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On the Chaotic Dynamics Generated by an Endogenous Growth Model

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1. Introduction

If for a given growth model there exists a continuum of equilibrium paths departing from the same initial condition, it is said that the model exhibits *indeterminacy of equilibria.* If each equilibrium path continues to stay in a small neighborhood of a balanced growth path, equilibrium is said to be *locally indeterminate.* Otherwise, it is said to be *globally indeterminate.*

A number of interesting phenomena have been demonstrated by introducing positive externalities into an *endogenous growth model* with infinitely-lived agents (Lucas [5] and Romer [8, 9]). The indeterminacy of equilibria is of particular relevance to this study. The existing literature is primarily concerned with the case of *local indeterminacy* (Benhabib and Farmer [2], Benhabib and Perli [3], and Boldrin and Rustichini [4]). As a result, the global properties of the equilibrium paths away from a balanced growth path have not yet been fully understood.

In the present study, we construct a simple, aggregate, endogenous growth model with a positive externality and develop a method of characterizing the global equilibrium dynamics for the case in which equilibrium paths do not, in general, converge to a balanced growth path. By using this method, we show that positive externalities can cause (i) an equilibrium to be globally indeterminate for any given initial capital stock and (ii) the growth rate along a particular equilibrium path to fluctuate chaotically. These results demonstrate that in models in which unbounded accumulation is possible and production externalities are important, trend and cycles in aggregate national output can be simultaneously generated by the same endogenous economic mechanism.

We achieve this by showing that an Euler path, which satisfies (1) the transversality condition and (2) the summability of the objective function, is an equilibrium path. We focus on chaotic equilibria such that (3) average growth rates are positive. When equilibria are locally indeterminate, there are relatively simple inequality constraints against which the conditions (1) to (3) can be tested along the balanced growth path. However, when we treat the *global* indeterminacy of equilibria, we do not necessarily have such simple inequality constraints. In this paper, we present a systematic method to overcome the analytical difficulties created by the simultaneous presence of chaotic solutions and unbounded state variables.

2. The Model

Our model is based on the two-sector model with externality that was developed in Boldrin and Rustichini [4]. There are two goods produced in two different sectors: the consumption good and the investment good. Let c_t and I_t be the amount of the consumption good and that of the investment good that are produced in period t. The consumption good is produced from both capital K_{1t} and labor L. The investment good is produced from capital K_{2t} alone. Let k_t be the amount of the capital good that is available at time t. Then

$$K_{1t} + K_{2t} = k_t. \tag{2.1}$$

The initial amount of capital input is given by $k_0 = \bar{k} > 0$. The economy is also endowed with a fixed amount of labor $L = 1$.

Externalities are at work in the production of the consumption good. Denote by e_t the source of this externality. The production function of the consumption good sector (sector 1) is

$$c_t = e_t^{\bar{\eta}} K_{1t}^{\bar{\alpha}} L^{1-\bar{\alpha}}, \tag{2.2}$$

where $0 < \bar{\alpha} < 1$ and $\bar{\eta} > 0$. The production function of the investment good sector (sector 2) is

$$I_t = bK_{2t}. \tag{2.3}$$

Let μ be the rate of depreciation of the capital good. Assume $0 < \mu \leq 1$. Then

$$k_{t+1} = I_t + (1-\mu)k_t. \tag{2.4}$$

Let $\theta = b + 1 - \mu$, and assume $\theta > 1$, which is necessary for sustained growth to be feasible.

Let $u(c_t)$ be the utility function of the representative consumer. Denote by δ, $0 < \delta < 1$, the discount factor of future utilities. Then, the representative agent's behavior is described by the following optimization problem. Given a sequence of externalities e_t, $t = 1, 2, \ldots$,

$$\max_{\{c_t, k_t, K_{1t}, K_{2t}, I_t\}_{t \geq 0}} \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{2.5}$$

$$\begin{aligned} \text{s.t. } c_t &= e_t^{\bar{\eta}} K_{1t}^{\bar{\alpha}} \\ I_t &= bK_{2t} \\ K_{1t} + K_{2t} &= k_t \\ k_{t+1} &= I_t + (1-\mu)k_t \\ k_0 &= \bar{k}. \end{aligned}$$

The level of the externality generated in period t is equal to the amount of the capital good employed in that period by the two sectors, *i.e.*,

$$e_t = k_t. \tag{2.6}$$

The general properties of this model have already been studied in Boldrin and Rustichini [4]. In particular, it was shown there that the model has a continuum of equilibria with unbounded growth for certain ranges of parameter values.

In this study, we will focus on the dynamic properties of this set of equilibria. To this end, we adopt the following utility function.

$$u(c) = c^{1-\sigma},\ 0 < \sigma < 1. \tag{2.7}$$

Since $c_t = e_t^{\bar{\eta}}[(\theta k_t - k_{t+1})/b]^{\bar{\alpha}}$, the above optimization problem, (2.5), gives rise to

$$\max_{\{k_t\}_{t \geq 0}} \sum_{t=0}^{\infty} \delta^t e_t^{\eta} (\theta k_t - k_{t+1})^{\alpha} \text{ s.t. } k_0 = \bar{k} \text{ and } (1-\mu)k_t \leq k_{t+1} \leq \theta k_t, \tag{2.8}$$

where $\eta = (1-\sigma)\bar{\eta}$ and $\alpha = (1-\sigma)\bar{\alpha}$. We may treat a path of accumulation $\{k_t\}_{t\geq 0}$ as an *equilibrium path* if (i) it solves the optimization problem (2.8) and (ii) $e_t = k_t$ for $t = 0, 1, 2, \ldots$.

We call a path $\{k_t\}_{t\geq 0}$ satisfying $k_0 = \bar{k}$ and $(1-\mu)k_t \leq k_{t+1} \leq \theta k_t$ a *feasible path* (from $k_0 = \bar{k}$). We call $\{k_t\}_{t\geq 0}$ a *balanced growth path* if it is in equilibrium and $k_{t+1}/k_t = \lambda$ for $t = 0, 1, \ldots$.

3. Equilibrium Dynamics

3.1. Characterization of Equilibria

In order to characterize the equilibrium in our model, we will first characterize the solution to the maximization problem (2.8).

Lemma 1. *Let $\{k_t\}_{t\geq 0}$ be a feasible path from $k_0 = \bar{k}$. Then, it solves the maximization problem* (2.8) *if the following three conditions are satisfied.*

$$\textit{Euler Equation:}\ -e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha-1} + \delta\theta e_{t+1}^{\eta}(\theta k_{t+1} - k_{t+2})^{\alpha-1} = 0; \tag{3.1}$$

$$\textit{Transversality Condition:}\ \lim_{t\to\infty} \delta^t e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha-1}k_{t+1} = 0; \tag{3.2}$$

$$\textit{Summability Condition:}\ \sum_{t=0}^{\infty} \delta^t e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha} < \infty. \tag{3.3}$$

Proof: We will demonstrate that if $\{k_t\}_{t\geq 0}$ satisfies (3.1), (3.2) and (3.3), then it solves the optimization problem (2.8). Define

$$p_{t+1} = \alpha\delta^t e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha-1}. \tag{3.4}$$

Since, by (3.1), $p_t = \alpha\delta^t\theta e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha-1}$, $(p_t, -p_{t+1})$ is the vector of partial derivatives of $\delta^t e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha}$. Therefore, it holds that

$$\begin{aligned} &\delta^t e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha} + p_{t+1}k_{t+1} - p_t k_t \\ &\geq \delta^t e_t^{\eta}(\theta x_t - x_{t+1})^{\alpha} + p_{t+1}x_{t+1} - p_t x_t \end{aligned}$$

for any feasible $\{x_t\}_{t\geq 0}$. Adding up this inequality over t, by (3.2) and (3.3),

$$\sum_{t=0}^{\infty} \delta^t e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha} \geq \limsup_{T\to\infty} \sum_{t=0}^{T} \delta^t e_t^{\eta}(\theta x_t - x_{t+1})^{\alpha}$$

for any feasible $\{x_t\}$ such that $x_0 = k_0$. This implies that $\{k_t\}$ solves (2.8). **Q.E.D.**

Corollary 1. *Suppose that a path* $\{k_t\}$ *satisfies* (3.1), (3.2) *and* (3.3). *Then* $\{k_t\}$ *is an equilibrium if and only if* $e_t = k_t$ *for* $t = 0, 1, \ldots$.

Define $\lambda_t = k_{t+1}/k_t$. By the feasibility condition $(1-\mu)k_t \leq k_{t+1} \leq \theta k_t$, it holds that $(1-\mu) \leq \lambda_t \leq \theta$. Note that $\log \lambda_t$ is the growth rate of capital. By (2.6), the Euler equation associated with externality path $\{e_t\} = \{k_t\}$ can be transformed into a recursive system in a single variable, λ_t, as follows.

$$\lambda_{t+1} = \theta - (\delta\theta)^{\frac{1}{1-\alpha}} \cdot (\theta - \lambda_t)(\lambda_t)^{\frac{\eta}{1-\alpha}-1}. \tag{3.5}$$

Let $z_t = \theta - \lambda_t$ and $\beta = \frac{\eta}{1-\alpha} - 1$. It is more convenient to write the difference equation (3.5) in terms of z_t. Assume $\beta \geq 0$, and define a function $f : [0, \theta] \to R_+$ as

$$f(z) = (\delta\theta)^{\frac{1}{1-\alpha}} \cdot z(\theta - z)^{\beta}. \tag{3.6}$$

Then, the difference equation (3.5) can be expressed as

$$z_{t+1} = f(z_t). \tag{3.7}$$

The function f maps $[0, \theta]$ into the set of non-negative real numbers, R_+. If $\beta > 0$, then f is unimodal and satisfies $f(0) = f(\theta) = 0$. We focus on this case by assuming

$$\beta > 0. \tag{3.8}$$

The function f achieves its maximum at

$$\hat{z} = \frac{\theta}{\beta + 1}. \tag{3.9}$$

The maximum value is

$$f^{\max} = f(\hat{z}) = (\delta\theta)^{\frac{1}{1-\alpha}} \beta^{\beta} \left(\frac{\theta}{\beta+1}\right)^{\beta+1}.$$

If this value does not exceed θ, the function f maps the interval $[0, \theta]$ into itself. Also, note that

$$f'(z) = (\delta\theta)^{\frac{1}{1-\alpha}}(\theta - z)^{\beta-1}(\theta - (1+\beta)z). \tag{3.10}$$

We guarantee that $f^{\max} < \theta$ and that $f'(0) > 1$ by assuming

$$\textbf{Condition 1: } 1 < (\delta\theta)^{\frac{1}{1-\alpha}}\theta^{\beta} < \frac{(\beta+1)^{\beta+1}}{\beta^{\beta}}.$$

Let $\underline{\mu}$ be defined as $\underline{\mu} = \max\{0, 1 - (\theta - f^{\max})\}$. We choose μ from $(\underline{\mu}, 1]$ so that $f^{\max} < \theta - (1-\mu)$. Let $J = [0, \theta - (1-\mu)]$. f maps J into itself and (J, f) constitutes a dynamical system.

Lemma 2. *Suppose Condition 1 holds. Let z_0 belong to $(0, \theta - (1-\mu))$, and let $\{z_t\}$ be the sequence generated by iterates of f, starting with z_0. Define $k_0 = \bar{k}$, and $k_{t+1} = (\theta - z_t)k_t$ for $t \geq 0$. Then $\{k_t\}$ is an equilibrium path from $\bar{k}$ if $\sum_{t=0}^{\infty} \delta^t k_t^{\eta+\alpha} < \infty$.*

Proof: By Lemma 1, it suffices to demonstrate that $\{k_t\}$ satisfies the transversality and summability conditions. The value of the objective function is

$$\sum_{t=0}^{\infty} \delta^t e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha} \leq \sum_{t=0}^{\infty} \delta^t k_t^{\eta}(\theta k_t)^{\alpha} = \theta^{\alpha} \sum_{t=0}^{\infty} \delta^t k_t^{\eta+\alpha}, \tag{3.11}$$

since $(1-\mu)k_t \leq k_{t+1} \leq \theta k_t$. Therefore, the summability condition is satisfied.

Note that $z_t > \varepsilon$ for some $\varepsilon > 0$, since $z_0 > 0$, $f(f^{\max}) > 0$, and $f'(0) > 1$.

$$\delta^t k_t^{\eta}(\theta k_t - k_{t+1})^{\alpha-1}k_{t+1} \leq \delta^t k_t^{\eta}(\theta k_t - (\theta - \varepsilon)k_t)^{\alpha-1}\theta k_t = \delta^t \theta \varepsilon^{\alpha-1} k_t^{\eta+\alpha}. \tag{3.12}$$

Since $\delta^t k_t^{\eta+\alpha}$ is summable, $\delta^t k_t^{\eta}(\theta k_t - k_{t+1})^{\alpha-1}k_{t+1} \to 0$ which implies the transversality condition is also satisfied. **Q.E.D.**

3.2. Balanced Growth Path and Invariant Set

Given $\beta > 0$, the dynamical system (J, f) has at most two fixed points. One is $z = 0$. The other is $z^* = \theta - (\delta\theta)^{\frac{-1}{\beta(1-\alpha)}}$, if it is positive. We call $\lambda^* = (\delta\theta)^{\frac{-1}{\beta(1-\alpha)}}$ a *balanced growth factor* and $\log \lambda^*$ a *balanced growth rate.* By Condition 1, we have:

Lemma 3. *The dynamical system (J, f) induces a unique balanced growth factor,*

$$\lambda^* = (\delta\theta)^{\frac{-1}{\beta(1-\alpha)}}. \tag{3.13}$$

Note that Condition 1 implies $\delta(\delta\theta)^{-(1+\frac{1}{(1-\alpha)\beta})} < 1$ which is equivalent to $\delta\lambda^{*\alpha+\eta} < 1$. Therefore, both the transversality and the summability conditions are satisfied on a balanced growth path. From the viewpoint of economics, of particular interest is the case in which $\log \lambda^*$ is positive so that the capital stock grows indefinitely when the economy is on the balanced growth path. Since $\lambda^* = \theta - z^* = (\delta\theta)^{\frac{-1}{\beta(1-\alpha)}}$, this condition is equivalent to

$$\textbf{Condition 2: } \theta < \delta^{-1}.$$

Persistent fluctuations are possible, when the steady state is locally unstable. By (3.10), local instability of the steady state can be obtained by strenghtening the left-hand side of Condition 1,

$$\textbf{Condition 3: } 1 < \frac{(\beta+2)^\beta}{\beta^\beta} < (\delta\theta)^{\frac{1}{1-\alpha}}\theta^\beta.$$

Conditions 1–3 force the graph of f to be unimodal over the segment $[0, \theta]$. To bound the invariant set of f (*i.e.*, the interval I such that $f(I) = I$) away from the origin, we assume that the image of the maximum value of f, $f(f^{\max})$, is smaller than the value of its pre-image, $\hat{z}$. This is equivalent to

$$\textbf{Condition 4: } (\delta\theta)^{\frac{2}{1-\alpha}}\left(\frac{\beta\theta}{\beta+1}\right)^\beta\left(\theta - (\delta\theta)^{\frac{1}{1-\alpha}}\left(\frac{\theta}{\beta+1}\right)\left(\frac{\beta\theta}{\beta+1}\right)^\beta\right)^\beta < 1.$$

Lemma 4. *Let $I = [f(f^{\max}), f^{\max}]$. Then, under Conditions 1 through 4, f is smooth and onto I. Moreover, f also satisfies:*

P1: *At $\hat{z} = \frac{\theta}{1+\beta}$, $f'(\hat{z}) = 0$ and $f''(\hat{z}) < 0$. Furthermore $f'(z) > 0$ for $z \in (0, \hat{z})$ and $f'(z) < 0$ for $z \in (\hat{z}, \theta)$.*

P2: *$f(z) > z$ for $z \in (0, \hat{z})$.*

Proof: Given $\beta > 0$, $f(z)$ is monotone increasing over $J' = [0, \hat{z}]$ and monotone decreasing over $I'' = [\hat{z}, f^{\max}]$. Moreover, since $f'(z^*) < -1$, it holds that, under Condition 4, $f(f^{\max}) < \hat{z} < f^{\max}$. Let $I' = [f(f^{\max}), \hat{z}] \subset J'$. Then, $f(I') = [f(f(f^{\max})), f^{\max}]$ and $f(I'') = [f(f^{\max}), f^{\max}]$. Since $f'(0) > 1$, and since $f(f^{\max}) < \hat{z} < z^*$ by $f'(z^*) < -1$, $f(f^{\max}) < f(f(f^{\max}))$. This implies $f(I'') \supset f(I')$, which implies $f(I) = I$. The remaining properties are straightforward. **Q.E.D.**

3.3. Topological Chaos

The concept of chaos that we discuss first is topological chaos. Consider a dynamical system generated by a continuous map $h : X \to X$, where X is a bounded closed interval. The dynamical system (X, h) is said to be *topologically chaotic* if it has positive topological entropy.[1] We say that x is a period-n point of h if x is a fixed point of the n-th iterate of h, *i.e.*, $x = h^n(x)$ but not a fixed point of the iterate of any order lower than n, *i.e.*, $x \neq h^i(x)$ for $i = 1, 2..., n-1$. If x is a period-n point, we call the path $x, h(x), ..., h^{n-1}(x)$, a period-n orbit or a period-n cycle. We have the following result (Theorem 4.4.19 in Alsedà et al. [1]):

Theorem 3.1. *Suppose that X is one-dimensional. Then the dynamical system (X, h) has positive topological entropy, if and only if it has a cycle of a period that is not a power of 2.*

Along a chaotic path, the equilibrium growth factor λ_t exceeds $\delta^{-1/(\alpha+\eta)}$ infinitely often, and it also becomes less than $\delta^{-1/(\alpha+\eta)}$ infinitely often. Consequently, in expressing the summability condition in terms of the growth factors λ_t along any given path, we need to take into account explicitly the relative frequency with which $\delta\lambda_t^{\alpha+\eta} > 1$ and $\delta\lambda_t^{\alpha+\eta} < 1$. To this end, define

$$s_t = \log\delta + (\eta + \alpha)\log\lambda_t$$

and

$$S_T = \log\delta + \frac{1}{T}(\eta + \alpha)\sum_{t=0}^{T-1}\log\lambda_t = \frac{1}{T}\sum_{t=0}^{T-1} s_t.$$

Lemma 5. *Let $z_{t+1} = f(z_t)$ for $t = 0, 1, 2, ...$ with $z_0 \in I$. Then $\limsup_{T\to\infty} S_T$ exists. If, in particular, $\limsup_{T\to\infty} S_T < 0$, then the path $\{k_t\}$, $k_{t+1} = (\theta - z_t)k_t$, is an equilibrium.*

[1] See Alsedà et al. [1] for the concept of topological entropy.

Proof: Since the objective function can be transformed into

$$\sum_{t=0}^{\infty} \delta^t e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha} = \sum_{t=0}^{\infty} \delta^t k_t^{\eta+\alpha}(\theta - \lambda_t)^{\alpha},$$

in order to demonstrate that its value is finite, it suffices to show that $\delta^t k_t^{\eta+\alpha}$ is summable. To this end, note that

$$\begin{aligned} \delta^{\tau} k_{\tau}^{\eta+\alpha} &= \delta^{\tau} k_0^{\eta+\alpha}(\prod_{t=0}^{\tau-1} \lambda_t)^{\eta+\alpha} = k_0^{\eta+\alpha} e^{\log(\delta^{\tau})}(\prod_{t=0}^{\tau-1} e^{(\eta+\alpha)\log\lambda_t}) \\ &= k_0^{\eta+\alpha} \exp\{\sum_{t=0}^{\tau-1}(\log\delta + (\eta+\alpha)\log\lambda_t)\} = k_0^{\eta+\alpha}\exp\{\tau S_{\tau}\}. \end{aligned}$$

Since $z_t \in I$, $0 < \theta - f^{\max} \le \lambda_t \le \theta - f(f^{\max})$. Therefore, there exist a and b, both finite, such that $a \le s_t \le b$. This implies that $a \le S_T \le b$. Hence, $\limsup_{T\to\infty} S_T$ exists. If $S^* = \limsup_{T\to\infty} S_T < 0$, then there exist a sufficiently large τ' and a sufficiently small $\nu > 0$ such that $S_{\tau} < S^* + \nu < 0$ for all $\tau > \tau'$. Hence,

$$\delta^{\tau} k_{\tau}^{\eta+\alpha} = k_0^{\eta+\alpha}\exp\{\tau S_{\tau}\} \le k_0^{\eta+\alpha}\exp\{\tau(S^* + \nu)\}.$$

This implies that $\delta^{\tau} k_{\tau}^{\eta+\alpha}$ is summable. Let $\bar{\varepsilon} = f(f^{\max})$. The transversality condition is given by

$$\lim_{t\to\infty} \delta^t k_t^{\eta}(\theta k_t - k_{t+1})^{\alpha-1} k_{t+1} = \lim_{t\to\infty} \delta^t k_t^{\eta+\alpha}(\theta - \lambda_t)^{\alpha-1}\lambda_t \le \lim_{t\to\infty} \delta^t k_t^{\eta+\alpha}\bar{\varepsilon}^{\alpha-1}\theta = 0,$$

which holds, since $\delta^{\tau} k_{\tau}^{\eta+\alpha}$ is summable. **Q.E.D.**

Example 1: Here, we fix parameters α, β and δ at $(\alpha, \beta, \delta) = (0.5, 5, 0.6)$ and allow the parameter θ to vary in the closed interval $[1.56, 1.66]$. This choice of parameters satisfies Conditions 1 through 4. In order to make explicit the dependence of the dynamical system on the varying parameter θ, we write $f(z) = f(z;\, \theta)$, $z_t = z_t(\theta)$, $s_t(\theta) = \log\delta + ((1-\alpha)\beta + 1)\log(\theta - z_t(\theta))$, and $S_T(\theta) = \frac{1}{T}\sum_{t=0}^{T-1} s_t(\theta)$. For each value of $\theta = 1.56 + 0.0005m$, $m = 0, 1, 2...$, a series of the sample average $\{S_T(\theta)\}_{T=4001}^{5000}$ has been calculated from a sample series $\{z_t(\theta)\}_{t=0}^{4999}$, where $z_0(\theta) = f^N(\frac{\theta}{1+\beta};\, \theta)$ and $N = 5000$. Let $\underline{S}(\theta, T_1, T_2) = \min\{S_T(\theta)\}_{T=T_1+1}^{T_1+T_2}$ and $\overline{S}(\theta, T_1, T_2) = \max\{S_T(\theta)\}_{T=T_1+1}^{T_1+T_2}$. In Figure 1, the horizontal axis measures the value of θ, and for a given θ, $\underline{S}(\theta, T_1, T_2)$ and $\overline{S}(\theta, T_1, T_2)$ are plotted along the vertical direction, where $T_1 = 4000$ and $T_2 = 1000$. Although the figure looks like a function of θ, it is a correspondence of θ, because two points are plotted for each value of θ. For a given θ, the time series $\{S_T(\theta)\}_{T=T_1+1}^{T_1+T_2}$ looks like a negative constant series, and is uniformly bounded away from 0. For each θ in $[1.56, 1.66]$, we can safely conclude that $\limsup_{T\to\infty} S_T(\theta) < 0$ on the path starting from $z_0 = f^N(\frac{\theta}{1+\beta}; \theta)$. Each of these paths is an equilibrium. The average growth rate is strictly positive on each path.

Example 2: Set $(\alpha, \beta, \delta, \theta) = (0.5, 5, 0.6, 1.599586)$. This choice of parameters satisfies Conditions 1 through 4 so that f has all the properties listed in Lemma 4. The graph of $f^5 : I \to I$ has 11 fixed points which implies that the dynamical system (I, f) has a single steady state and two cyclical orbits of period 5. Therefore, by Theorem 3.1, (I, f) is topologically chaotic, and the same is true for the stock of capital defined as $k_{t+1} = (\theta - z_t)k_t$. If $\limsup_{T\to\infty} S_T < 0$ on a chaotic path, then the path constitutes an equilibrium. Since topological chaos is not necessarily observable, we might not be able to check the condition $\limsup_{T\to\infty} S_T < 0$ on a chaotic path by means of *numerical* methods. Mitra [7] has devised an *analytical* method that overcomes this difficulty. He derives sufficient conditions under which $\limsup_{T\to\infty} S_T < 0$ holds for all initial values z_0 in I.[2] This example satisfies E.1 and E.2 in Mitra [7], hence topological chaos is an equilibrium.

If the system exhibits strong nonlinearity, then the condition E.1 in Mitra [7] is not satisfied, and one can not appeal to his method. For $(\alpha,\ \beta,\ \delta) = (0.5,\ 5,\ 0.6)$, E.1 does not hold for $\theta_E < \theta \le 1.66$, where $\theta_E = 1.600097368...$. In contrast, our statistical method developed below is applicable to each θ in $[1.56, 1.66]$.

3.4. Ergodic Dynamical System and Ergodic Chaos

Consider a continuous map $h : X \to X$, where X is a bounded closed interval. Let B_X be the set of all Lebesgue measurable subsets of X. Let ν be a measure defined on the measurable space (X, B_X) with $\nu(X) = 1$. Then, $[(X, B_X, \nu), h]$ constitutes a measurable dynamical system, and the mapping h is said to be *ergodic* relative

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[2] Let z' be a unique point in $(\hat{z}, f(\hat{z}))$ such that $f(z') = f^3(\hat{z})$. The conditions derived by Mitra [7] are as follows.

(E.1) $\delta^{\frac{1}{1-\alpha}}(\theta - f^3(\hat{z}))^{\frac{1}{1-\alpha}+\beta} < 1$.

(E.2) $\delta^{\frac{2}{1-\alpha}}(\theta - y)^{\frac{1}{1-\alpha}+\beta}(\theta - f(y))^{\frac{1}{1-\alpha}+\beta} < 1$, for all y in $[z', f(\hat{z})]$.

to v, (1) if it is measure-preserving and (2) if it is indecomposable. If h is ergodic, then ν is said to be an *ergodic invariant measure.* The dynamical system (X, h) is often said to be *ergodically chaic* if it has *a unique ergodic invariant measure that is absolutely continuous with respect to the Lebesgue measure.*

Lemma 6. *If the dynamical system (I, f) has an ergodic invariant measure v, then there exists a constant $\bar{S}$ such that $\bar{S} = \lim_{T\to\infty} S_T$ for v-almost all z_0 in I. Furthermore, if $\bar{S} < 0$, and if $\{z_t\}_{t\geq 0}$ is generated by the ergodic dynamical system $[(I, B_I, v), f]$, then the path $\{k_t\}$, $k_{t+1} = (\theta - z_t)k_t$, is an equilibrium.*

Proof: Let $s = s(z)$ be defined as $s(z) = \log\delta + (\alpha+\eta)\log(\theta - z)$. Since $f^{\max} < \theta$ by Condition 1, $s : I \to R$ is a continuous function defined on the compact set I, and it is v-integrable. Let $\bar{S}$ be defined as $\bar{S} = \int_I s(z)v(dz)$. By the pointwise ergodic theorem, $\bar{S} = \lim_{T\to\infty} S_T$ for v-almost all z_0 in I. Note that if $\lim_{T\to\infty} S_T$ exists, then $\limsup_{T\to\infty} S_T = \lim_{T\to\infty} S_T$. Therefore, if $\bar{S} < 0$, and if $\{z_t\}_{t\geq 0}$ is generated by $[(I, B_I, v), f]$, then the path $\{k_t\}$ is an equilibrium by Lemma 5. **Q.E.D.**

As shown by the proof of the lemma, the sample average S_T is a *consistent estimator* of the constant $\bar{S}$.

A period-n cycle is ergodic with $1/n$ assigned to each periodic point as the invariant measure. If there exists a *stable cycle* with a short period, then we can estimate $\bar{S}$ corresponding to the stable cycle from a sample series $\{z_t\}_{t\geq 0}$ by discarding an initial data set of sufficiently large length, before taking the sample average S_T. In this case, the system is asymptotically ergodic. If $\bar{S} < 0$, then both the periodic orbit and each path converging to it constitute an equilibrium.

Suppose that the dynamical system (I, f) is ergodically chaotic, and let v^* be an invariant measure. Since the support of v^* has positive Lebesgue measure, there is *a positive probability* with which a sample series $\{z_t\}_{t\geq 0}$ is ergodically chaotic, and the sample average S_T is a consistent estimator of $\bar{S}$. If S_T is negative for a sufficiently large T, then it holds, with *this positive probability*, that $\bar{S}$ is negative and that ergodic chaos is an equilibrium.

This study adopts a concept of ergodic chaos based on the Schwartzian derivative. The Schwartzian derivative of $h : X \to X$, $Sh(z)$, is defined for $z \in X$ such that $h'(z) \neq 0$ as follows:

$$Sh(z) = \frac{h'''(z)}{h'(z)} - \frac{3}{2}\left(\frac{h''(z)}{h'(z)}\right)^2.$$

The following proposition is due to Misiurewicz [6].

Proposition 1. *Let $a < b$ and $X = [a, b]$. Suppose that $h : X \to X$ satisfies the following conditions.*

1. h is of class C^3.
2. There is $c \in (a, b)$ such that $h'(c) = 0$ and $h''(c) < 0$. Furthermore, $a < x < c$ implies $h'(x) > 0$ and $c < x < b$ implies $h'(x) < 0$.
3. $h(x) > x$ for $x \in (a, c)$.
4. $Sh(x) < 0$ for $x \in (a, b)$, $x \neq c$.
5. There is an unstable periodic point $y \in X$ such that $y = h^n(c)$ for some $n \geq 2$.

Then, h has a unique ergodic invariant measure that is absolutely continuous with respect to the Lebesgue measure.

We have shown in Lemma 4 that f, when restricted to I, satisfies the properties 1–3 of Proposition 1. We shall now consider the properties 4 and 5. Given the form of f derived above, we have

$$Sf(z) = -\frac{\beta[(\beta + 2)x^2 + 2(\beta - 1)\theta^2]}{2[\theta - (1 + \beta)z]^2(\theta - z)^2},$$

where $x = 2\theta - (1 + \beta)z$. Therefore, given $f'(z) \neq 0$, $Sf(z) < 0$ if the following condition is satisfied.

Condition 5: $\beta \geq 1$.

To $P1$ and $P2$ in Lemma 4 we can now add that, under Conditions 1–5:

P3. $Sf(z) < 0$ for $z \in I \backslash \{\hat{z}\}$.

Lemma 7. *Under Conditions 1 through 5, if there is $n \geq 2$ such that $f^n(\frac{\theta}{1+\beta}) = z^* = f(z^*)$, then the dynamical system (I, f) exhibits ergodic chaos. Let v^* be the invariant measure associated with ergodic chaos. There exists a constant $\bar{S}$ such that $\bar{S} = \lim_{T \to \infty} S_T$ for v^*-almost all z_0 in I. Furthermore, if $\bar{S} < 0$, and if $\{z_t\}_{t \geq 0}$ is generated by the ergodic dynamical system $[(I, B_I, v^*), f]$, then the path $\{k_t\}$, $k_{t+1} = (\theta - z_t)k_t$, is an equilibrium.*

Proof: Under Conditions 1 through 5, the unique non-trivial steady state z^* is unstable. Therefore, $f : I \to I$ satisfies the properties 1–5 of Proposition 1 and the dynamical system (I, f) is ergodically chaotic. The remaining properties directly result from Lemma 6. **Q.E.D.**

Example 3: As in Example 1, fix parameters α, β and δ at $(\alpha, \beta, \delta) = (0.5, 5, 0.6)$ and allow parameter θ to vary in the closed interval $[1.56, 1.66]$. Note that this choice of parameters satisfies Conditions 1-5; therefore, the properties $P1$-$P3$ hold. To apply Lemma 7, it suffices to check that $f^n(\frac{\theta}{1+\beta}; \theta) = z^*$ for some $n \geq 2$, and then that $\bar{S} < 0$. To check the first condition, write $g(\theta; n) = f^n(\frac{\theta}{1+\beta}; \theta)$ and $z^*(\theta) = \theta - (\delta\theta)^{\frac{-1}{\beta(1-\alpha)}}$. Figure 2.a reports values of $\theta \in [1.56, 1.66]$ along the horizontal axis and the corresponding values of $g(\theta; 20)$ and $z^*(\theta)$ along the vertical one. $g(\theta; 20) = z^*(\theta)$ for all those values of θ at which the two curves intersect. For any such θ, (I, f) is ergodically chaotic, and there is a positive probability with which a sample series $\{z_t\}_{t \geq 0}$ is ergodic and the sample average S_T is a consistent estimator of $\bar{S}$. As Example 1 demonstrates, for a given θ, the time series $\{S_T\}_{T=4001}^{5000}$ takes negative values, which are very close to each other, and uniformly bounded away from 0. Hence, for any θ such that the hypotheses of Lemma 7 are satisfied, $\bar{S} < 0$ with positive probability and ergodic chaos is an equilibrium.

Figure 2.b illustrates the bifurcation diagram of $z_{t+1} = f(z_t; \theta)$. For each value of $\theta = 1.56 + 0.0005m$, $m = 0, 1, 2...$, we compute 2000 iterates of the critical point $\theta/(1 + \beta)$ and then, corresponding to any such θ, we report in the graph the values of $f^n(\theta/(1 + \beta); \theta)$ for $n = 1001, ..., 2000$. For the values of θ corresponding to stable cycles, both the periodic orbit and each path converging to it constitute an equilibrium, since the time series $\{S_T\}_{T=4001}^{5000}$ takes negative values, and is uniformly bounded away from 0.

Figure 2.c depicts the value of a sample Liapunov exponent $\frac{1}{T} \sum_{n=0}^{T-1} \log |f'(z_n; \theta)|$ from $z_0 = f^{N+1}(\frac{\theta}{1+\beta}; \theta)$ at different values of θ, where $N = T = 1000$. The Liapunov exponent is a measure of the complexity of the dynamical system. A positive Liapunov exponent implies sensitive dependence on initial conditions. For the values of θ at which the two curves in Figure 2.a intersect, (I, f) is ergodically chaotic. For θ close to such values, the Liapunov exponent in Figure 2.c are positive. For the values of θ at which at least one periodic point is stable, a window appears in the bifurcation diagram, Figure 2.b. For any such θ, the

Liapunov exponent is negative and the two curves in Figure 2.a do not intersect.

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Figure 1

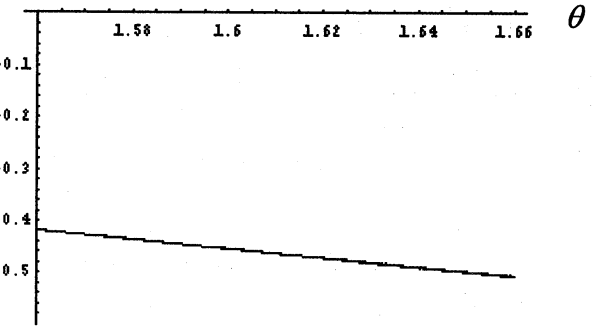

Figure 2.a

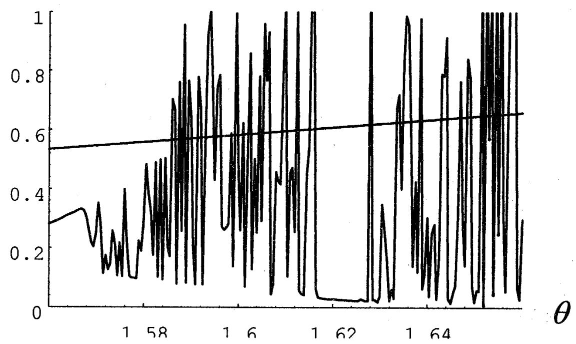

Figure 2.b

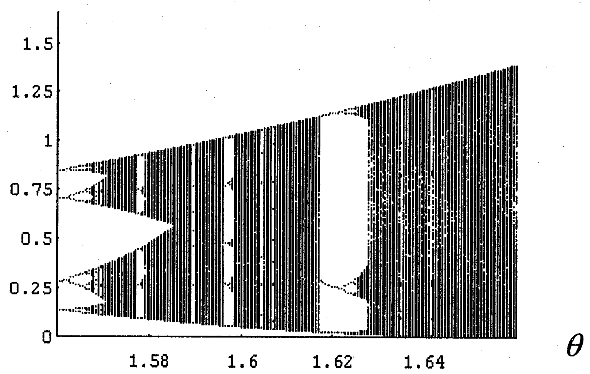

Figure 2.c