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<th>Title</th>
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</thead>
<tbody>
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Cantor family of superstable manifolds of a double root in the dynamics of Newton's method *†

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Abstract

In the local dynamics of Newton’s method, a generic double root of a holomorphic function of two variables has a Cantor family of holomorphic superstable manifolds.

1 Introduction

The aim of this paper is to give a geometric description on the local convergence of Newton’s method toward a generic multiple root $z_0$, in the case of a holomorphic function of two variables.

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic function defined locally on a neighborhood of a point $z_0$. Newton’s method of $F$ is the mapping $NF(z) = z - (DF)^{-1}F(z)$ where $z = (x, y) \in \mathbb{C}^2$. If $L : \mathbb{C}^2 \to \mathbb{C}^2$ is a linear automorphism, then we have $N(L \circ F) = NF$ and $N(F \circ L) = L^{-1} \circ NF \circ L$. The point $z_0$ is called a multiple root of $F$ if $F(z_0) = (0, 0)$ and $\det(DF)_{z_0} = 0$.

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Suppose that $z_0 = (0,0)$ is a 'non-degenerate' multiple root, that is, $F$ is written, after a linear coordinate change, by

$$F(z) = \left(x + a_2 x^2 + a_1 xy + a_0 y^2 + O(||z||^3), y^2 - x^2 + O(||z||^3)\right)$$

where $||z|| = \max(|x|, |y|)$ is the box norm. Suppose furthermore that

$$a_2 + a_0 \neq \pm a_1.$$  

We are going to show the followings. There exists a neighborhood $K \ni z_0$ that is divided into three subsets

$$K \setminus \{z_0\} = A \cup B \cup C$$

where

- $A$ is called an attracting set. $NF(A) \subset A$. For each $z \in A$, we have $\frac{\|(NF)^{n+1}(z)\|}{\|(NF)^n(z)\|} \to \frac{1}{2}$ as $n \to \infty$.

- $B$ is called a bursting set. $B = \bigcup_{n=0}^{\infty} B_n$ where $B_0 = (NF|_K)^{-1}(C^2 \setminus K)$, $B_n = (NF|_K)^{-n}(B_0)$. The image $(NF)^{n+1}(B_n)$ is unbounded. Each $B_n$ consists of $2^n$ components.

- $C$ is called a chaotic set, or a Cantor family of holomorphic superstable manifolds. There exist constants $0 < c_1 < c_2$ such that $c_1 |x|^2 \leq ||NF(z)|| \leq c_2 |x|^2$ for each $z \in C$.

Section 2 gives the decomposition (3). A keypoint is that the multiple root $z_0$ of $F$ is an indeterminate point of $NF$. By choosing appropriate coordinates, we find a local blow-up transformation that is defined on a pair of polydiscs and is mapped to an unbounded region transversing themselves. Section 3 studies such a mapping, which we call a critical 'dango' (or 'barbecue') transformation.

By the $C^r$ center manifold theorem (see [2]), we know that there exists a $C^r$ invariant manifold of $z_0$ in the attracting set $A$, but its analyticity is not known. In section 4 we consider this problem in a general situation.

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2 A multiple root in Newton’s method

Here we give the decomposition (3).

Newton’s method of $F$ as in (1) is written by

$$(X, Y) = NF(z) = \left( \frac{h_1(z)}{2y + h_0(z)}, \frac{y^2 - x^2 + h_2(z)}{2y + h_0(z)} \right)$$

where $h_0 = O(||z||^2)$, $h_1 = a_1x^3 + 2(a_2 + a_0)x^2y + a_1xy^2 + O(||z||^4)$, and $h_2 = O(||z||^3)$. Let $c_{20}$ be the coefficient of $x^2$ in $h_0$.

Given $0 < \epsilon < 1$, let $A_0 = \{|x| < \epsilon |y|\}$, $B_0' = \{|y| < \epsilon |x|\}$, and $C_0 = C_0^+ \cup C_0^- = \{|y-x| < \epsilon |x|\} \cup \{|y+x| < \epsilon |x|\}$. Given $\epsilon' > 0$ and $0 < \delta < \epsilon^2/4$, there exists a $0 < \rho \leq 1/(3\epsilon')$ such that (i) $|h_0| < \delta ||z||$, $|h_1| < \delta ||z||^2$, and $|h_2| < \delta ||z||^2$ in $K = \{(x, y) \in C^2 \mid |x| < \delta \rho, |y| < \rho\}$, and (ii) $|y^2 + h_2| < \frac{1}{3} |x|^2$ and $|h_0 - c_{20}x^2| < \epsilon' |x|^2$ in $B_0'' = \{(x, y) \in K \mid 2y + c_{20}x^2 < \epsilon' |x|^2\}$.

Lemma 1 $B_0'' \subset B_0 \subset B_0' \cap K$.

(proof) If $(x, y) \in K \setminus B_0'$, we have

$$|X| = \left| \frac{h_1}{2y + h_0} \right| < \frac{\delta \rho^2}{2\rho - \delta \rho} < \delta \rho,$$

$$|Y| = \left| \frac{y^2 - x^2 + h_2}{2y + h_0} \right| < \frac{\rho^2 + \delta^2 \rho^2 + \delta \rho^2}{2\rho - \delta \rho} < \rho.$$ 

Thus $NF(x, y) \in K$. For $(x, y) \in B_0''$, we have

$$|Y| > \frac{|x|^2 - \frac{1}{3} |x|^2}{\epsilon' |x|^2 + \epsilon' |x|^2} \geq \rho.$$ 

(qed)

Lemma 2 $NF(K \setminus C_0) \subset A_0$.

(proof) If $(x, y) \notin C_0$, we have $|y^2 - x^2| > \frac{1}{2} \varepsilon ||z||^2$ and

$$\frac{|X|}{|Y|} = \left| \frac{h_1}{y^2 - x^2 + h_2} \right| < \frac{\delta ||z||^2}{\frac{1}{2} \varepsilon ||z||^2 - \delta ||z||^2} < \varepsilon.$$
By this Lemma, $B_n \subset C_0$ for $n \geq 1$. Define $C = \bigcap_{n=0}^{\infty} C_n$, $C_n = (NF|_{K})^{-n}(C_0)$, and $A = N \setminus (B \cup C) = \bigcup_{n=0}^{\infty} (NF|_{K})^{-n}(A_0)$.

In the following three subsections, we describe the sub-dynamics in $A_0$, $B_0$, and $C_0$.

### 2.1 Attracting set

Here we consider the dynamics in $A_0$. Let $(x, y) = \phi(u, v) = (uv, v)$, $(U, V) = (U_1, V_1)$ where $(U_n, V_n) = (\phi^{-1} \circ NF \circ \phi)^n(u, v)$. Both $U$ and $V$ are divisible by $v$, and $(U/v, V/v)|_{(u,v)=(0,0)} = (0, 1/2)$. Thus by the standard argument similar to Schröder's equation (see [1]), $\varphi = \varphi(u, v) = \lim_{n\to\infty} 2^n V_n = v + \cdots$ is uniformly convergent in a neighborhood of $(u, v) = (0, 0)$. Since $\varphi/v$ is holomorphic around the origin $(u, v) = (0, 0)$, $U$ is divisible by $\varphi$, and $\psi = U/\varphi$ is also holomorphic. By the new coordinates $(\xi, \eta) = (u, \varphi)$, we obtain the dynamics

$$(\xi, \eta) \mapsto (\eta\psi(\xi, \eta), \eta/2).$$

By the $C^r$ center manifold theorem (see [2], Appendix III), we know that there exists a $C^r$ differentiable function $\xi = \sigma(\eta) = \sigma(\text{re}(\eta), \text{im}(\eta))$ around the origin, whose graph is invariant under the dynamics (5). In section 4, we consider the problem whether this invariant manifold is holomorphic, in a general context.

### 2.2 Bursting set

**Lemma 3** The image $NF(B'_0) \subset NF(B_0)$ is unbounded.

(proof) Given any $0 < \varepsilon'' < \varepsilon'$, take a point $z \in B'_0$ such that $|2y + c_{20}x^2| < \varepsilon'' |x|^2$ and $|h_0 - c_{20}x^2| < \varepsilon'' |x|^2$. Then we have

$$|Y| \geq \frac{|x|^2 - \frac{1}{3} |x|^2}{\varepsilon'' |x|^2 + \varepsilon'' |x|^2} = \frac{1}{3\varepsilon''}.$$ 

(qed)

As a description by coordinate geometry, let $(u, v) = (x, y/x^2)$ and $(\tilde{X}, \tilde{Z}) = (X/Y, 1/Y)$. Then $(u, v) = (0, v)$ is mapped to $(\tilde{X}, \tilde{Z}) = (0, -2v - c_{20})$. If $a_1 \neq 0$, this is a local diffeomorphism around each $(u, v) = (0, v)$.
2.3 Chaotic set

In (4), choose the coordinates \((u, v) = (x, y/x), (U, V) = (X, X/Y)\). Let \(K_1\) and \(K_2\) be neighborhoods of \((u, v) = q_1 = (0, 1)\) and \(q_2 = (0, -1)\) respectively. Let \(K\) be a neighborhood of the line \(u = 0\). Around each \(q_i\), the mapping \((u, v) \mapsto (\sqrt{U}, \sqrt{U}V)\) is a local diffeomorphism with

\[
\frac{\partial (\sqrt{U}, \sqrt{U}V)}{\partial (u, v)} \bigg|_{(u,v)=(0,\pm 1)} = \begin{pmatrix}
\pm \frac{1}{2}(a_2 + a_0 \pm a_1) & 0 \\
0 & \pm 2(a_2 + a_0 \pm a_1)^{-1}
\end{pmatrix}
\]

where \(\sqrt{U}\) is any branch. Thus we can apply Theorem 4, given in Section 3, to the local dynamics \(K_1 \cup K_2 \to K\) to obtain the Cantor family of holomorphic curves \(\sigma : \Sigma(2) \to H_1 \cup H_2\). By re-choosing \(\delta\) sufficiently small if necessary, we obtain the chaotic set \(C\) as the graph \(G(\sigma)\).

3 Cantor family of superstable manifolds

Here we give a prototype of a local dynamics that makes a Cantor family of holomorphic superstable manifolds. Let \(i, j = 1, 2\) throughout this section.

Let \(\pi(u,v) = (u, uv)\) and \(\text{sq}(u,v) = (u^2, v)\) be mappings of \(\mathbb{C}^2\). Let \(K_0\) be a neighborhood of the origin in \(\mathbb{C}^2\), and let \(K = \pi^{-1}(K_0)\). Consider two points \(q_i = (0, \alpha_i)\) and their neighborhoods \(K_i \ni q_i\). Let \(g_i : K_0 \to K_i, g_i(0,0) = q_i,\) be a biholomorphic map with its linear part \(S_i(u, v) = (a_i u + b_i v, \alpha_i + c_i u + d_i v)\). We consider the local dynamics

\[f : K_1 \cup K_2 \to K, \quad \text{where } f|_{K_i} = \text{sq} \circ \pi^{-1} \circ g_i^{-1}.
\]

(Note that the dynamics of a mapping like \(\pi^{-1} \circ g_i^{-1} : K_i \to K\) was studied in [3].)

Let \(B_0 = \overline{D}(0, \rho) \times \overline{D}(0, r_0) \subset \overline{D}(0, \sqrt{\rho}) \times \overline{D}(0, r_0) \subset K_0\) be closed polydisks where \(0 < \rho < 1\) and \(B_i = \overline{D}(0, \rho) \times \overline{D}(\alpha_i, r) \subset K_i\). Let \(L_i = \text{Lip}_M(\overline{D}(0, \rho), \overline{D}(\alpha_i, r))\) be the set of Lipschitz functions of \(\overline{D}(0, \rho)\) to \(\overline{D}(\alpha_i, r)\) with Lipschitz constant \(\leq M\), and its subset

\[H_i = \{ \tau_i \in L_i \mid \tau_i|_{\overline{D}(0, \rho)} \text{ is holomorphic} \}.
\]

Let \(\Sigma(2) = \{1, 2\}^\mathbb{N} \ni w = w_0 w_1 \cdots\) be a Cantor set.
Theorem 4 Suppose that $|a_i + b_i \alpha_j| \neq 0$, $i, j = 1, 2$. There exist $r, r_0, M > 0$, $0 < \rho < 1$, and a unique embedding (homeomorphism onto its image) $\sigma: \Sigma(2) \to H_1 \cup H_2$ such that

1. $\text{graph}(\sigma(w)) \cap \text{graph}(\sigma(w')) = \{q_{w_0}\}$ for any $w, w' \in \Sigma(2)$ with $w_0 = w_0'$.

2. $\sigma$ is invariant under $f$: $\text{graph}(\sigma(w)) = B_{w_0} \cap f^{-1}(\text{graph}(\sigma(s(w))))$ for each $w \in \Sigma(2)$.

3. The graph $G(\sigma) = \bigcup_{w \in \Sigma(2)} \text{graph}(\sigma(w))$ is the maximal local invariant set in $B_1 \cup B_2$: $G(\sigma) = \bigcap_{n=0}^{\infty} f^{-n}(B_1 \cup B_2)$.

4. $G(\sigma)$ is the local stable set of $\{q_1, q_2\}$, written by $W_{10}^{s}(\{q_1, q_2\})$: $f^n(z) \to \{q_1, q_2\}$ as $n \to \infty$ for each $z \in G(\sigma) \setminus \{q_1, q_2\}$.

5. $G(\sigma)$ is the local 'superstable' set of $\{q_1, q_2\}$: there exist constants $0 < c_1 < c_2$ such that $c_1 |x|^2 \leq |p_1 f(z)| \leq c_2 |x|^2$ for each $z = (x, y) \in G(\sigma) \setminus \{q_1, q_2\}$.

The remainder of this section is a proof of this theorem.

Let $b = \max(|b_1|, |b_2|, |d_1|, |d_2|)$. Given $r > 0$ and $M > \frac{|c_i + d_i \alpha_j|}{|a_i + b_i \alpha_j|}$, there exist $r_0 > 0$ and $0 < \rho < 1$ that satisfy the followings: $\rho(\sqrt{p} |\alpha_j| + r) \leq r_0$, $\rho M \leq r$, $\delta + \rho \leq |a_i + b_i \alpha_j|$, $\frac{|c_i + d_i \alpha_j| + \delta}{|a_i + b_i \alpha_j| - \delta} \leq M$, $\delta_2 = (\ell + b) \sqrt{\rho} (1 + M) < 1$ where $\ell = \text{Lip}(g_i - s_i)$ is the Lipschitz constant as a mapping of $\overline{D}(0, \sqrt{\rho}) \times \overline{D}(0, r_0)$ and $\delta = \ell \max(1, |\alpha_j| + r + 2\rho M) + b(r + 2\rho M)$.

Denote by $\tau_j^*(u) = \pi(u, \tau_j(u^2))$ for $\tau_j \in L_j$. We are going to define the graph transform $\Gamma_{\tau_j} = p_2 g_i \tau_j^*(p_1 g_i \tau_j^*)^{-1}$.

Lemma 5 $\Gamma_{\tau_j}: \overline{D}(0, \rho) \to C$ is well-defined.

(proof) As a function of $\overline{D}(0, \sqrt{\rho})$, we have

$$\text{Lip}(u \mapsto u(\tau_j(u^2) - \alpha_j)) \leq r + 2\rho^2 M.$$
\[ p_k S_i \tau_{j0}^* \leq b(r + 2\rho^2 M), \]

and

\[
\text{Lip}(p_1 g_i \tau_j^* - p_1 S_i \tau_{j0}^*) \\
\leq \text{Lip}(p_1 g_i \tau_j^* - p_1 S_i \tau_j^*) + \text{Lip}(p_k S_i \tau_j^* - p_k S_i \tau_{j0}^*) \\
= \delta.
\]

Since \( p_1 S_i \tau_{j0}^*(u) = (a_i + b_i \alpha_j)u \) is a linear mapping with \( |a_i + b_i \alpha_j| > \delta \), the Lipschitz Inverse Function Theorem ([2], Appendix I) can be applied. The mapping \( p_1 g_i \tau_j^* \) is a homeomorphism of \( \overline{\text{D}}(0, \sqrt{\rho}) \) onto its image, with

\[
\text{Lip}([p_1 g_i \tau_j^*]^{-1}) \leq (|a_i + b_i \alpha_j| - \delta)^{-1}.
\]

Thus the image contains \( \overline{\text{D}}(0, \sqrt{\rho}(|a_i + b_i \alpha_j| - \delta)) \supset \overline{\text{D}}(0, \rho). \) (qed)

**Lemma 6** \( \Gamma_{g_i} : L_1 \cup L_2 \rightarrow L_i \) is well-defined.

(proof) As a mapping on \( \overline{\text{D}}(0, \rho) \), we have

\[
\text{Lip}([p_1 g_i \tau_j^*]^{-1} - [p_1 S_i \tau_{j0}^*]^{-1}) \\
\leq \text{Lip}([p_1 g_i \tau_j^*]^{-1}) \text{Lip}(p_1 g_i \tau_j^* - p_1 S_i \tau_{j0}^*) \text{Lip}([p_1 S_i \tau_{j0}^*]^{-1}) \\
\leq \frac{\delta}{(|a_i + b_i \alpha_j| - \delta) \ |a_i + b_i \alpha_j|}.
\]

Then

\[
\text{Lip}(\Gamma_{g_i}(\tau_j) - \Gamma_{S_i}(\tau_{j0})) \leq \text{Lip}(p_2 g_i \tau_j^* - p_2 S_i \tau_{j0}^*) \text{Lip}([p_1 g_i \tau_j^*]^{-1}) \\
+ \text{Lip}(p_2 S_i \tau_{j0}^*) \text{Lip}([p_1 g_i \tau_j^*]^{-1} - [p_1 S_i \tau_{j0}^*]^{-1}) \\
\leq \frac{\delta}{|a_i + b_i \alpha_j| - \delta} \left( 1 + \frac{|c_i + d_i \alpha_j|}{|a_i + b_i \alpha_j|} \right).
\]

Since \( \Gamma_{S_i}(\tau_{j0})(u) = \alpha_j + (c_i + d_i \alpha_j)(a_i + b_i \alpha_j)^{-1}u \), we have

\[
\text{Lip}(\Gamma_{g_i}(\tau_j)) \leq \text{Lip}(\Gamma_{g_i}(\tau_j) - \Gamma_{S_i}(\tau_{j0})) + \text{Lip}(\Gamma_{S_i}(\tau_{j0})) \leq M.
\]

We also have \( \Gamma_{g_i}(\tau_j)(0) = \alpha_j \) and \( \rho M \leq r \), so \( \Gamma_{g_i}(\tau_j) \in L_i \). (qed)

**Lemma 7** \( \Gamma_{g_i} : L_1 \cup L_2 \rightarrow L_i \) is a contraction:

\[
\left\| \Gamma_{g_i}(\tau_j') - \Gamma_{g_i}(\tau_j) \right\| \leq \delta_2 \left\| \tau_j' - \tau_j \right\|, \quad \tau_j, \tau_j' \in L_j,
\]

where \( \| \cdot \| \) denotes the sup norm of a function on \( \overline{\text{D}}(0, \rho) \).
(proof) For a point \((u, v) \in \overline{D}(0, \sqrt{\rho}) \times \overline{D}(\alpha_i, r)\) we have

\[
\left| p_k g_i \pi(u, v) - p_k g_i \pi(u, \tau_j(u^2)) \right|
\leq \operatorname{Lip}(p_k) \operatorname{Lip}(g_i - S_i) \left| \pi(u, v) - \pi(u, \tau_j(u^2)) \right|
+ \left| p_k S_i \pi(u, v) - p_k S_i \pi(u, \tau_j(u^2)) \right|
\leq (\ell + b) \sqrt{\rho} |v - \tau_j(u^2)|.
\]

Since \(p_2 g_i \pi(u, \tau_j(u^2)) = \Gamma_{\sigma_i}(\tau_j)(p_1 g_i \pi(u, \tau_j(u^2)))\) we obtain

\[
\left| p_2 g_i \pi(u, v) - \Gamma_{\sigma_i}(\tau_j)(p_1 g_i \pi(u, \tau_j(u^2))) \right|
\leq \left| p_2 g_i \pi(u, v) - p_2 g_i \pi(u, \tau_j(u^2)) \right|
+ \operatorname{Lip}(\Gamma_{\sigma_i}(\tau_j)) \left| p_1 g_i \pi(u, \tau_j(u^2)) - p_1 g_i \pi(u, v) \right|
\leq \delta_2 |v - \tau_j(u^2)|.
\]

Let \(v = \tau_j'(u^2)\) and \(u' = p_1 g_i \pi(u, \tau_j'(u^2))\) to obtain

\[
\left| \Gamma_{\sigma_i}(\tau_j')(u') - \Gamma_{\sigma_i}(\tau_j)(u') \right| \leq \delta_2 \left| \tau_j'(u^2) - \tau_j(u^2) \right|.
\]

If \(u^2\) runs in \(\overline{D}(0, \rho)\), \(u'\) runs in a region containing \(\overline{D}(0, \rho)\). (qed)

Two contraction mappings \(\Gamma_{g_i} : L_1 \cup L_2 \to L_i\) makes a homeomorphism (onto its image) \(\sigma : \Sigma(2) \to L_1 \cup L_2\) by defining

\[
\sigma(w) = \prod_{n=1}^{\infty} \Gamma_{g_{w_0}} \cdots \Gamma_{g_{w_{n-1}}} (L_{w_n}).
\]

Since \(\Gamma_{g_i}(H_1 \cup H_2) \subset H_i\), we have \(\sigma(\Sigma(2)) \subset H_1 \cup H_2\). All the properties 1–5 are now clear from the construction of \(\sigma\).

### 4 Invariant curve in the attracting set

In this section we consider the local dynamics \(z = (x, y) \mapsto F(z) = (y f(z), \lambda y)\) where \(f(0) = 0\) and \(0 < |\lambda| < 1\), defined in a neighborhood of the origin. Our problem is the existence of a local holomorphic curve \(x = \sigma(y)\) passing through the origin, forward invariant under \(F\). If there exists such a \(x = \sigma(y) = \sum_{n=1}^{\infty} c_n y^n\), then it has to satisfy the functional equation

\[
y f(\sigma(y), y) = \sigma(\lambda y)
\]
so that the coefficients $c_n$ are uniquely determined.

**Proposition 8** If $f(z) = ax + by$ is a linear function with $b \neq 0$, there exists no invariant holomorphic curve $x = \sigma(y)$ that passes through the origin.

(proof) From (6), we obtain $c_1 = 0$, $c_2 \lambda = b$ and $c_{n+1} \lambda^n = ac_n$, $n \geq 2$. Thus $c_n = \lambda^{-n(n-1)/2} a^{n-2} b$, and the radius of convergence of $\sigma$ is equal to 0. (qed)

**Proposition 9** For any holomorphic function $\sigma(y) = \sum_{n=2}^{\infty} c_n y^n$ there exists an $f$ such that the graph $x = \sigma(y)$ is invariant under $F$.

(proof) $f(x, y) = x - \sigma(y) + \sigma(\lambda y)/y$ for instance. (qed)

**References**

