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On the topological orbit equivalence in a class of substitution minimal systems

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In this note, a partial answer to the problem to characterize the topological orbit equivalence class of substitution minimal systems. The characterization is given in terms of the Perron-Frobenius eigenvalue of a matrix associated with a substitution.

1. Topological Orbit Equivalence in Cantor Systems

A topological dynamical system \((X, \phi)\) is called a Cantor system if \(X\) is a Cantor set and \(\phi\) is a minimal homeomorphism on \(X\).

Definition 1.1. Let \((X, \phi)\) be a Cantor system. We put
\[
\tilde{K}^0(\phi) = C(X, \mathbb{Z})/Z_{\phi}
\]
\[
\tilde{K}_+^0(\phi) = C(X, \mathbb{Z}_+)/Z_{\phi}
\]
\[
\tilde{u}_\phi = [1]
\]
where \(C(X, \mathbb{Z})\) is the abelian group of continuous functions on \(X\) with integer values, \([1]\) is the equivalence class of the constant function 1 by the subgroup \(Z_{\phi}\) and
\[
Z_{\phi} = \{ f \in C(X, \mathbb{Z}) | \int_X f d\mu = 0 \text{ for every } \phi^{-}\text{inv. prob. meas. on } X \}.
\]

Definition 1.2. Let \((X_i, \phi_i)\) be Cantor systems for \(i = 1, 2\). \(\phi_1\) and \(\phi_2\) are said to be topologically orbit equivalent if there exists a homeomorphism \(F : X_1 \rightarrow X_2\) such that \(F(\text{Orb}_{\phi_1}(x)) = \text{Orb}_{\phi_2}(F(x))\) for every \(x \in X_1\) where \(\text{Orb}_{\phi_i}(y)\) is the orbit of \(y\) by \(\phi_i\).

Theorem 1.3 ([GPS]). The triple \((\tilde{K}^0(\phi), \tilde{K}_+^0(\phi), \tilde{u}_\phi)\) is a complete invariant of the topological orbit equivalence in the class of Cantor systems.

Put
\[
X = \prod_{i=1}^\infty \{0, 1, \ldots, n_i\}, \quad n_i \geq 2,
\]
\[
\phi : X \rightarrow X, \text{ the addition of } (1, 0, 0, \ldots) \text{ with carries.}
\]

Then, \((X, \phi)\) is a Cantor systems and called the odometer system with base \((n_1, n_2, n_3, \ldots)\). The invariant \(\tilde{K}^0\) of \(\phi\) is the group \(\{l/m | l \in \mathbb{Z}, m \text{ divides some } \prod_{i=1}^\infty n_i\}\) and we denote the group of this form by \(\mathbb{Z}(q)\) where \(q = \prod_{i=1}^\infty n_i\) as a formal product. The invariant \(\tilde{K}^0\) of \(\phi\) is \((\mathbb{Z}(q), \mathbb{Z}(q) \cap \mathbb{R}_+, 1)\) as the triple.
2. Definition of substitution systems

Let $A$ be an alphabet, i.e. a finite set, and $A^+$ be the set of words on $A$.

**Definition 2.1.** A map $\sigma : A \to A^+$ is called a substitution on $A$.

Let $\sigma$ be a substitution on $A$. A substitution $\sigma$ is naturally extended on $A^+$ and $A^\mathbb{Z}$. We put $L(\sigma) = \{ u \in A^+ | u \text{ occurs in some } \sigma^k(a), k \geq 1, a \in A \}$ and denote by $M(\sigma)$ the $A \times A$ matrix whose $(a, b)$-entry is the number of occurrences of $b$ in $\sigma(a)$ and call it the composition matrix of $\sigma$. A substitution $\sigma$ is said to be of constant length if the length of $\sigma(a)$ does not depend on the choice of $a$ and to be primitive if there exists an integer $k \geq 1$ such that for every $a, b \in A$, $a$ occurs in $\sigma^k(b)$, equivalently $M(\sigma)$ is a primitive matrix.

**Remark 2.2.** As the alphabet $A$ is a finite set, there exist an integer $k \geq 1$ and letters $a, b$ such that

1. $a$ is a prefix of $\sigma^{k}(a)$;
2. $b$ is a suffix of $\sigma^{k}(b)$;
3. $ba \in L(\sigma)$.

Then $x = \lim_{n \to \infty} \sigma^{kn}(b) \cdot \sigma^{kn}(a)$ converges in $A^\mathbb{Z}$ where the dot means the separation between the $-1$-st coordinate and the $0$-th one.

**Remark 2.3.** We always assume that every substitution $\sigma$ in this note satisfies the following conditions:

1. there exists a letter $a$ such that $\lim_{n \to \infty} \sigma^n(a) = \infty$;
2. a point $x$ given as above is aperiodic.

Let $T$ be a bilateral shift on $A^\mathbb{Z}$ and $X_\sigma$ be the closure of the orbit of $x$ by $T$. Put $T_\sigma = T|_{X_\sigma}$.

**Definition 2.4.** The substitution system arising from a substitution $\sigma$ is $(X_\sigma, T_\sigma)$.

**Proposition 2.5** ([Qu]). If a substitution $\sigma$ is primitive, then $T_\sigma$ is uniquely ergodic and minimal.

We always assume that every substitution in this note is primitive.

3. The invariant $\tilde{K}^0(T_\sigma)$.

**Definition 3.1.** A substitution $\sigma$ is said to be proper if there exist an integer $k \geq 1$ and letters $a, b$ such that for every letter $c$, $a$ is a prefix of $\sigma^k(c)$ and $b$ is a suffix of $\sigma^k(c)$.

**Remark 3.2** ([DHS]). A proper substitution is not a special one from the view point of dynamical systems because for every substitution $\sigma$ there exists a proper substitution $\zeta$ such $T_\zeta$ is topologically conjugate to $T_\sigma$.

We first consider the case where a substitution $\sigma$ is proper.
Definition 3.3. We put
\[ K^0(T_{\sigma}) = \lim_{n \to \infty} (M(\sigma) : \mathbb{Z}^r \to \mathbb{Z}^r) \text{ where } s = |A|, \]
\[ K_+^0(T_{\sigma}) = \bigcup_{n=1}^{\infty} \varphi_n(\mathbb{Z}^r_+), \]
\[ u_{T_{\sigma}} = (1, \ldots, 1), \]
where \( \varphi_n \) is a natural homomorphism, which satisfies that \( \varphi_n = \varphi_{n+1} M(\sigma) \) and \( K^0(T_{\sigma}) = \bigcup_{n=1}^{\infty} \varphi_n(\mathbb{Z}^r) \). Define \( p_\sigma : K^0(T_{\sigma}) \to \mathbb{R} \) by \( p_\sigma(\varphi_n(a)) = \lambda^{-(n-1)} \alpha(a) \) for \( a \in \mathbb{Z}^r \) where \( \lambda \) is the Perron-Frobenius eigenvalue of \( M(\sigma) \) and \( \alpha \) is the left eigenvector corresponding to \( \lambda \) such that \( \sum_i \alpha_i = 1 \).

Theorem 3.4 (From a result of [DHS]). The invariant \( \tilde{K}^0(T_{\sigma}), \tilde{K}_+^0(T_{\sigma}), \tilde{u}_{T_{\sigma}} \) defined in Definition 1.1 of the topologically orbit equivalence for a substitution minimal system \((X_{\sigma}, T_{\sigma})\) is \((K^0(T_{\sigma})/\ker(p_\sigma), K_+^0(T_{\sigma})/\ker(p_\sigma), p_\sigma(u_{T_{\sigma}})) = (\mathrm{Im}(p_\sigma), \mathrm{Im}(p_\sigma) \cap \mathbb{R}_+, 1)\).

Therefore, if \( \lambda \) is rational, i.e., integral, then \( \tilde{K}^0(T_{\sigma}) = \mathbb{Z}_{(d \cdot \lambda^\infty)} \) for some integer \( d \geq 1 \).

Next, we consider the case where a substitution \( \sigma \) is not proper.

Definition 3.5. A word \( u \in \mathcal{L}(\sigma) \) is a return word to \( ba \), where \( a \) and \( b \) are letters, if
1. \( a \) is a prefix of \( u \).
2. \( b \) is a suffix of \( u \).
3. \( bua \in \mathcal{L}(\sigma) \).
4. \( ba \) occurs in \( bua \) only twice.

Remark 3.6. The number of return words is finite because of the minimality of \( T_{\sigma} \). The length of a return word \( u \) to \( ba \) is the first return time to the cylinder set \([b.a]\) of the points in the cylinder set \([b.ua]\) where \([u.v] = \{ y \in X_\sigma | y_{[-|u|,|v|]} = uv \} \) for words \( u, v \).

Fix an integer \( k \geq 1 \) and letters \( a, b \) such that the conditions of Remark 2.2 hold. Put \( W = \{ w_1, \ldots, w_r \} \) indexed in order of occurrence without multiplicities in \( x_{[0, +\infty)} \). Define a substitution \( \tau \) on the alphabet \( R = \{1, \ldots, r \} \) by
\[ \tau(i) = i_1 \ldots i_l \text{ if } \sigma^k(w_i) = w_i_1 \ldots w_i_l. \]

Proposition 3.7 ([DHS]). The substitution \( \tau \) defined as above is primitive and proper. The substitution system arising from \( \tau \) is topologically conjugate to the induced transformation on \([b.a]\) by \( T_{\sigma} \).

Definition 3.8. We put
\[ K^0(T_{\sigma}) = \lim_{n \to \infty} (M(\tau) : \mathbb{Z}^r \to \mathbb{Z}^r), \]
\[ K_+^0(T_{\sigma}) = \bigcup_{n=1}^{\infty} \psi_n(\mathbb{Z}^r_+), \]
\[ u_{T_{\sigma}} = (|w_1|, \ldots, |w_r|), \]
where \( \psi_n \) is a natural homomorphism as in Definition 3.3. Define \( p_\sigma : K^0(T_{\sigma}) \to \mathbb{R} \) by \( p_\sigma(\psi_n(a)) = \mu^{-(n-1)} \beta(a), \) \( a \in \mathbb{Z}^r \), where \( \mu \) is the Perron-Frobenius eigenvalue of \( M(\tau) \) and \( \beta \) is the left eigenvector corresponding to \( \beta \) such that \( \sum_i \beta_i |w_i| = 1 \).
Theorem 3.9 (From a result of [DHS]). The invariant \( \tilde{K}^0(T_\sigma), \tilde{K}^0(T_\sigma), \tilde{u}_\sigma \) defined in Definition 1.1 of the topologically orbit equivalence for a substitution minimal system \((X_\sigma, T_\sigma)\) is \((K^0(T_\sigma)/\ker(p_\sigma), K^0(T_\sigma)/\ker(p_\sigma), p_\sigma(\tau_{T_\sigma})) = (\Im(p_\sigma), \Im(p_\sigma) \cap \mathbb{R}_+, 1)\).

Therefore, if \( \mu \) is integral, then \( \tilde{K}^0(T_\sigma) = \mathbb{Z}(d', \mu^\infty) \) for some integer \( d' \geq 1 \).

Remark 3.10. Given a substitution \( \sigma \), there exist an infinite graph and a partial order on the edge set of the graph which induces a minimal homeomorphism on the infinite path space which is topologically conjugate to \( T_\sigma \). If \( \sigma \) is proper, then the connection rule between vertices in the corresponding graph is given by \( M(\sigma) \). If \( \sigma \) is not proper, then the connection rule is given by \( M(\tau) \). This is the reason why the way to compute the invariant \( \tilde{K}^0(T_\sigma) \) is different between the case where \( \sigma \) is proper and the case where \( \sigma \) is not proper. See [DHS] for more details.

Theorem 3.11 ([Yu]). Let \( \sigma \) be a substitution whose \( M(\sigma) \) has an integral Perron-Frobenius eigenvalue \( \lambda \). Then, the substitution system arising from the substitution \( \sigma \) is topologically orbit equivalent to the odometer system with base \((d, \lambda, \lambda, \ldots)\) (called a stationary odometer system) for some integer \( d \geq 1 \). In particular, every substitution system arising from a substitution of constant length is topologically orbit equivalent to a stationary odometer system.

Key lemma for the proof is the following.

Lemma 3.12. \( \mu = \lambda^k \).

Proof. Let \( S \) be an \( R \times A \) matrix whose \((a, i)\)-entry is the number of occurrences of \( a \) in \( w_i \). Then \( SM(\sigma)^k = M(\tau)S \). Therefore, \( \mu = \lambda^k \) because of the Perron-Frobenius Theorem.

Remark 3.13. When \( \sigma \) is proper, \( d = \sum \alpha_i \) where \( \alpha = (\alpha_1, \ldots, \alpha_s) \) is the left Perron-Frobenius eigenvector of \( M(\sigma) \) such that every \( \alpha_i \) is integral and \((\alpha_i, \alpha_j) = 1 \) if \( i \neq j \). When \( \sigma \) is not proper, \( d' = \sum \beta_i |w_i| \) where \( \beta = (\beta_1, \ldots, \beta_r) \) is the left Perron-Frobenius eigenvector of \( M(\tau) \) such that each \( \beta_i \) is integral and \((\beta_i, \beta_j) = 1 \) if \( i \neq j \).

The converse of Theorem 3.11:

Theorem 3.14 ([Yu]). Let \((X, \phi)\) be an arbitrary stationary odometer system and its base be \((d, \lambda, \lambda, \ldots)\). Then, there exists a proper and primitive substitution \( \sigma \) of constant length such that \( T_\sigma \) is topologically orbit equivalent to \( \phi \).

Proof. We may assume that \( d > 1 \) and \( \lambda > 1 \). It is enough to show that there exists a proper and primitive substitution \( \sigma \) of constant length \( \lambda^n \) on the alphabet \( \{1, \ldots, d\} \) for some integer \( n \geq 1 \). Take \( n \geq 1 \) such that \( \lambda^n > 3 \vee d \). Put \( v = (\lambda^m, \lambda^m, \ldots, \lambda^m, (\lambda^m - d + 1)\lambda^m) \). Let \( M \) be the integral \( d \times d \) matrix whose \((i, j)\)-entry is the \( \kappa^{j-1}(i) \)-th entry of \( v \) for \( 1 \leq i, j \leq d \) where \( \kappa \) is the permutation on \( \{1, 2, \ldots, d\} \) defined by \( \kappa(d) = 1 \) and \( \kappa(i) = i + 1 \) if \( 1 \leq i < d \). We can find a proper and primitive substitution \( \sigma \) such that \( M(\sigma) = M \).
REFERENCES


[Yu] H.Yuasa, On the topological orbit equivalence in a class of substitution minimal systems, Preprint.