right|$, there exist $r_0 > 0$ and $0 < \rho < 1$ that satisfy the followings: $\sqrt{\rho}(|\alpha_i| + r) \leq r_0$, $\rho M \leq r$, $\delta + \sqrt{\rho} \leq |a_i + b_i\alpha_j|$, $\frac{|c_i + d_i\alpha_j| + \delta}{|a_i + b_i\alpha_j| - \delta} \leq M$, $\delta_2 = (\ell + b)\sqrt{\rho}(1 + M) < 1$ where $\ell = \mathrm{Lip}(g_i - S_i)$ is the Lipschitz constant as a mapping of $\overline{\mathbf{D}}(0, \sqrt{\rho}) \times \overline{\mathbf{D}}(0, r_0)$ and $\delta = \ell \max(1, |\alpha_j| + r + 2\rho^2 M) + b(r + 2\rho^2 M)$.

Denote by $\tau_j^*(u) = \pi(u, \tau_j(u^2))$ for $\tau_j \in \mathbf{L}_j$. We are going to define the graph transform

$$\Gamma_{g_i}(\tau_j) = p_2 g_i \tau_j^* (p_1 g_i \tau_j^*)^{-1}.$$

Lemma 5 $\Gamma_{g_i}(\tau_j) : \overline{\mathbf{D}}(0, \rho) \to \mathbf{C}$ *is well-defined.*

(proof) As a function of $\overline{\mathbf{D}}(0, \sqrt{\rho})$, we have

$$\mathrm{Lip}(u \mapsto u(\tau_j(u^2) - \alpha_j)) \leq r + 2\rho^2 M.$$

Let $\tau_{j0} \in \mathbf{L}_j$ be the constant function $\tau_{j0}(u) = \alpha_j$, $k = 1, 2$. Then, as a function of $\overline{\mathbf{D}}(0, \sqrt{\rho})$, we have $\mathrm{Lip}(\tau_j^*) \leq \max(1, |\alpha_j| + r + 2\rho^2 M)$, $\mathrm{Lip}(p_k S_i \tau_j^* -$

$p_k S_i \tau_{j0}^*) \le b(r + 2\rho^2 M)$, and

$$\begin{aligned} & \mathrm{Lip}(p_k g_i \tau_j^* - p_k S_i \tau_{j0}^*) \\ \le\ & \mathrm{Lip}(p_k g_i \tau_j^* - p_k S_i \tau_j^*) + \mathrm{Lip}(p_k S_i \tau_j^* - p_k S_i \tau_{j0}^*) \\ =\ & \delta. \end{aligned}$$

Since $p_1 S_i \tau_{j0}^*(u) = (a_i + b_i \alpha_j) u$ is a linear mapping with $|a_i + b_i \alpha_j| > \delta$, the Lipschitz Inverse Function Theorem ([2], Appendix I) can be applied. The mapping $p_1 g_i \tau_j^*$ is a homeomorphism of $\overline{\mathbf{D}}(0, \sqrt{\rho})$ onto its image, with

$$\mathrm{Lip}([p_1 g_i \tau_j^*]^{-1}) \le (|a_i + b_i \alpha_j| - \delta)^{-1}.$$

Thus the image contains $\overline{\mathbf{D}}(0, \sqrt{\rho}(|a_i + b_i \alpha_j| - \delta)) \supset \overline{\mathbf{D}}(0, \rho)$. (qed)

Lemma 6 $\Gamma_{g_i} : \mathbf{L}_1 \cup \mathbf{L}_2 \to \mathbf{L}_i$ *is well-defined.*

(proof) As a mapping on $\overline{\mathbf{D}}(0, \rho)$, we have

$$\begin{aligned} & \mathrm{Lip}([p_1 g_i \tau_j^*]^{-1} - [p_1 S_i \tau_{j0}^*]^{-1}) \\ \le\ & \mathrm{Lip}([p_1 g_i \tau_j^*]^{-1}) \mathrm{Lip}(p_1 g_i \tau_j^* - p_1 S_i \tau_{j0}^*) \mathrm{Lip}([p_1 S_i \tau_{j0}^*]^{-1}) \\ \le\ & \frac{\delta}{(|a_i + b_i \alpha_j| - \delta)\, |a_i + b_i \alpha_j|}. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Lip}(\Gamma_{g_i}(\tau_j) - \Gamma_{S_i}(\tau_{j0})) \ \le\ & \mathrm{Lip}(p_2 g_i \tau_j^* - p_2 S_i \tau_{j0}^*) \mathrm{Lip}([p_1 g_i \tau_j^*]^{-1}) \\ & + \mathrm{Lip}(p_2 S_i \tau_{j0}^*) \mathrm{Lip}([p_1 g_i \tau_j^*]^{-1} - [p_1 S_i \tau_{j0}^*]^{-1}) \\ \le\ & \frac{\delta}{|a_i + b_i \alpha_j| - \delta} \left(1 + \left| \frac{c_i + d_i \alpha_j}{a_i + b_i \alpha_j} \right| \right). \end{aligned}$$

Since $\Gamma_{S_i}(\tau_{j0})(u) = \alpha_j + (c_i + d_i \alpha_j)(a_i + b_i \alpha_j)^{-1} u$, we have

$$\mathrm{Lip}(\Gamma_{g_i}(\tau_j)) \le \mathrm{Lip}(\Gamma_{g_i}(\tau_j) - \Gamma_{S_i}(\tau_{j0})) + \mathrm{Lip}(\Gamma_{S_i}(\tau_{j0})) \le M.$$

We also have $\Gamma_{g_i}(\tau_j)(0) = \alpha_i$ and $\rho M \le r$, so $\Gamma_{g_i}(\tau_j) \in \mathbf{L}_i$. (qed)

Lemma 7 $\Gamma_{g_i} : \mathbf{L}_1 \cup \mathbf{L}_2 \to \mathbf{L}_i$ *is a contraction:*

$$\left\| \Gamma_{g_i}(\tau_j') - \Gamma_{g_i}(\tau_j) \right\| \le \delta_2 \left\| \tau_j' - \tau_j \right\|, \quad \tau_j, \tau_j' \in \mathbf{L}_j,$$

where $\|\cdot\|$ *denotes the sup norm of a function on* $\overline{\mathbf{D}}(0, \rho)$.

(proof) For a point $(u,v) \in \overline{\mathbf{D}}(0,\sqrt{\rho}) \times \overline{\mathbf{D}}(\alpha_i, r)$ we have

$$\begin{aligned}
& \left|p_k g_i \pi(u,v) - p_k g_i \pi(u, \tau_j(u^2))\right| \\
\leq\ & \mathrm{Lip}(p_k)\mathrm{Lip}(g_i - S_i)\left|\pi(u,v) - \pi(u,\tau_j(u^2))\right| \\
& + \left|p_k S_i \pi(u,v) - p_k S_i \pi(u,\tau_j(u^2))\right| \\
\leq\ & (\ell + b)\sqrt{\rho}\left|v - \tau_j(u^2)\right|.
\end{aligned}$$

Since $p_2 g_i \pi(u, \tau_j(u^2)) = \Gamma_{g_i}(\tau_j)(p_1 g_i \pi(u, \tau_j(u^2)))$ we obtain

$$\begin{aligned}
& |p_2 g_i \pi(u,v) - \Gamma_{g_i}(\tau_j)(p_1 g_i \pi(u,v))| \\
\leq\ & \left|p_2 g_i \pi(u,v) - p_2 g_i \pi(u,\tau_j(u^2))\right| \\
& + \mathrm{Lip}(\Gamma_{g_i}(\tau_j))\left|p_1 g_i \pi(u,\tau_j(u^2)) - p_1 g_i \pi(u,v)\right| \\
\leq\ & \delta_2 \left|v - \tau_j(u^2)\right|.
\end{aligned}$$

Let $v = \tau_j'(u^2)$ and $u' = p_1 g_i \pi(u, \tau_j'(u^2))$ to obtain

$$\left|\Gamma_{g_i}(\tau_j')(u') - \Gamma_{g_i}(\tau_j)(u')\right| \leq \delta_2 \left|\tau_j'(u^2) - \tau_j(u^2)\right|.$$

If u^2 runs in $\overline{\mathbf{D}}(0,\rho)$, u' runs in a region containing $\overline{\mathbf{D}}(0,\rho)$. (qed)

Two contraction mappings $\Gamma_{g_i} : \mathbf{L}_1 \cup \mathbf{L}_2 \to \mathbf{L}_i$ makes a homeomorphism (onto its image) $\sigma : \Sigma(2) \to \mathbf{L}_1 \cup \mathbf{L}_2$ by defining

$$\sigma(w) = \bigcap_{n=1}^{\infty} \Gamma_{g_{w_0}} \cdots \Gamma_{g_{w_{n-1}}}(\mathbf{L}_{w_n}).$$

Since $\Gamma_{g_i}(\mathbf{H}_1 \cup \mathbf{H}_2) \subset \mathbf{H}_i$, we have $\sigma(\Sigma(2)) \subset \mathbf{H}_1 \cup \mathbf{H}_2$. All the properties 1–5 are now clear from the construction of σ.

4 Invariant curve in the attracting set

In this section we consider the local dynamics $z = (x,y) \mapsto F(z) = (yf(z), \lambda y)$ where $f(0) = 0$ and $0 < |\lambda| < 1$, defined in a neighborhood of the origin. Our problem is the existence of a local holomorphic curve $x = \sigma(y)$ passing through the origin, forward invariant under F. If there exists such a $x = \sigma(y) = \sum_{n=1}^{\infty} c_n y^n$, then it has to satisfy the functional equation

$$y f(\sigma(y), y) = \sigma(\lambda y) \tag{6}$$

so that the coefficients c_n are uniquely determined.

Proposition 8 *If $f(z) = ax + by$ is a linear function with $b \neq 0$, there exists no invariant holomorphic curve $x = \sigma(y)$ that passes through the origin.*

(proof) From (6), we obtain $c_1 = 0$, $c_2\lambda = b$ and $c_{n+1}\lambda^n = ac_n$, $n \geq 2$. Thus $c_n = \lambda^{-n(n-1)/2}a^{n-2}b$, and the radius of convergence of σ is equal to 0. (qed)

Proposition 9 *For any holomorphic function $\sigma(y) = \sum_{n=2}^{\infty} c_n y^n$ there exists an f such that the graph $x = \sigma(y)$ is invariant under F.*

(proof) $f(x, y) = x - \sigma(y) + \sigma(\lambda y)/y$ for instance. (qed)

References

[1] Morosawa Shunsuke, et al., Holomorphic dynamics, Cambridge (2000).

[2] Michael Shub, Global Stability of Dynamical Systems, Springer (1987).

[3] Yamagishi Yoshikazu, Cantor bouquet of holomorphic stable manifolds for a periodic indeterminate point, preprint.