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The turning orbit is dense in the attractor for almost all Lozi families

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Abstract

This is the summary of my new paper [8]. We claim in the paper that attractors for almost every Lozi family are filled with forward orbits beginning from its singularity set, and that such a property is not special for this family. In fact, we find an open set of one-dimensional C^2 maps such that, for almost every point of the parameter arc defined by any element of this open set, the orbit of a turning point is dense in the Lozi attractor.

1 Introduction

When one studies dynamical systems for piecewise differentiable mappings, one can not escape the influence of singularity even if the systems are ruled in the nonsingular parts by some hyperbolicity. We can not directly apply the theory of smooth dynamical systems classed as Axiom A to these singular dynamics. Therefore the role of singularity is essential in the theory of piecewise hyperbolic dynamical systems. For example, the stochastic stability of piecewise expanding maps on the interval depends on the existence of their singularities which are called turning periodic orbits [1].

The simplest example of such singularity may be a *turning point* $x=0$ of the tent map $f_a(x)=1-a|x|$ on the interval. Brucks et al showed that there exists a dense set of parameters $a\in[\sqrt{2},\ 2]$ such that an orbit of the turning point is dense in its dynamical core [2]. Moreover, Brucks and Misiurewicz showed that for *almost every* $a\in[\sqrt{2},\ 2]$ the turning orbit is dense in its dynamical core, and presented some problems concerning with the turning orbit in its core [3]. Bruin gave an affirmative result to one of them, that is, for almost every parameter value, such a turning orbit of the tent map is typical for an absolutely continuous invariant probability measure [4].

The aim of this paper is to extend the results of the tent map by Brucks and Misiurewicz to some two-dimensional maps having both strange attractor and singularity set. An appropriate extension of the tent map would be the two-parameter family of piecewise diffeomorphisms in $\mathbb{R}^2$ which is called the *Lozi family*:

$$L_{a,b}(x,\ y)=(1-a|x|+y,\ bx).$$

This family was introduced and its apparent resemblance to Hénon family was pointed out by Lozi [10]. Let us consider a nonempty open subset $\mathcal{M}$ of the parameter space such that each $(a,\ b) \in \mathcal{M}$ satisfies the following conditions:

$$\begin{cases} 0 < b < 1,\ a > b+1,\ 2a+b < 4, \\ a\sqrt{2} > b+2,\ b < (a^2-1)/(2a+1). \end{cases} \tag{1}$$

Misiurewicz showed that for any $(a,\ b) \in \mathcal{M}$ there exists a bounded trapping region in $\mathbb{R}^2$ containing a $L_{a,b}$-invariant and topologically mixing subset called the *Lozi attractor* where the unstable set of a saddle fixed point in the first quadrant is dense [11]. Under somewhat weaker conditions than (1), similar attractors were found [9]. When b is negative, Cao and Liu showed that this family also has nontrivial attractors so that the union of its transversal and weak transversal homoclinic points are dense [5].

For the Lozi family, the y-axis $\{(x,\ y) \in \mathbb{R}^2 \mid x = 0\}$ is a *singularity set* corresponding to the turning point $x = 0$ for the tent maps. That is, under the condition that b is non-zero, this family maps the complement of the y-axis in some neighborhood of Lozi attractor diffeomorphically onto its image.

From a statistical standpoint some people tried to give descriptions of some good characteristics based on the SRB-measure for such type attractors, and in some unified framework these were clarified by Young recently [12]. The Hausdorff dimension of the Lozi attractor was estimated by applying Ishii's pruning par method, and moreover, the monotonicity of the topological entropy and bifurcations on the attractors were also elucidated for the case where the Lozi family is equipped with some dissipative condition, that is, a value of b is sufficiently close to zero [6, 7]. In this paper we also study the same family under some dissipative conditions but we do not need any rigid condition for parameter arcs. We have obtained the following main result about the singularity orbits in the Lozi attractor in some flexibility.

Main Theorem. *There exists a C^2-open set $\mathcal{A}$ of parameter arcs in $\mathcal{M}$ and a singularity subset $\mathcal{S}$ on the y-axis such that, for almost every parameters on the arc of $\mathcal{A}$, the forward orbit of the Lozi family beginning from $\mathcal{S}$ is dense in the Lozi attractor.*

After the notation for the one-dimensional case, we call a point in $\mathcal{S}$ a *turning point* for the Lozi family.

At the end of this introduction, it is important to know that there exist some pathological arcs in in the *complement* of $\mathcal{A}$. For example, let us consider the arc $\mathrm{graph}(\psi) = \{(a,\ \psi(a)) \in \mathcal{M} \mid 2\psi(a) = -1 - a + \sqrt{5a^2 - 2a - 3}\}$. It is not difficult to check that $L^3_{a,b}(0,\ y) = (0,\ y)$ for any parameter $(a,\ b) \in \mathrm{graph}(\psi)$, that is, the orbit of any turning point is not dense in the Lozi attractor.

2 Estimations of parameter dependence

We always require the conditions (1) to study the Lozi attractors. Then, we have $\sqrt{2} < a < 2$. The other parameter b takes values in the arc given by any $\varphi \in C^2(I)$, i.e., $b = \varphi(a)$, so that $(a,\ \varphi(a)) \in \mathcal{M}$ and

$$|\varphi'(a)|, \quad |\varphi''(a)| < \varepsilon,$$

where $I = [\sqrt{2},\ 2]$. For each $n \geq 1$, let us write

$$\xi_n(a) = L^n_{a,\varphi(a)}(0,\ y).$$

That is, we regard ξ_n as a map from the a-axis to $\mathbb{R}^2$. For each $n \geq 1$, we define

$$\gamma_n = \left\{\xi_n(a) \in \mathbb{R}^2 \ \middle|\ (a,\ \varphi(a)) \in \mathcal{M}\right\}.$$

Note that $\xi_1(a) = (1+y,\ 0)$. Then, trivially, the *tangent vector* of γ_1 is given by

$$\tau_1 = \frac{d\xi_1(a)}{da} = (0,\ 0).$$

Since $\xi_2(a) = (1 - a|1+y|,\ \varphi(a)(1+y))$, γ_2 is a smooth segment. The *tangent vector* $\tau_2(a) = \tau_2$ of γ_2 is given by

$$\tau_2 = \frac{d\xi_2(a)}{da} = (-|1+y|,\ \varphi'(a)(1+y)).$$

Hereafter, we write $\xi_n(a) = (x_n,\ y_n)$. For each $n \geq 2$, the tangent vector $\tau_n(a) = \tau_n$ of γ_n, if it exists, is obtained as follows:

$$\tau_n = \frac{d\xi_n(a)}{da} = \left(DL_{a,\varphi(a)}\right)_{n-1}\tau_{n-1} + (-|x_{n-1}|,\ \varphi'(a)x_{n-1}) \tag{2}$$

where $\left(DL_{a,\varphi(a)}\right)_{n-1}$ is the linearization of $L_{a,\varphi(a)}$ at $\xi_{n-1}(a)$. Let C^u be the unstable cone filed which is given in [11].

Lemma 1. *(i) There exist $\sqrt{2} < a_1 < 2$ and $n_1 \geq 2$ such that $\tau_n(a) \in C^u$ and*

$$\frac{|\tau_{n+1}(a)|}{|\tau_n(a)|} \geq \alpha > 1$$

for any $a_1 \leq a < 2$ and $n \geq n_1$, where $\alpha = \alpha_{a,\varepsilon}$ is a constant independently of n.

(ii) For any $\gamma > 0$ there exists $n_2 \geq 2$ such that if ξ_n is differentiable on a neighborhood I_n of $a \in (a_2,\ 2)$ for $n \geq n_2$, then

$$\frac{|\tau_n(a)|}{|\tau_n(a')|} \leq 1 + \gamma$$

for any $a' \in I_n$.

Proof. The both proofs are obtained from basic calculations, see [8]. □

3 Usefulness and maturity of parameter arcs

We extend the concepts of usefulness and order of intervals [3] to arcs in the parameter space. Let us consider an arc $\mathcal{J}$ which is given by the above φ. We say that ξ_n is differentiable on $\mathcal{J}$ if ξ_n is differentiable for any a satisfying $(a,\ \varphi(a)) \in \mathcal{J}$. The arc $\mathcal{J}$ is called *k-useful* if ξ_k is differentiable on $\mathcal{J}$ and there exists an a_0 such that the point $(a_0,\ \varphi(a_0))$ is one of the endpoints of $\mathcal{J}$ and $\xi_k(a_0)$ is located on y-axis. If there exist several k's for which $\mathcal{J}$ is k-useful, we call the largest one *order* of $\mathcal{J}$ and denote it by $\mathrm{Ord}(\mathcal{J})$.

Next we apply the concept of maturity to a subset of the arc $\mathcal{J}$ of order N. Let us consider a sub-arc $\widetilde{\mathcal{J}} \subset \mathcal{J}$. We say that $\widetilde{\mathcal{J}}$ is *mature* if there exists some $k \geq N$ such that $\widetilde{\mathcal{J}}$ is also k-useful and that for some $(\tilde{a},\ \varphi(\tilde{a})) \in \widetilde{\mathcal{J}}$ and for some $(0,\ \tilde{y}) \in \{y\text{-axis}\}$

$$\xi_k(\tilde{a}) = L^k_{\tilde{a},\varphi(\tilde{a})}(0,\ y) = L^m_{\tilde{a},\varphi(\tilde{a})}(0,\ \tilde{y})$$

for some $m \in \{1,\ 2,\ 3,\ 4\}$. The points of $\mathcal{J}$ which do not belong to any mature subset of $\mathcal{J}$ are called *bad*, and the set of such points are denoted by $\mathcal{B}$.

We last define partitions of a curve inductively. Let $\mathcal{J}$ be an arc given by the above φ and $k > 0$ be an integer such that $\xi_k(\mathcal{J}) \cap \{y\text{-axis}\} = \varnothing$ where

$$\xi_k(\mathcal{J}) = \left\{ \xi_k(a) \;\middle|\; (a,\ \varphi(a)) \in \mathcal{J} \right\}.$$

ξ_{k+n} cannot be differentiable on entire $\mathcal{J}$ for any n, see [8]. Then, there is the smallest integer $h > 0$ such that $\xi_{k+h}(\mathcal{J})$ intersects transversely at one point of the y-axis. Note that ξ_{k+h} is still differentiable on $\mathcal{J}$. So, by such a transverse intersection, $\mathcal{J}$ is divided into two $(k+h)$-useful subsets J_1 and J_2. We now get the first *partition* $\mathcal{P}_1 = \{J_1,\ J_2\}$. We let J_1 and J_2 share the dividing point. For a mature $J_i \in \mathcal{P}_1$ we define $\rho(J_i) = \{J_i\}$: otherwise, by similar steps, we can divide J_i into two $(k+h')$-useful, $h' > h$, arcs J_{i1} and J_{i2}, and set $\rho(J_i) = \{J_{i1},\ J_{i2}\}$. We let the two arcs share the dividing point. Then we get the next *partition* $\mathcal{P}_2 = \bigcup_{J_i \in \mathcal{P}_1} \rho(J_i)$. Thus, for every $n \geq 3$, we obtain the *partition* $\mathcal{P}_n = \bigcup_{J \in \mathcal{P}_{n-1}} \rho(J)$ of $\mathcal{J}$. We claim the following:

Theorem 2. *There exists an open set $\mathcal{A} \subset C^2(I)$ such that for almost every point $(a,\ \varphi(a))$ on $\mathcal{J}$ defined by any $\varphi \in \mathcal{A}$, there is a k-mature arc $\mathcal{I} \subset \mathcal{J}$ containing $(a,\ \varphi(a))$ so that one of the endpoints of the image of $\mathcal{I}$ by ξ_k keeps away from S at least by δ.*

Proof. The proof is obtained from some technical steps, see [8], but these are essentially same as [3]. □

4 Proof of the main theorem

Finally we can prove the main theorem given in the introduction. The trapping region of $L_{a,\varphi(a)}$ is denoted by $\mathcal{T}_{a,\varphi(a)}$ where $\varphi \in \mathcal{A}$, and it was shown [11] that there is an $L_{a,b}$-invariant set given by

$$\Lambda_a = \bigcap_{n \geq 0} L^n_{a,\varphi(a)}(\mathcal{T}_{a,\varphi(a)}),$$

and the unstable set W_a^u of the saddle fixed point in the first quadrant is dense in Λ_a, that is, $\Lambda_a = \mathrm{cl}(W_a^u)$.

Theorem 3. *Let $\mathcal{J}$ be an arc given by any $\varphi \in \mathcal{A}$. For almost every $(a,\ \varphi(a)) \in \mathcal{J}$, the forward orbit of any turning point in $\mathcal{S}$ for $L_{a,\varphi(a)}$ is dense in Λ_a.*

Proof. Let $\{O_i\}_{i\in\mathbb{N}}$ be a countable open base of $\mathbb{R}^2$, where each element O_i is an open ball centered at the rational point in $\mathbb{Q}^2$. For each O_i and $(0,\ y) \in \mathcal{S}$, we define a set of parameters on $\mathcal{J}$ so that the forward orbit beginning from $(0,\ y)$ never visits O_i, as follows:

$$\chi_i = \left\{(a,\ \varphi(a)) \in \mathcal{J} \ \middle|\ \Lambda_a \cap O_i \neq \varnothing,\ L^n_{a,\varphi(a)}(0,\ y) \cap O_i = \varnothing \text{ for } \forall n \geq 1\right\}.$$

Moreover, we set

$$\chi = \bigcup_{i\in\mathbb{N}} \chi_i.$$

We want to prove that

$$\mu(\chi) = 0.$$

Since $\mu(\chi) = \sum_{i\in\mathbb{N}} \mu(\chi_i)$, we show that $\mu(\chi_i) = 0$, as follows.

Suppose that $\mu(\chi_i) > 0$ for some $i \in \mathbb{N}$. Then, there exists a $(a,\ \varphi(a)) \in \chi_i$ which is a Lebesegue density point of χ_i, and such that the property of the statement of Theorem 2 holds for $(a,\ \varphi(a))$. That is, there exists a k-mature arc $\mathcal{I} \subset \mathcal{J}$ such that $(a,\ \varphi(a)) \in \mathcal{I}$. Note that $\xi_k(\mathcal{I})$ is contained in the bounded region $\mathcal{T}_{a,\varphi(a)}$, and from Lemma 1, $\tau_k \in T\xi_k(\mathcal{I})$ is contained in the unstable cone C^u. Then, there exist $m \geq 0$ and $l \subset \xi_k(\mathcal{I})$ such that

$$L^m_{\tilde{a},\varphi(\tilde{a})}(l) \subset O_i$$

for any $(\tilde{a},\ \varphi(\tilde{a})) \in \xi_k^{-1}(l)$. Since $(a,\ \varphi(a))$ is the Lebesegue density point of χ_i, for every arc $U \subset \mathcal{I}$ containing $(a,\ \varphi(a))$, we have

$$\frac{\mu(U \cap \chi_i)}{\mu(U)} > 1 - \frac{\mu(l)}{\delta}, \tag{3}$$

for any positive constant δ.

Note that

$$\mu(l) = \int_{\xi_k^{-1}(l)} |\tau_k| d\mu, \qquad \mu(\xi_k(U)) = \int_U |\tau_k| d\mu.$$

Then, from the Mean Value Theorem and Lemma 1, we have

$$\frac{\mu(l)/\mu(\xi_k^{-1}(l))}{\mu(\xi_k(U))/\mu(U)} < 1 + \gamma.$$

Then we get

$$\frac{\mu(l)}{(1+\gamma)\mu(\xi_k(U))} < \frac{\mu(\xi_k^{-1}(l))}{\mu(U)}.$$

For every $(\tilde{a},\ \varphi(\tilde{a})) \in \xi_k^{-1}(l)$, since $\xi_k(\tilde{a}) \in l$, we have

$$L^m_{\tilde{a},\ \varphi(\tilde{a})}(\xi_k(\tilde{a})) = L^m_{\tilde{a},\ \varphi(\tilde{a})}(l) \subset O_i.$$

Then,

$$\xi_{m+k}(\tilde{a}) \cap O_i \neq \varnothing.$$

Therefore, we have $(\tilde{a},\ \varphi(\tilde{a})) \in U \setminus \chi_i$. Then we get

$$\frac{\mu(\xi_k^{-1}(l))}{\mu(l)} < \frac{\mu(U \setminus \chi_i)}{\mu(U)} = 1 - \frac{\mu(U \cap \chi_i)}{\mu(U)}.$$

that is,

$$\frac{\mu(l)}{(1+\gamma)\mu(\xi_k(U))} < 1 - \frac{\mu(U \cap \chi_i)}{\mu(U)}$$

which contradicts (3). $\square$

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