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# On the topological orbit equivalence in a class of substitution minimal systems

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In this note, a partial answer to the problem to characterize the topological orbit equivalence class of substitution minimal systems. The characterization is given in terms of the Perron-Frobenius eigenvalue of a matrix associated with a substitution.

## 1. Topological orbit equivalence in Cantor systems

A topological dynamical system $(X, \phi)$ is called a Cantor system if $X$ is a Cantor set and $\phi$ is a minimal homeomorphism on $X$.

**Definition 1.1.** Let $(X, \phi)$ be a Cantor system. We put

$$\begin{aligned} \tilde{K}^0(\phi) &= C(X, \mathbb{Z})/Z_\phi \\ \tilde{K}^0_+(\phi) &= C(X, \mathbb{Z}_+)/Z_\phi \\ \tilde{u}_\phi &= [1] \end{aligned}$$

where $C(X, \mathbb{Z})$ is the abelian group of continuous functions on $X$ with integer values, [1] is the equivalence class of the constant function 1 by the subgroup $Z_\phi$ and

$$Z_\phi = \{f \in C(X, \mathbb{Z}) | \int_X f d\mu = 0 \text{ for every } \phi\text{-inv. prob. meas. on } X\}.$$

**Definition 1.2.** Let $(X_i, \phi_i)$ be Cantor systems for $i = 1, 2$. $\phi_1$ and $\phi_2$ are said to be topologically orbit equivalent if there exists a homeomorphism $F : X_1 \to X_2$ such that $F(\mathrm{Orb}_{\phi_1}(x)) = \mathrm{Orb}_{\phi_2}(F(x))$ for every $x \in X_1$ where $\mathrm{Orb}_{\phi_i}(y)$ is the orbit of $y$ by $\phi_i$.

**Theorem 1.3** ([GPS]). *The triple $(\tilde{K}^0(\phi), \tilde{K}^0_+(\phi), \tilde{u}_\phi)$ is a complete invariant of the topological orbit equivalence in the class of Cantor systems.*

Put

$$X = \prod_{i=1}^{\infty} \{0, 1, \dots, n_i\}, \quad n_i \geq 2,$$
$$\phi : X \to X, \quad \text{the addition of } (1, 0, 0, \dots) \text{ with carries.}$$

Then, $(X, \phi)$ is a Cantor systems and called the odometer system with base $(n_1, n_2, n_3, \dots)$. The invariant $\tilde{K}^0$ of $\phi$ is the group $\{l/m | l \in \mathbb{Z},\ m \text{ divides some } \prod_{i=1}^k n_i\}$ and we denote the group of this form by $\mathbb{Z}_{(q)}$ where $q = \prod_{i=1}^{\infty} n_i$ as a formal product. The invariant $\tilde{K}^0$ of $\phi$ is $(\mathbb{Z}_{(q)}, \mathbb{Z}_{(q)} \cap \mathbb{R}_+, 1)$ as the triple.

## 2. Definition of substitution systems

Let $A$ be an alphabet, i.e. a finite set, and $A^+$ be the set of words on $A$.

**Definition 2.1.** A map $\sigma : A \to A^+$ is called a substitution on $A$.

Let $\sigma$ be a substitution on $A$. A substitution $\sigma$ is naturally extended on $A^+$ and $A^{\mathbb{Z}}$. We put $\mathcal{L}(\sigma) = \{u \in A^+ | u \text{ occurs in some } \sigma^k(a), k \geq 1, a \in A\}$ and denote by $M(\sigma)$ the $A \times A$ matrix whose $(a, b)$-entry is the number of occurrences of $b$ in $\sigma(a)$ and call it the composition matrix of $\sigma$. A substitution $\sigma$ is said to be of constant length if the length of $\sigma(a)$ does not depend on the choice of $a$ and to be primitive if there exists an integer $k \geq 1$ such that for every $a, b \in A$, $a$ occurs in $\sigma^k(b)$, equivalently $M(\sigma)$ is a primitive matrix.

**Remark 2.2.** As the alphabet $A$ is a finite set, there exist an integer $k \geq 1$ and letters $a, b$ such that

1. $a$ is a prefix of $\sigma^k(a)$;
2. $b$ is a suffix of $\sigma^k(b)$;
3. $ba \in \mathcal{L}(\sigma)$.

Then $x = \lim_{n\to\infty} \sigma^{kn}(b).\sigma^{kn}(a)$ converges in $A^{\mathbb{Z}}$ where the dot means the separation between the $-1$-st coordinate and the 0-th one.

**Remark 2.3.** We always assume that every substitution $\sigma$ in this note satisfies the following conditions:

1. there exists a letter $a$ such that $\lim_{n\to\infty} \sigma^n(a) = \infty$;
2. a point $x$ given as above is aperiodic.

Let $T$ be a bilateral shift on $A^{\mathbb{Z}}$ and $X_\sigma$ be the closure of the orbit of $x$ by $T$. Put $T_\sigma = T|_{X_\sigma}$.

**Definition 2.4.** The substitution system arising from a substitution $\sigma$ is $(X_\sigma, T_\sigma)$.

**Proposition 2.5** ([Qu]). *If a substitution $\sigma$ is primitive, then $T_\sigma$ is uniquely ergodic and minimal.*

We always assume that every substitution in this note is primitive.

## 3. The invariant $\tilde{K}^0(T_\sigma)$.

**Definition 3.1.** A substitution $\sigma$ is said to be proper if there exist an integer $k \geq 1$ and letters $a, b$ such that for every letter $c$, $a$ is a prefix of $\sigma^k(c)$ and $b$ is a suffix of $\sigma^k(c)$.

**Remark 3.2** ([DHS]). A proper substitution is not a special one from the view point of dynamical systems because for every substitution $\sigma$ there exists a proper substitution $\zeta$ such $T_\zeta$ is topologically conjugate to $T_\sigma$.

We first consider the case where a substitution $\sigma$ is proper.

**Definition 3.3.** We put

$$K^0(T_\sigma) = \varinjlim(M(\sigma) : \mathbb{Z}^s \to \mathbb{Z}^s) \ \text{ where } s = |A|,$$

$$K^0_+(T_\sigma) = \bigcup_{n=1}^{\infty} \varphi_n(\mathbb{Z}^s_+),$$

$$u_{T_\sigma} = {}^t(1, \ldots, 1),$$

where $\varphi_n$ is a natural homomorphism, which satisfies that $\varphi_n = \varphi_{n+1}M(\sigma)$ and $K^0(T_\sigma) = \bigcup_{n=1}^{\infty} \varphi_n(\mathbb{Z}^s)$. Define $p_\sigma : K^0(T_\sigma) \to \mathbb{R}$ by $p_\sigma(\varphi_n(a)) = \lambda^{-(n-1)}\alpha(a)$ for $a \in \mathbb{Z}^s$ where $\lambda$ is the Perron- Frobenius eigenvalue of $M(\sigma)$ and $\alpha$ is the left eigenvector corresponding to $\lambda$ such that $\sum_i \alpha_i = 1$.

**Theorem 3.4** (From a result of [DHS]). *The invariant $(\tilde{K}^0(T_\sigma), \tilde{K}^0_+(T_\sigma), \tilde{u}_{T_\sigma})$ defined in Definition 1.1 of the topologically orbit equivalence for a substitution minimal system $(X_\sigma, T_\sigma)$ is $(K^0(T_\sigma)/\ker(p_\sigma), K^0_+(T_\sigma)/\ker(p_\sigma), p_\sigma(u_{T_\sigma})) = (\mathrm{Im}(p_\sigma), \mathrm{Im}(p_\sigma) \cap \mathbb{R}_+, 1)$.*

Therefore, if $\lambda$ is rational,i.e. integral, then $\tilde{K}^0(T_\sigma) = \mathbb{Z}_{(d\cdot\lambda^\infty)}$ for some integer $d \geq 1$.

Next, we consider the case where a substitution $\sigma$ is not proper.

**Definition 3.5.** A word $u \in \mathcal{L}(\sigma)$ is a return word to $ba$, where $a$ and $b$ are letters, if

1. $a$ is a prefix of $u$.
2. $b$ is a suffix of $u$.
3. $bua \in \mathcal{L}(\sigma)$.
4. $ba$ occurs in $bua$ only twice.

**Remark 3.6.** The number of return words is finite because of the minimality of $T_\sigma$. The length of a return word $u$ to $ba$ is the first return time to the cylider set $[b.a]$ of the points in the cylider set $[b.ua]$ where $[u.v] = \{y \in X_\sigma | y_{[-|u|,|v|)} = uv\}$ for words $u, v$.

Fix an integer $k \geq 1$ and letters $a, b$ such that the conditions of Remark 2.2 hold. Put $W = \{w_1, \ldots, w_r\}$ indexed in order of occurence without multiplicities in $x_{[0,+\infty)}$. Define a substitution $\tau$ on the alphabet $R = \{1, \ldots, r\}$ by

$$\tau(i) = i_1 \ldots i_l \ \text{ if } \ \sigma^k(w_i) = w_{i_1} \ldots w_{i_l}.$$

**Proposition 3.7** ([DHS]). *The substitution $\tau$ defined as above is primitive and proper. The substitution system arising from $\tau$ is topologically conjugate to the induced transformation on $[b.a]$ by $T_\sigma$.*

**Definition 3.8.** We put

$$K^0(T_\sigma) = \varinjlim(M(\tau) : \mathbb{Z}^r \to \mathbb{Z}^r),$$

$$K^0_+(T_\sigma) = \bigcup_{n=1}^{\infty} \psi_n(\mathbb{Z}^r_+),$$

$$u_{T_\sigma} = {}^t(|w_1|, \ldots, |w_r|),$$

where $\psi_n$ is a natural homomorphism as in Definition 3.3. Define $p_\sigma : K^0(T_\sigma) \to \mathbb{R}$ by $p_\sigma(\psi_n(a)) = \mu^{-(n-1)}\beta(a)$, $a \in \mathbb{Z}^r$, where $\mu$ is the Perron-Frobenius eigenvalue of $M(\tau)$ and $\beta$ is the left eigenvector corresponding to $\beta$ such that $\sum_i \beta_i |w_i| = 1$.

**Theorem 3.9** (From a result of [DHS]). *The invariant* $(\tilde{K}^0(T_\sigma), \tilde{K}^0_+(T_\sigma), \tilde{u}_{T_\sigma})$ *defined in Definition 1.1 of the topologically orbit equivalence for a substitution minimal system* $(X_\sigma, T_\sigma)$ *is* $(K^0(T_\sigma)/\ker(p_\sigma), K^0_+(T_\sigma)/\ker(p_\sigma), p_\sigma(u_{T_\sigma})) = (\mathrm{Im}(p_\sigma), \mathrm{Im}(p_\sigma) \cap \mathbb{R}_+, 1)$.

Therefore, if $\mu$ is integral, then $\tilde{K}^0(T_\sigma) = \mathbb{Z}_{(d' \cdot \mu^\infty)}$ for some integer $d' \geq 1$.

**Remark 3.10.** Given a substitution $\sigma$, there exist an infinite graph and a partial order on the edge set of the graph which induces a minimal homeomorphism on the infinite path space which is topologically conjugate to $T_\sigma$. If $\sigma$ is proper, then the connection rule between vertices in the corresponding graph is given by $M(\sigma)$. If $\sigma$ is not proper, then the connection rule is given by $M(\tau)$. This is the reason why the way to compute the invariant $\tilde{K}^0(T_\sigma)$ is different between the case where $\sigma$ is proper and the case where $\sigma$ is not proper. See [DHS] for more details.

**Theorem 3.11** ([Yu]). *Let* $\sigma$ *be a substitution whose* $M(\sigma)$ *has an integral Perron-Frobenius eigenvalue* $\lambda$. *Then, the substitution system arising from the substitution* $\sigma$ *is topologically orbit equivalent to the odometer system with base* $(d, \lambda, \lambda, \ldots)$ *(called a stationary odometer system ) for some integer* $d \geq 1$. *In particular, every substitution system arising from a substitution of constant length is topologically orbit equivalent to a stationary odometer system.*

Key lemma for the proof is the following.

**Lemma 3.12.** $\mu = \lambda^k$.

*Proof.* Let $S$ be an $R \times A$ matrix whose $(a, i)$-entry is the number of occurrences of $a$ in $w_i$. Then $SM(\sigma)^k = M(\tau)S$. Therefore, $\mu = \lambda^k$ because of the Perron-Frobenius Theorem. □

**Remark 3.13.** When $\sigma$ is proper, $d = \sum_i \alpha_i$ where $\alpha = (\alpha_1, \ldots, \alpha_s)$ is the left Perron-Frobenius eigenvector of $M(\sigma)$ such that every $\alpha_i$ is integral and $(\alpha_i, \alpha_j) = 1$ if $i \neq j$. When $\sigma$ is not proper, $d' = \sum_i \beta_i |w_i|$ where $\beta = (\beta_1, \ldots, \beta_r)$ is the left Perron-Frobenius eigenvector of $M(\tau)$ such that each $\beta_i$ is integral and $(\beta_i, \beta_j) = 1$ if $i \neq j$.

The converse of Theorem 3.11:

**Theorem 3.14** ([Yu]). *Let* $(X, \phi)$ *be an arbitrary stationary odometer system and its base be* $(d, \lambda, \lambda, \ldots)$. *Then, there exists a proper and primitive substitution* $\sigma$ *of constant length such that* $T_\sigma$ *is topologically orbit equivalent to* $\phi$.

*Proof.* We may assume that $d > 1$ and $\lambda > 1$. It is enough to show that there exists a proper and primitive substitution $\sigma$ of constant length $\lambda^n$ on the alphabet $\{1, \ldots, d\}$ for some integer $n \geq 1$. Take $n \geq 1$ such that $\lambda^n > 3 \vee d$. Put $v = {}^t(\lambda^m, \lambda^m, \ldots, \lambda^m, (\lambda^m - d + 1)\lambda^m)$. Let $M$ be the integral $d \times d$ matrix whose $(i, j)$-entry is the $\kappa^{j-1}(i)$-th entry of $v$ for $1 \leq i, j \leq d$ where $\kappa$ is the permutation on $\{1, 2, \ldots, d\}$ defined by $\kappa(d) = 1$ and $\kappa(i) = i + 1$ if $1 \leq i < d$. We can find a proper and primitive substitution $\sigma$ such that $M(\sigma) = M$. □

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