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The inverse problem of the Birkhoff-Gustavson normalization and *ANFER*, Algorithm of Normal Form Expansion and Restoration[1]

京都大学情報学研究科 上野 嘉夫 (UWANO, Yoshio)

Department of Applied Mathematics and Physics
Kyoto University
Kyoto 606-8501, Japan

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Abstract

In the series of papers [1-4], the inverse problem of the Birkhoff-Gustavson normalization was posed and studied. To solve the inverse problem, the symbolic-computing program named ANFER (Algorithm of Normal Form Expansion and Restoration) is written up, with which a new aspect of the Bertrand and Darboux integrability condition is found [1]. In this paper, the procedure in ANFER is presented in mathematical terminology, which is organized on the basis of the composition of canonical transformations.

Keywords: Birkhoff-Gustavson normal form, Inverse problem, Computer algebra

1 Introduction

It has been recognized that the Birkhoff-Gustavson normalization [5] works effectively in studying various nonlinear dynamical systems. When a two-degree-of-freedom Hamiltonian system with a 1:1-resonant equilibrium point is given, for example, the BG-normalization of its Hamiltonian around the equilibrium point provides an 'approximate' Hamiltonian system whose Hamiltonian is the truncation of the normalized Hamiltonian up to a finite degree: The approximate Hamiltonian system provides a good account of the surface of section with sufficiently small energies of the given system [6, 7]. Such a good approximation thereby implies that to find the family of Hamiltonian systems sharing the same BG-normalization up to a finite-degree amounts to find a family of Hamiltonian systems admitting the surface of section similar to each other. The following question has been posed by the author as the *inverse problem* of the BG-normalization [1, 2, 3, 4]: *What kind of polynomial Hamiltonian can be brought into a given polynomial Hamiltonian in BG-normal form ?*

The Birkhoff-Gustavson normalization to be dealt with this paper is outlined as follows (cf. [8, 1]): Let $\mathbf{R}^n \times \mathbf{R}^n$ be the phase space endowed with the Cartesian coordinates

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(q,p) working as canonical coordinates. Let $K(q,p)$ be a Hamiltonian function defined on $\mathbf{R}^n \times \mathbf{R}^n$ (or a certain domain of it) which admits the power-series expansion

$$K(q,p) = \sum_{j=1}^{n} \frac{\nu_j}{2}(p_j^2 + q_j^2) + \sum_{k=3}^{\infty} K_k(q,p), \tag{1}$$

around the origin of $\mathbf{R}^n \times \mathbf{R}^n$, where each $K_k(q,p)$ $(k = 3, 4, \cdots)$ is a homogeneous polynomial of degree-k in (q,p), and $\{\nu_j\}$ non-vanishing constants.

Remark 1 *The convergent radius of the power series (1) may vanish [8]; this happens to any K that is not analytic but differentiable around the origin, for example. In such a case, the power series (1) is considered only in a formal sense. We will, however, often eliminate the word 'formal' from such formal power series henceforth.*

The BG-normalization of $K(q,p)$ is made as follows: Let (ξ,η) be another canonical coordinates of $\mathbf{R}^n \times \mathbf{R}^n$ and let the power series (see Remark 1)

$$W(q,\eta) = \sum_{j=1}^{n} q_j\eta_j + \sum_{k=3}^{\infty} W_k(q,\eta) \tag{2}$$

be a second-type generating function [9], a function in the *old* position variables q and the *new* momentum variables η, associated with the canonical transformation

$$(q,p) \to (\xi,\eta) \quad \text{with} \quad p = \frac{\partial S}{\partial q} \quad \text{and} \quad \xi = \frac{\partial S}{\partial \xi}, \tag{3}$$

where each $W_k(q,\eta)$ $(k = 3, 4, \cdots)$ is the homogeneous polynomials of degree k in (q,η). Note that the transformation (3) leaves the origin of $\mathbf{R}^n \times \mathbf{R}^n$ invariant. Through the transformation (3), the Hamiltonian $K(q,p)$ is brought into a power series, say $G(\xi,\eta)$, subject to

$$G(\frac{\partial W}{\partial \eta}, \eta) = H(q, \frac{\partial W}{\partial q}), \tag{4}$$

It is easy to see from (1), (2) and (4) that $G(\xi,\eta)$ takes the form

$$G(\xi,\eta) = \sum_{j=1}^{n} \frac{\nu_j}{2}\left(\eta_j^2 + \xi_j^2\right) + \sum_{k=3}^{\infty} G_k(\xi,\eta), \tag{5}$$

where each $G_k(\xi,\eta)$ $(k = 3, 4, \cdots)$ is a homogeneous polynomial of degree-k in (ξ,η).

Definition 1.1 *The power series $G(\xi,\eta)$ is said to be in the Birkhoff-Gustavson (BG-) normal form up to degree-r if $G(\xi,\eta)$ satisfies the Poisson-commuting relation,*

$$\left\{ \sum_{j=1}^{n} \frac{\nu_j}{2}\left(\eta_j^2 + \xi_j^2\right),\ G_k(\xi,\eta) \right\}_{\xi,\eta} = 0 \quad (k = 3, \cdots, r), \tag{6}$$

where $\{\cdot,\cdot\}_{\xi,\eta}$ denotes the canonical Poisson bracket to the coordinates (ξ,η) (see [10]).

The inverse problem of the BG-normalization has been hence posed as follows [1, 2]: *For a given power series Hamiltonian $G(\xi,\eta)$ in the BG-normal form (5) with (6, $r=\infty$), identify all the possible power-series Hamiltonians whish share $G(\xi,\eta)$ as their BG-normalization.* In contrast with the inverse prblem, we will refer to the problem of normalizing Hamiltonians into BG-normal form as the *ordinary* problem.

Since elementary algebraic operations, differentiation, and integration of polynomials have to be repeated many times to solve the inverse problem, computer algebra is worth applying to the inverse problem; see [1, 2, 11] for 'ANFER' and [3, 4] for 'GITA^{-1}'. The procedure in ANFER is based deeply on the composition of canonical transformation, which is published in [1] only for the two-degree-of-freedom in 1:1 resonance case.

The aim of this paper is to present the procedure in ANFER in mathematical terminology. After the procedure, a new aspect of the Bertrand-Darboux integrability found through the inverse problem of certain perturbed oscillator Hamiltonians [1] is presented briefly as an application of ANFER. The contents of this paper is organized as follows.

Section 2 sets up the inverse problem of the BG-normalization: For a better understanding of the inverse problem, the solution of the ordinary problem is given first. After that, the inverse problem is posed and solved. Section 3 is devoted to studying the composition of canonical transformations which provides a key of organizing ANFER. Those who have taken a course of analytical mechanics might think that the composition is well-known already. However, it seems that almost all the things known of are on the composition of *infinitesimal* transformations. Since the transformations dealt with here are not infinitesimal, section 3 is indeed important to organize ANFER. In section 4, the procedure in ANFER is described. Section 5 is for an application of ANFER: a new aspect of the Bertrand-Darboux integrability condition for certain the perturbed harmonic oscillators [1] is presented. Section 6 is for the concluding remarks.

2 The inverse problem of the BG-normalization

The major part of this section is devoted to the inverse problem of the BG-normalization posed in [1, 2, 3, 4]. Before the inverse problem, however, we show the way solve the ordinary problem, which will promote a better understanding of the inverse problem.

2.1 Solving the ordinary problem

Let us start with equating the homogeneous-polynomial part of degree-k ($k=3,4,\cdots$) in (4). Then equation (4) is put into the series of equations,

$$G_k(q,\eta)+(D_{q,\eta}W_k)=K_k(q,\eta)+\Phi_k(q,\eta)\quad(k=3,4,\cdots),\tag{7}$$

where $D_{q,\eta}$ is the differential operator,

$$D_{q,\eta} = \sum_{j=1}^{n} \nu_j \left(q_j \frac{\partial}{\partial \eta_j} - \eta_j \frac{\partial}{\partial q_j} \right). \tag{8}$$

The $\Phi_k(q,\eta)$ in (7) is the homogeneous polynomial of degree-k in (q,η) which is uniquely determined by $W_3, \cdots, W_{k-1}$, $K_3, \cdots, K_{k-1}$, $G_3, \cdots, G_{k-1}$ given: In particular, we have $\Phi_3(q,\eta) = 0$ and

$$\Phi_4(q,\eta) = \sum_{j=1}^{n} \left(\frac{1}{2} \left(\frac{\partial W_3}{\partial q_j} \right)^2 + \left. \frac{\partial K_3}{\partial p_j} \right|_{(q,\eta)} \frac{\partial W_3}{\partial q_j} - \frac{1}{2} \left(\frac{\partial W_3}{\partial \eta_j} \right)^2 - \left. \frac{\partial G_3}{\partial \xi_j} \right|_{(q,\eta)} \frac{\partial W_3}{\partial \eta_j} \right), \tag{9}$$

while Φ_k with larger k would be of very complicated form to be described.

To solve equation (7), the direct-sum decomposition induced by $D_{q,\eta}$ of the spaces of homogeneous polynomials are of great use. Let us denote by $V_k(q,\eta)$ the vector space of homogeneous polynomials of degree-k in (q,η) with real-valued coefficients $(k = 0, 1, \cdots)$. Since the differential operator $D_{q,\eta}$ acts linearly on each $V_k(q,\eta)$, the action of $D_{q,\eta}$ naturally induces the direct-sum decomposition,

$$V_k(q,\eta) = \operatorname{image} D_{q,\eta}^{(k)} \oplus \ker D_{q,\eta}^{(k)} \quad (k = 0, 1, \cdots), \tag{10}$$

of $V_k(q,\eta)$, where $D_{q,\eta}^{(k)}$ denotes the restriction,

$$D_{q,\eta}^{(k)} = D_{q,\eta}|_{V_k(q,\eta)} \quad (k = 0, 1, \cdots). \tag{11}$$

For $D_{q,\eta}$ and $D_{q,\eta}^{(k)}$ $(k = 3, 4, \cdots)$, we have the following easy to be shown.

Lemma 2.1 *Equation (6) is equivalent to*

$$(D_{q,\eta}(G_k|_{(q,\eta)}))(q,\eta) = (D_{q,\eta}^{(k)}(G_k|_{(q,\eta)}))(q,\eta) = 0 \quad (k = 3, \cdots, r). \tag{12}$$

Namely,

$$G_k|_{(q,\eta)} \in \ker D_{q,\eta}^{(k)} \quad (k = 3, \cdots, r). \tag{13}$$

Lemma 2.2 *Let the coefficients, $\{\nu_j\}_{j=1,\cdots,n}$, of $\sum_{j=1}^{n} (\nu_j/2)(p_j^2 + q_j^2)$ in $K(q,p)$ be said to be independent over $\mathbf{Z}$ (integers) if and only if*

$$\sum_{j=1}^{n} \gamma_j \nu_j = 0 \quad (\gamma_j \in \mathbf{Z}) \quad \Leftrightarrow \quad \gamma_j = 0 \; (j = 1, \cdots, n). \tag{14}$$

holds true. Then, $\ker D_{q,\eta}$ *is spanned by the even degree polynomials,*

$$P_m = \prod_{j=1}^{n} (\eta_j^2 + q_j^2)^{m_j} \qquad (m_j : \textit{non-negative integer},\ j = 1, \cdots, n), \tag{15}$$

if and only if $\{\nu_j\}$ are independent over $\mathbf{Z}$ (cf. [8]). In particular, if $\{\nu_j\}$ are independent over $\mathbf{Z}$ then $\ker D_{q,\eta}^{(k)} = \{0\}$ *for every odd k.*

We proceed to solving (7) now. According to (10), let us decompose $K_k(q,\eta)$ and $\Phi_k(q,\eta)$ $(k=3,4,\cdots)$ to be

$$\begin{aligned} K_k(q,\eta) &= K_k^{\text{image}}(q,\eta) + K_k^{\text{ker}}(q,\eta), \\ \Phi_k(q,\eta) &= \Phi_k^{\text{image}}(q,\eta) + \Phi_k^{\text{ker}}(q,\eta), \end{aligned} \tag{16}$$

where

$$\begin{aligned} &K_k^{\text{image}}(q,\eta), \Phi_k^{\text{image}}(q,\eta) \in \operatorname{image} D_{q,\eta}^{(k)}, \\ &K_k^{\text{ker}}(q,\eta), \Phi_k^{\text{ker}}(q,\eta) \in \ker D_{q,\eta}^{(k)}, . \end{aligned} \tag{17}$$

Since $G_k(q,\eta) \in \ker D_{q,\eta}^{(k)}$ by Lemma 2.1 and since $D_{q,\eta}W_k \in \operatorname{image} D_{q,\eta}^{(k)}$, we obtain

$$G_k(q,\eta) = K_k^{\text{ker}}(q,\eta) + \Phi_k^{\text{ker}}(q,\eta) \quad (k=3,4,\cdots), \tag{18}$$

as a solution of (7), where W_k is chosen to be

$$W_k(q,\eta) = \left(D_{q,\eta}^{(k)}\Big|_{\operatorname{image} D_{q,\eta}^{(k)}}^{-1} (K_k^{\text{image}}|_{(q,\eta)} + \Phi_k^{\text{image}}) \right)(q,\eta). \tag{19}$$

What is crucial of (19) is that $W_k(q,\eta) \in \operatorname{image} D_{q,\eta}^{(k)}$ $(k=3,4,\cdots)$, which ensures the uniqueness of $G(\xi,\eta)$: For a certain integer $\kappa \geq 3$, let us consider the sum, $\tilde{W}_\kappa = W_\kappa +$ (any polynomial in $\ker D_{q,\eta}^{(\kappa)}$), where G_k and W_k with $k<\kappa$ are given by (18,19). Even after such a modification, $\tilde{W}_\kappa$ satisfy (19) still, which will lead another series of solutions of (7) with $k>\kappa$. Therefore, under the restriction, $W_k \in \operatorname{image} D_{q,\eta}^{(k)}$ $(k=3,4,\cdots)$, (18) with (19) is said to be the unique solution of (4). To summarize, the ordinary problem is defined as follows:

Definition 2.3 (The ordinary problem) *For a given Hamiltonian $K(q,p)$ in power series (1), bring $K(q,p)$ into the BG-normal form $G(\xi,\eta)$ in power series (5) which satisfy (4) and (6), where the second-type generating function W of the form (2) is chosen to satisfy (4) and*

$$W_k(q,\eta) \in \operatorname{image} D_{q,\eta}^{(k)} \quad (k=3,4,\cdots). \tag{20}$$

Theorem 2.4 *The BG-normal form $G(\xi,\eta)$ for the Hamiltonian $K(q,p)$ is given by (5) with (18), where the second-type generating function $W(q,\eta)$ in power series (2) is chosen to satisfy (19) and (20).*

2.2 The inverse problem

To pose the inverse problem appropriately, we start with looking the key equation, (4), of the ordinary problem into more detail from a viewpoint of canonical transformations. With $W(q,\eta)$, let us associate the inverse canonical transformation,

$$(\xi,\eta) \to (q,p) \qquad \text{with} \qquad \xi = -\frac{\partial(-W)}{\partial \eta} \quad \text{and} \quad p = -\frac{\partial(-W)}{\partial q}, \tag{21}$$

of (3), so that $-W(q,\eta)$ is regarded as a *third-type* generating function (Goldstein 1950), a function of the *old* momentum variables η and the *new* position variables q. Equation (4) rewritten as

$$K(q, -\frac{\partial(-W)}{\partial q}) = G(-\frac{\partial(-W)}{\partial \eta}, \eta), \tag{22}$$

is then combined with (21) to show the following.

Lemma 2.5 *Let $G(\xi,\eta)$ of (5) be the BG-normal form for the Hamiltonian $K(q,p)$ of (1), which satisfies (4) with a second-type generating function $W(q,\eta)$. The Hamiltonian $K(q,p)$ is restored from $G(\xi,\eta)$ through the canonical transformation (21) associated with the third-type generating function $-W(q,\eta)$.*

Now we can pose the inverse problem in the following way: Let the Hamiltonian $H(q,p)$ be written in the form,

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{n}\left(p_j^2 + q_j^2\right) + \sum_{k=3}^{\infty} H_k(q,p), \tag{23}$$

where each $H_k(q,p)$ $(k = 3, 4, \cdots)$ is a homogeneous polynomial of degree-k in (q,p). Further, let a third-type generating function $S(q,\eta)$ be written in the form,

$$S(q,\eta) = -\sum_{j=1}^{n} q_j\eta_j - \sum_{k=3}^{\infty} S_k(q,\eta), \tag{24}$$

where each $S_k(q,p)$ $(k = 3, 4, \cdots)$ is a homogeneous polynomial of degree-k in (q,p).

Definition 2.6 (The inverse problem) *For a given power series, $G(\xi,\eta)$, in BG normal form (5,6), identify all the Hamiltonians $H(q,p)$ in power series (23) which satisfy*

$$H(q, -\frac{\partial S}{\partial q}) = G(-\frac{\partial S}{\partial \eta}, \eta), \tag{25}$$

where the third-type generating function $S(q,\eta)$ in power series (24) is chosen to satisfy (25) and

$$S_k(q,\eta) \in \text{image}\, D_{q,\eta}^{(k)} \quad (k = 3, 4, \cdots). \tag{26}$$

2.3 Solving the inverse problem

We solve the inverse problem in the following way. On equating the homogeneous-polynomial part of degree-k in (25), equation (25) is put into the series of equations,

$$H_k(q,\eta) - (D_{q,\eta}S_k)(q,\eta) = G_k(q,\eta) - \Psi_k(q,\eta) \qquad (k = 3, 4, \cdots), \tag{27}$$

where $D_{q,\eta}$ is given by (8). The $\Psi_k(q,\eta)$ is the homogeneous polynomial of degree-k in (q,η) determined uniquely by H_3, $\cdots$, H_{k-1}, G_3, $\cdots$, G_{k-1}, S_3, $\cdots$, S_{k-1} given. In particular, we have $\Psi_3(q,\eta)=0$ and

$$\Psi_4(q,\eta)=\sum_{j=1}^{n}\left(\frac{1}{2}\left(\frac{\partial S_3}{\partial q_j}\right)^2+\left.\frac{\partial H_3}{\partial p_j}\right|_{(q,\eta)}\frac{\partial S_3}{\partial q_j}-\frac{1}{2}\left(\frac{\partial S_3}{\partial \eta_j}\right)^2-\left.\frac{\partial G_3}{\partial \xi_j}\right|_{(q,\eta)}\frac{\partial S_3}{\partial \eta_j}\right), \tag{28}$$

while Ψ_k with larger k would be of very complicated form to be described. Like in the ordinary problem, we solve (27) for H_k and S_k by using the direct-sum decomposition (10) of $V_k(q,\eta)$, the vector spaces of homogeneous polynomials of degree-k ($k=3,4,\cdots$). Let us decompose H_k and Ψ_k to be

$$\begin{aligned} H_k(q,\eta) &= H_k^{\text{image}}(q,\eta)+H_k^{\text{ker}}(q,\eta),\\ \Psi_k(q,\eta) &= \Psi_k^{\text{image}}(q,\eta)+\Psi_k^{\text{ker}}(q,\eta), \end{aligned} \tag{29}$$

where

$$\begin{aligned} &H_k^{\text{image}}(q,\eta), \Psi_k^{\text{image}}(q,\eta)\in \operatorname{image} D_{q,\eta}^{(k)},\\ &H_k^{\text{ker}}(q,\eta), \Psi_k^{\text{ker}}(q,\eta)\in \ker D_{q,\eta}^{(k)}, \end{aligned} \tag{30}$$

Then on equating $\ker D_{q,\eta}^{(k)}$-part in (27), H_k^{ker} is determined to be

$$H_k^{\text{ker}}(q,\eta)=G_k(q,\eta)-\Psi_k^{\text{ker}}(q,\eta). \tag{31}$$

Equating $\operatorname{image} D_{q,\eta}$-part of (27), we have

$$H_k^{\text{image}}(q,\eta)-(D_{q,\eta}S_k)(q,\eta)=-\Psi_k^{\text{image}}(q,\eta). \tag{32}$$

Since we have the pair of unidentified polynomials, H_k^{image} and S_k in (32), H_k^{image} is not determined uniquely in contrast with H_k^{ker}; such a non-uniqueness is of the very nature of the inverse problem. Accordingly, equation (32) is solved to as

$$H_k^{\text{image}}(q,\eta)\in \operatorname{image} D_{q,\eta}^{(k)}\ :\ \text{chosen arbitrarily}, \tag{33}$$

$$S_k(q,\eta)=\left(\left.D_{q,\eta}^{(k)}\right|^{-1}_{\operatorname{image} D_{q,\eta}^{(k)}}(H_k^{\text{image}}|_{(q,\eta)}+\Psi_k^{\text{image}})\right)(q,\eta), \tag{34}$$

($k=3,4,\cdots$). Thus we have the following.

Theorem 2.7 *For a given BG-normal form $G(\xi,\eta)$ in power series (5), the solution $H(q,p)$ of the inverse problem is given by (23) subject to (31) and (33), where the third-type generating function $S(q,\eta)$ in (25) is chosen to be (24) subject to (34).*

Remark 2 *We have another expression of and S_k and H_k^{image} equivalent to (31, 33, 34);*

$$H_k^{\text{image}}(q,\eta)=\left(D_{q,\eta}^{(k)}S_k\right)(q,\eta)+\Psi_k^{\text{image}}(q,\eta), \tag{35}$$

with

$$S_k(q,\eta)\in \operatorname{image} D_{q,\eta}^{(k)}\ :\ \textit{chosen arbitrarily}. \tag{36}$$

Since we usually pay much more interests in H than in S, in the inverse problem, it would be better to take the expression (31, 33, 34).

2.4 The degree-ρ ordinary and inverse problems

From a practical point of view, we usually deal with the BG-normal form Hamiltonians not in power series but in polynomial. Indeed, as mentioned in section 1, when we utilize the BG-normalization to provide an approximate system for a given system, we truncate the normalized Hamiltonian up to a finite degree. Hence we naturally come to think of a 'finite-degree version' of both the ordinary and the inverse problems [1, 2].

Definition 2.8 (The degree-ρ ordinary problem) *For a given Hamiltonian, $K(q,p)$, of the form (1) (possibly in polynomial form) and an integer $\rho \geq 3$, bring $K(q,p)$ into the polynomial $G(\xi,\eta)$ of degree-ρ in BG-normal form which satisfy (4) up to degree-ρ, where the second-type generating function $W(q,\eta)$ in (4) is chosen to be the polynomial of degree-ρ subject to (20) with $k = 3,\cdots,\rho$.*

Remark 3 *In the papers [1, 2], the degree, ρ, in Definition 2.8 is restricted to be an even integers, because all the ν_j's are set to be 1 there. (cf. Lemmas 2.1 and 2.2) However, since we are dealing with the general $\{\nu_j\}$ in this paper, the degree ρ has no restriction.*

Definition 2.9 (The degree-ρ inverse problem) *For a given degree-ρ polynomial ($\rho \geq 3$: an integer), $G(\xi,\eta)$, in BG-normal form, identify all the Hamiltonians, $H(q,p)$s, in degree-ρ polynomial which satisfy (25) up to degree-ρ, where the third-type generating function $S(q,\eta)$ is chosen to be the polynomial of degree-ρ subject to (26) with $k = 3,\cdots,\rho$.*

3 Composition of canonical transformations

In Section 3, we have shown the way to solve the inverse problem which is theoretically complete. From a practical point of view, however, it is rather difficult to realize the solution in the present form, (31, 33, 34), since the calculations required will be highly combinatorial: Let us take the 1:1 resonant two-degree-of-freedon case ($n = 2$, $\nu_1 = \nu_2 = 2$), for example. In order to calculate Φ_k (resp. Ψ_k), the polynomials $G_3,\cdots,G_{k-1}$, $S_3,\cdots,S_{k-1}$, and $K_3,\cdots,K_{k-1}$ (resp. $H_3,\cdots,H_{k-1}$) have to be kept each of which is in V_ℓ of combinatorially rising dimension, $\sum_{h=1}^{4}\binom{4}{h}\binom{\ell-1}{h-1}$, where the symbol $\binom{\cdot}{\cdot}$ indicates the binomial coefficient.

To get rid of such a difficulty, we break the needed calculations into a series of calculation of simpler form, which is realized in ANFER (see [1, 2, 11]). This section is hence devoted to the mathematical basis for organizing ANFER.

A key idea in writing-up ANFER is to realize (31, 33, 34) up to degree-ρ by applying not the canonical transformation associated with $S(q,\eta)$ but the composition of canonical transformations

$$\tau_h : (\xi^{(h-1)},\eta^{(h-1)}) \to (\xi^{(h)},\eta^{(h)}) \quad \text{with} \quad (\xi^{(2)},\eta^{(2)}) = (\xi,\eta) \quad (r = 3,\cdots,\rho), \tag{37}$$

associated with the third-type generating functions,

$$S^{(h)}(\xi^{(h)},\eta^{(h-1)}) = -\sum_{j=1}^{n}\xi_j^{(h)}\eta_j^{(h-1)} - S_h(\xi^{(h)},\eta^{(h-1)}) \qquad (h=3,\cdots,\rho), \tag{38}$$

where $S_h(\xi^{(h)},\eta^{(h-1)})$ is the homogeneous polynomial part of degree-h of the generating function $S(q,\eta)$ (see (24)) with $(\xi^{(h)},\eta^{(h-1)})$ in place of (q,η).

We start by showing the following on the generating function associated with the composition of a pair of canonical transformations. It should be pointed out that little is known explicitly of the generating function of the composition of non-infinitesimal canonical transformations like the following lemma, while well-known is that of infinitesimal ones (see [9], for example).

Lemma 3.1 *Let the function $f^{(h)}(u^{(h)},v^{(h-1)})$ $(h=1,2)$ be the third-type generating functions of canonical transformations,*

$$\sigma_h : (u^{(h-1)},v^{(h-1)}) \to (u^{(h)},v^{(h)}) \quad (h=1,2), \tag{39}$$

with

$$u^{(h-1)} = -\frac{\partial f^{(h)}}{\partial v^{(h-1)}}, \quad v^{(h)} = -\frac{\partial f^{(h)}}{\partial u^{(h)}} \quad (h=1,2), \tag{40}$$

each of which leaves the origin of $\mathbf{R}^n\times\mathbf{R}^n$ invariant. Then the composition, $\sigma_2\circ\sigma_1$: $(u^{(0)},v^{(0)}) \to (u^{(2)},v^{(2)})$, is generated by the third-type generating function,

$$f_{1,2}(u^{(2)},v^{(0)}) = \sum_{j=1}^{n}\tilde{u}_j\tilde{v}_j + f^{(1)}(\tilde{u},v^{(0)}) + f^{(2)}(u^{(2)},\tilde{v}), \tag{41}$$

where $\tilde{u}^{(1)}$ and $\tilde{v}^{(1)}$ on the rhs of (41) are the C^∞-functions of $(u^{(2)},v^{(0)})$ uniquely determined around $(u^{(2)},v^{(0)})=(0,0)$ to satisfy

$$\tilde{u} = -\frac{\partial f^{(2)}}{\partial v^{(1)}}(u^{(2)},\tilde{v}), \qquad \tilde{v} = -\frac{\partial f^{(1)}}{\partial u^{(1)}}(\tilde{u},v^{(0)}). \tag{42}$$

Proof: Since the canonical relation (40) yields the identities [9]

$$-\sum_{j=1}^{n}(u_j^{(h-1)}dv_j^{(h-1)} + v_j^{(h)}du_j^{(h)}) = d(f^{(h)}(u^{(h)},v^{(h-1)})) \quad (h=1,2), \tag{43}$$

we have

$$-\sum_{j=1}^{n}(u_j^{(0)}dv_j^{(0)} + v_j^{(2)}du_j^{(2)}) = d\left(\sum_{j=1}^{n}u_j^{(1)}v_j^{(1)} + f^{(1)}(u^{(1)},v^{(0)}) + f^{(2)}(u^{(2)},v^{(1)})\right). \tag{44}$$

Further, the canonical relation (40) restricts $\{(u^{(h)},v^{(h)})\}_{h=0,1,2}$ to the inverse image, $\sigma^{-1}(0,0,0,0)$, of the C^∞-map

$$\begin{aligned}&\sigma : (u^{(0)},v^{(0)},u^{(1)},v^{(1)},u^{(2)},v^{(2)}) \in \mathbf{R}^{6n}\\ &\mapsto \left(u^{(0)} + \frac{\partial f^{(1)}}{\partial v^{(0)}}, u^{(1)} + \frac{\partial f^{(2)}}{\partial v^{(1)}}, v^{(1)} + \frac{\partial f^{(1)}}{\partial u^{(1)}}, v^{(2)} + \frac{\partial f^{(2)}}{\partial u^{(2)}}\right) \in \mathbf{R}^{4n}.\end{aligned} \tag{45}$$

Applying the implicit function theorem [12] to the map σ, we obtain uniquely the functions, $\tilde{u}$, $\tilde{v}$, $\widehat{u}$ and $\widehat{v}$, of $(u^{(2)}, v^{(0)})$ subject to $\sigma(\widehat{u}, v^{(0)}, \tilde{u}, \tilde{v}, u^{(2)}, \widehat{v}) = 0$ around $(0,0)$. The differentiability comes from that of σ due to the implicit function theorem. The substitution of $(\tilde{u}, \tilde{v})$ into $(u^{(1)}, v^{(1)})$ in (44) thereby shows (41) with (42). This ends the proof.

We are now in a position to compose the series, $\{\tau_h\}_{h=3,\cdots,\rho}$, of canonical transformations given by (37) and (38) by using Lemma 3.1. We show the following for the generating function of the composition, $\tau_\rho \circ \cdots \circ \tau_3$.

Lemma 3.2 *For any fixed integer $\rho \geq 4$, the composition, $\tau_\rho \circ \cdots \circ \tau_3$, of the canonical transformations $\{\tau_h\}_{h=3,\cdots,\rho}$ given by*

$$\xi^{(h-1)} = -\frac{\partial S^{(h)}}{\partial \eta^{(h-1)}}(\xi^{(h)}, \eta^{(h-1)}), \quad \eta^{(h)} = -\frac{\partial S^{(h)}}{\partial \xi^{(h)}}(\xi^{(h)}, \eta^{(h-1)}) \quad (h = 3, 4, \cdots). \tag{46}$$

with (38) and (37) is associated with the third-type-3 generating function $\mathcal{S}^{(\rho)}(\xi^{(\rho)}, \eta^{(2)})$ defined recursively by

$$\mathcal{S}^{(3)}(\xi^{(3)}, \eta^{(2)}) = S^{(3)}(\xi^{(3)}, \eta^{(2)}), \tag{47}$$

$$\begin{aligned} &\mathcal{S}^{(h)}(\xi^{(h)}, \eta^{(2)}) \\ = \; &\sum_{j=1}^{n} \tilde{\xi}_j^{(h-1)} \tilde{\eta}_j^{(h-1)} + \mathcal{S}^{(h-1)}(\tilde{\xi}^{(h-1)}, \eta^{(2)}) + S^{(h)}(\xi^{(h)}, \tilde{\eta}^{(h-1)}) \quad (h = 4, \cdots, \rho), \end{aligned} \tag{48}$$

where $S^{(h)}$ are the third-type generating functions given by (38), and $\tilde{\xi}^{(h-1)}$ and $\tilde{\eta}^{(h-1)}$ the C^∞-functions of $(\xi^{(h)}, \eta^{(2)})$ determined uniquely to satisfy

$$\tilde{\xi}^{(h-1)} = -\frac{\partial S^{(h)}}{\partial \eta^{(h-1)}}(\xi^{(h)}, \tilde{\eta}^{(h-1)}), \quad \tilde{\eta}^{(h-1)} = -\frac{\partial \mathcal{S}^{(h-1)}}{\partial \xi^{(h-1)}}(\tilde{\xi}^{(h-1)}, \eta^{(2)}) \quad (h = 3, 4, \cdots), \tag{49}$$

around $(\xi^{(h)}, \eta^{(2)}) = (0,0)$.

Proof: We prove the lemma by induction: The starting case, (48) with $h = 4$, immediately follows due to Lemma 3.1. We move on to show the case of $h = r(\geq 5)$ in turn under the assumption that (48) and (49) hold true for $h = r - 1$. Applying Lemma 3.1 to the pair of transformations, τ_r with $S^{(r)}$ and $\tau_{r-1} \circ \cdots \circ \tau_3$ with $\mathcal{S}^{(r-1)}$, we obtain $\mathcal{S}^{(r)}(\xi^{(r)}, \eta^{(2)})$ given by (48) and (49) with $h = r$ as the third-type generating function of $\tau_r \circ (\tau_{r-1} \circ \cdots \circ \tau_3)$. The differentiability of $\tilde{\xi}^{(r-1)}$ and $\tilde{\eta}^{(r-1)}$ follows from that of the generating functions $\mathcal{S}^{(r-1)}$ and $S^{(r)}$ (see the proof of Lemma 3.1). This ends the proof.

Expanding $\mathcal{S}^{(h)}$ in terms of $(\xi^{(h)}, \eta^{(2)})$, we have the following, a key theorem of ANFER.

Theorem 3.3 *The third-type generating functions, $\mathcal{S}^{(h)}(\xi^{(h)},\eta^{(2)})$ $(h=4,5,\cdots)$, admit the power series expansion,*

$$\begin{aligned} &\mathcal{S}^{(h)}(\xi^{(h)},\eta^{(2)}) = -\sum_{j=1}^{n}\xi_j^{(h)}\eta_j^{(2)} - \sum_{k=3}^{h} S_k(\xi^{(h)},\eta^{(2)}) + o_h(\xi^{(h)},\eta^{(2)}), \\ &\frac{o_h(\xi^{(h)},\eta^{(2)})}{(\sum_{j=1}^{n}(\xi_j^{(h)^2}+\eta_j^{(2)^2}))^{h/2}} \to 0 \qquad ((\sum_{j=1}^{n}(\xi_j^{(h)^2}+\eta_j^{(2)^2}))^{1/2}\to 0), \end{aligned} \tag{50}$$

where $S_k(\xi^{(h)},\eta^{(2)})$ are the homogeneous part of degree-k in $S(q,\eta)$ (see (24)) with $(\xi^{(h)},\eta^{(2)})$ in place of (q,η).

Proof: We show (50) by induction. Expanding (49) with $h=4$ in terms of $(\xi^{(4)},\eta^{(2)})$, we have

$$\tilde{\xi}^{(3)} = \xi^{(4)} + \frac{\partial S_4}{\partial\eta^{(3)}}(\xi^{(4)},\tilde{\eta}^{(3)}), \tag{51}$$

$$\tilde{\eta}^{(3)} = \eta^{(2)} + \frac{\partial S_3}{\partial\xi^{(3)}}(\tilde{\xi}^{(3)},\eta^{(2)}). \tag{52}$$

Since the second term on the rhs of (51) is of degree higher than two and since that of (52) are of degree higher than one, (51) and (52) are put together to provide a further expansion,

$$\begin{aligned} \tilde{\xi}^{(3)} &= \xi^{(4)} + \frac{\partial S_4}{\partial\eta^{(3)}}(\xi^{(4)},\eta^{(2)}) + o_3'(\xi^{(4)},\eta^{(2)}), \\ \tilde{\eta}^{(3)} &= \eta^{(2)} + \frac{\partial S_3}{\partial\xi^{(3)}}(\xi^{(4)},\eta^{(2)}) + o_3''(\xi^{(4)},\eta^{(2)}), \end{aligned} \tag{53}$$

where o_3' and o_3'' indicate the terms in $(\sqrt{\sum_{j=1}^{n}(\xi_j^{(4)^2}+\eta_j^{(2)^2})})$ of degree higher than *three*. Equations (38,48,53) are then put together to yield

$$\begin{aligned} &\mathcal{S}^{(4)}(\xi^{(4)},\eta^{(2)}) \\ &= \sum_{j=1}^{n}\left(\xi_j^{(4)} + \left.\frac{\partial S_4}{\partial\eta_j^{(3)}}\right|_{(\xi^{(4)},\eta^{(2)})} + o_3'\right)\left(\eta_j^{(2)} + \left.\frac{\partial S_3}{\partial\xi_j^{(3)}}\right|_{(\xi^{(4)},\eta^{(2)})} + o_3''\right) \\ &\quad -\left\{\sum_{j=1}^{n}\left(\xi_j^{(4)} + \left.\frac{\partial S_4}{\partial\eta_j^{(3)}}\right|_{(\xi^{(4)},\eta^{(2)})} + o_3'\right)\eta_j^{(2)} + S_3(\xi^{(4)} + \left.\frac{\partial S_4}{\partial\eta^{(3)}}\right|_{(\xi^{(4)},\eta^{(2)})} + o_3', \eta_j^{(2)})\right\} \\ &\quad -\left\{\sum_{j=1}^{n}\xi_j^{(4)}\left(\eta_j^{(2)} + \left.\frac{\partial S_3}{\partial\xi_j^{(3)}}\right|_{(\xi^{(4)},\eta^{(2)})} + o_3''\right) + S_4(\xi^{(4)},\eta^{(2)} + \left.\frac{\partial S_3}{\partial\xi^{(3)}}\right|_{(\xi^{(4)},\eta^{(2)})} + o_3'')\right\} \\ &= -\sum_{j=1}^{n}\xi_j^{(4)}\eta_j^{(2)} - \sum_{k=3}^{4} S_k(\xi^{(4)},\eta^{(2)}) + o_4(\xi^{(4)},\eta^{(2)}), \end{aligned} \tag{54}$$

which shows (50) with $h = 4$. We move on to show (50) with $h = r \geq 5$ in turn under the assumption that (50) with $h = r - 1$ holds true. In a similar way to get (53), we obtain

$$\begin{aligned} \tilde{\xi}^{(r-1)} &= \xi^{(r)} + \frac{\partial S_r}{\partial \eta^{(r-1)}}(\xi^{(r)}, \eta^{(2)}) + o'_{r-1}(\xi^{(r)}, \eta^{(2)}), \\ \tilde{\eta}^{(r-1)} &= \eta^{(2)} + \sum_{k=3}^{r-1} \frac{\partial S_k}{\partial \xi^{(k)}}(\xi^{(r)}, \eta^{(2)}) + o''_{r-1}(\xi^{(r)}, \eta^{(2)}), \end{aligned} \tag{55}$$

where o'_{r-1} and o''_{r-1} indicate the terms in $(\sqrt{\sum_{j=1}^{n}(\xi_j^{(r)^2} + \eta_j^{(2)^2})})$ of degree higher than $r-1$. Like (54), equations (38) with $h = r$, (48) with $h = r - 1$ and (53) are thereby put together to yield

$$\begin{aligned} & \mathcal{S}^{(r)}(\xi^{(r)}, \eta^{(2)}) \\ = & \sum_{j=1}^{n} \left(\xi_j^{(r)} + \left.\frac{\partial S_r}{\partial \eta_j^{(r-1)}}\right|_{(\xi^{(r)},\eta^{(2)})} + o'_{r-1} \right) \left(\eta_j^{(2)} + \sum_{k=3}^{r-1} + \left.\frac{\partial S_k}{\partial \xi_j^{(k)}}\right|_{(\xi^{(r)},\eta^{(2)})} + o''_{r-1} \right) \\ & - \left\{ \sum_{j=1}^{n} \left(\xi_j^{(r)} + \left.\frac{\partial S_r}{\partial \eta_j^{(r-1)}}\right|_{(\xi^{(r)},\eta^{(2)})} + o'_{r-1} \right) \eta_j^{(2)} \right. \\ & \qquad \left. + \sum_{k=3}^{r-1} S_k(\xi^{(r)} + \left.\frac{\partial S_r}{\partial \eta^{(3)}}\right|_{(\xi^{(r)},\eta^{(2)})} + o'_{r-1}, \eta_j^{(2)}) \right\} \\ & - \left\{ \sum_{j=1}^{n} \xi_j^{(r)} \left(\eta_j^{(2)} + \sum_{k=3}^{r-1} \left.\frac{\partial S_k}{\partial \xi_j^{(k)}}\right|_{(\xi^{(r)},\eta^{(2)})} + o''_{r-1} \right) \right. \\ & \qquad \left. + S_r(\xi^{(r)}, \eta^{(2)} + \sum_{k=3}^{r-1} \left.\frac{\partial S_k}{\partial \xi^{(3)}}\right|_{(\xi^{(r)},\eta^{(2)})} + o''_{r-1}) \right\} \\ & + \tilde{o}_r(\xi^{(r)}, \eta^{(2)}) \\ = & -\sum_{j=1}^{n} \xi_j^{(r)} \eta_j^{(2)} - \sum_{k=3}^{r} S_k(\xi^{(r)}, \eta^{(2)}) + o_r(\xi^{(r)}, \eta^{(2)}), \end{aligned} \tag{56}$$

which proves our assertion.

Corollary 3.4 *$\mathcal{S}^{(h)}(q, \eta)$ coincides with $S(q, \eta)$ up to degree-h if expanded.*

4 The procedure in ANFER

In the previous section, we have studied the composition of canonical transformations necessary to organize ANFER. This section is devoted to the solving procedure in ANFER in which the composition of transformations is utilized effectively.

4.1 The solution of the degree-ρ inverse problem

We show that the solution of the degree-ρ inverse problem is obtained by applying the composition $\tau_\rho \circ \cdots \circ \tau_3$ of canonical transformations studied in section. 3. Let us fix a BG-normal form power series (possibly a polynomial) Hamiltonian $G(\xi,\eta)$ of the form (5). For $G(\xi,\eta)$ fixed, we define the power-series Hamiltonians $H^{(h)}(\xi^{(h)},\eta^{(h)})$ $(h=3,\cdots,\rho)$ by the reccurent formula

$$H^{(h)} \circ \tau_h = H^{(h-1)} \quad (h=3,4,\cdots) \quad \text{with} \quad H^{(2)}(\xi,\eta) = G(\xi,\eta). \tag{57}$$

Namely, $H^{(h)}$ satisfy the equation

$$(H^{(h)} \circ \tau_h \circ \cdots \circ \tau_3)(\xi,\eta) = H^{(2)}(\xi,\eta)(= G(\xi,\eta)) \quad (h=3,4,\cdots), \tag{58}$$

which is put together with Lemma 3.2 to show

$$H^{(h)}(\xi^{(h)}, -\frac{\partial \mathcal{S}^{(h)}}{\xi^{(h)}}) = H^{(2)}(-\frac{\partial \mathcal{S}^{(h)}}{\partial \eta^{(2)}}, \eta^{(2)}) \quad (h=3,4,\cdots), \tag{59}$$

where $\mathcal{S}^{(h)}(\xi^{(h)},\eta^{(2)})$ is given by (48) with (49).

Let us recall the equation (27) equivalent to the defining equation (25) of the inverse problem. From (27), we see that only the homogeneous parts in S and H of degree lower than h, and those in G of degree lower than or equal to h (cf. (28)) are necessary to determine the homogeneous part H_h of degree-h in H This observation is thereby put together with Corollary 3.4 to show the following.

Theorem 4.1 *For a given BG-normal form power-series, $G(\xi,\eta)$, the power-series Hamiltonian $H^{(h)}(q,p)$ coincides up to degree-h ($h=3,4,\cdots$) with the solution of the inverse problem, $H(q,p)$.*

Corollary 4.2 *The solution of the degree-ρ inverse problem is given by $H^{(\rho)}(q,p)$ truncated up to degree-ρ.*

4.2 The solving procedure in ANFER

We are now in a position to present the procedure of solving equation (27) with $h = 3,\cdots\rho$ (equivalently, (27) up to degree-ρ) for the degree-ρ inverse problem. In ANFER, the series of equations (57) are solved as follows: To be more precise, what we solve is

$$H^{(h)}(\xi^{(h)}, -\frac{\partial \mathcal{S}^{(h)}}{\partial \xi^{(h)}}) = H^{(h-1)}(-\frac{\partial \mathcal{S}^{(h)}}{\partial \eta^{(h-1)}}, \eta^{(h-1)}) \quad (h=3,\cdots,\rho) \tag{60}$$

with

$$H^{(2)}(\xi^{(2)},\eta^{(2)}) = G(\xi^{(2)},\eta^{(2)}) \tag{61}$$

equivalent to (57). For convenience, we will refer to the stage of solving (60) with $h = r$ as the 'Stage-r' hence force. Before starting the Stage-three, (61) is assumed to have been proceeded already. Further, it is convenient to express $H^{(h)}(\xi^{(h)}, \eta^{(h)})$ to be

$$H^{(h)}(\xi^{(h)}, \eta^{(h)}) = \frac{1}{2}\sum_{j=1}^{n}({\eta_j^{(h)}}^2 + {\xi_j^{(h)}}^2) + \sum_{k=3}^{\infty} H_k^{(h)}(\xi^{(h)}, \eta^{(h)}) \qquad (62)$$

like $H(q,p)$ (see (23)), where each $H_k^{(h)}(\xi^{(h)}, \eta^{(h)})$ $(k = 3, 4, \cdots)$ is the homogeneous polynomial part of degree-k in $(\xi^{(h)}, \eta^{(h)})$.

[**Stage**-r]:

At the Stage-r, (60) with $h = r$ is solved. Equating the homogeneous part of degree-k $(k = 3, \cdots, \rho)$ in (60) with $h = r$, we have the series of equation,

$$H_k^{(r)}(\xi^{(r)}, \eta^{(r-1)}) = H_k^{(r-1)}(\xi^{(r)}, \eta^{(r-1)}) \qquad (k = 3, \cdots, r-1), \qquad (63)$$

$$H_r^{(r)}(\xi^{(r)}, \eta^{(r-1)}) = \left(D_{\xi^{(r)},\eta^{(r-1)}} S_r\right)(\xi^{(r)}, \eta^{(r-1)}) + H_r^{(r-1)}(\xi^{(r)}, \eta^{(r-1)}) \quad (k = r), (64)$$

$$H_k^{(r)}(\xi^{(r)}, \eta^{(r-1)}) = H_k^{(r-1)}(\xi^{(r)}, \eta^{(r-1)}) + \Theta_k^{(r)}(\xi^{(r)}, \eta^{(r-1)}) \qquad (k = r+1, \cdots, \rho) (65)$$

Here $\Theta_k^{(r)}(\xi^{(r)}, \eta^{(r-1)})$ $(k = r+1, \cdots, \rho)$ in (65) are the homogeneous polynomials of degree-k given by

$$\begin{aligned}
&\Theta_k^{(r)}(\xi^{(r)}, \eta^{(r-1)}) \\
&= \sum_{|\alpha|=1}^{\left[\frac{k-2}{r-2}\right]} \frac{1}{\alpha!}\left[\left(\left.\frac{\partial S_r}{\partial \eta}\right|_{(\xi^{(r)},\eta^{(r-1)})}\right)^{\alpha}\left(\left.\left(\frac{\partial}{\partial \xi^{(r-1)}}\right)^{\alpha} H_{k-(r-2)|\alpha|}^{(r-1)}\right)\right|_{(\xi^{(r)},\eta^{(r-1)})}\right. \qquad (66)\\
&\qquad \left. - \left(\left.\frac{\partial S_r}{\partial \xi}\right|_{(\xi^{(r)},\eta^{(r-1)})}\right)^{\alpha}\left(\left.\left(\frac{\partial}{\partial \eta^{(r)}}\right)^{\alpha} H_{k-(r-2)|\alpha|}^{(r)}\right)\right|_{(\xi^{(r)},\eta^{(r-1)})}\right],
\end{aligned}$$

where $[(k-2)/(r-2)]$ stands for the integer-part of $(k-2)/(r-2)$, and $\alpha = (\alpha_1, \cdots, \alpha_n)$ does the multi-index with nonnegative-integer valued components associated with the notations,

$$\begin{aligned}
&|\alpha| = \sum_{j=1}^{n} \alpha_j, & &\alpha! = \alpha_1! \cdots \alpha_n!, \\
&\left(\frac{\partial S_r}{\partial \xi}\right)^{\alpha} = \left(\frac{\partial S_r}{\partial \xi_1}\right)^{\alpha_1} \cdots \left(\frac{\partial S_r}{\partial \xi_n}\right)^{\alpha_n}, & &\left(\frac{\partial S_r}{\partial \eta}\right)^{\alpha} = \left(\frac{\partial S_r}{\partial \eta_1}\right)^{\alpha_1} \cdots \left(\frac{\partial S_r}{\partial \eta_n}\right)^{\alpha_n}, \qquad (67)\\
&\left(\frac{\partial}{\partial \xi}\right)^{\alpha} = \left(\frac{\partial}{\partial \xi_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \xi_n}\right)^{\alpha_n}, & &\left(\frac{\partial}{\partial \eta}\right)^{\alpha} = \left(\frac{\partial}{\partial \eta_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \eta_n}\right)^{\alpha_n}.
\end{aligned}$$

Remark 4 *We do not have to deal with the homogeneous part in (60) of the degree higher than ρ taking the observation to (27) into account made above Theorem 4.1.*

Since Theorem 4.1 and (63) are put together to imply that each $H_k^{(r)}$ with $k < r$ has been identified already as $H_k^{(k)}$ at the stage-k, we do not have to deal with (63) a lot: Only the replacement of $H_k^{(r-1)}$ to $H_k^{(r)}$ $(k < r)$ is made.

We solve (64) and (65) in turn by using the decomposition,

$$H_k^{(r)}(\xi^{(r)},\eta^{(r-1)}) = H_k^{(r)^{\text{image}}}(\xi^{(r)},\eta^{(r-1)}) + H_k^{(r)^{\text{ker}}}(\xi^{(r)},\eta^{(r-1)}) \qquad (k=r,\cdots,\rho)$$
$$\Theta_k^{(r)}(\xi^{(r)},\eta^{(r-1)}) = \Theta_k^{(r)^{\text{image}}}(\xi^{(r)},\eta^{(r-1)}) + \Theta_k^{(r)^{\text{ker}}}(\xi^{(r)},\eta^{(r-1)}) \qquad (k=r+1,\cdots,\rho) \tag{68}$$

where

$$\begin{aligned} &H_k^{(r)^{\text{image}}}(\xi^{(r)},\eta^{(r-1)}), \Theta_k^{(r)^{\text{image}}}(\xi^{(r)},\eta^{(r-1)}) \in \text{image}\, D^{(k)}_{\xi^{(r)},\eta^{(r-1)}}, \\ &H_k^{(r)^{\text{ker}}}(\xi^{(r)},\eta^{(r-1)}), \Theta_k^{(r)^{\text{ker}}}(\xi^{(r)},\eta^{(r-1)}) \in \ker D^{(k)}_{\xi^{(r)},\eta^{(r-1)}}. \end{aligned} \tag{69}$$

Like in the way to solve (27) in subsection 2.3, (64) is solved to be

$$H_r^{(r)^{\text{ker}}}(\xi^{(r)},\eta^{(r-1)}) = H_r^{(r-1)^{\text{ker}}}(\xi^{(r)},\eta^{(r-1)}) \tag{70}$$
$$H_r^{(r)^{\text{image}}}(\xi^{(r)},\eta^{(r-1)})) \in \text{image}\, D^{(r)}_{\xi^{(r)},\eta^{(r-1)}} : \text{chosen arbitrarily}, \tag{71}$$
$$\begin{aligned} &S_r(\xi^{(r)},\eta^{(r-1)}) \\ &= \left(D^{(r)}_{\xi^{(r)},\eta^{(r-1)}}\Big|^{-1}_{\text{image}\, D^{(r)}_{\xi^{(r)},\eta^{(r-1)}}} (H_r^{(r)^{\text{image}}} - H_r^{(r-1)^{\text{image}}}|_{(\xi^{(r)},\eta^{(r-1)})}) \right)(\xi^{(r)},\eta^{(r-1)}). \end{aligned} \tag{72}$$

We move on to (65) with $k = s\,(> r)$: Since we have identified $H_k^{(r)}$s with $k < s$ and S_r already before solving (65) with $k = s(> r)$, $\Theta_s^{(r)}$ can be identified completely by (66) with $k = s$. After the calculation of $\Theta_s^{(r)}$, the $H_k^{(r)}$ with $k = s(> r)$ is thereby identified by (65).

In ANFER, the solving procedure described above is realized in the symbolic computing language, REDUCE3.6 or later. Till to now, a prototype of ANFER for the 1:1 resonant two-degree-of-freedom systems ($n = 2$, $\nu_1 = \nu_2 = 1$) has been written up by the author. The source-code of this prototype is available on the web page, http://yang.amp.i.kyoto-u.ac.jp/~uwano/, [11].

In closing this section, we wish to compare the procedures, (27)-(34) with (63)-(72). Through (27)-(34) with $k = r$, $S_3,\cdots,S_{r-1}$, $G_3,\cdots,G_r$, and $H_3,\cdots,H_r$ have to be kept. In contrast with this, S_{r-1}, $H_3^{(r-1)},\cdots,H_\rho^{(r-1)}$, and $H_3^{(r)},\cdots,H_\rho^{(r)}$ have to be kept. Hence, the procedure given in this section would contributes to the memory-saving.

5 Application

Although only a prototype of ANFER exists for a 1:1 resonant case, ANFER has worked very effectively to find a new deep relation between the Bertrand-Darboux integrability condition (BDIC) (see [13, 18, 19], for example) for the perturbed harmonic oscillators with homogeneous-*cubic* potentials (PHOCPs) and for the oscillators with homogeneous-*quartic* potentials (PHOQPs) [1]. The BDIC provides not only a sufficient condition for the integrability but also a separability, which will be reviewed briefly in Appendix.

5.1 The degree-4 ordinary and inverse problems for PHOCP

We start by solving the degree-4 ordinary problem for the PHOCP-Hamiltonian

$$\mathcal{F}^{(3)}(q,p) = \frac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) + (f_1 q_1^3 + f_2 q_1^2 + f_3 q_1 q_2^2 + f_4 q_2^3), \tag{73}$$

where (f_h) are real-valued parameters. Note that if we choose (f_h) to be

$$f_1 = f_3 = 0, \quad f_2 = 1, \quad f_4 = \mu, \tag{74}$$

$\mathcal{F}^{(3)}(q,p)$ becomes the well-known one-parameter Hénon-Heiles Hamiltonian [6, 7].

Through ANFER ([11]), we obtain the BG-normal form $\mathcal{G}$ of the following form as the solution of the degree-4 ordinary problem [1];

$$\begin{aligned}
\mathcal{G}(\xi,\eta) = {} & \frac{1}{2}\left(\zeta_1\overline{\zeta}_1 + \zeta_2\overline{\zeta}_2\right) \\
& - \frac{15}{16}(f_1^2\zeta_1^2\overline{\zeta}_1^2 + f_4^2\zeta_2^2\overline{\zeta}_2^2) - \frac{3}{4}(f_1 f_3\zeta_1\zeta_2\overline{\zeta}_1\overline{\zeta}_2 + f_2 f_4\zeta_1\zeta_2\overline{\zeta}_1\overline{\zeta}_2) \\
& - \frac{5}{8}(f_1 f_2\zeta_1^2\overline{\zeta}_1\overline{\zeta}_2 + f_1 f_2\zeta_1\zeta_2\overline{\zeta}_1^2 + f_3 f_4\zeta_1\zeta_2\overline{\zeta}_2^2 + f_3 f_4\zeta_2^2\overline{\zeta}_1\overline{\zeta}_2) \\
& - \frac{5}{24}(f_2 f_3\zeta_1^2\overline{\zeta}_1\overline{\zeta}_2 + f_2 f_3\zeta_1\zeta_2\overline{\zeta}_1^2 + f_2 f_3\zeta_1\zeta_2\overline{\zeta}_2^2 + f_2 f_3\zeta_2^2\overline{\zeta}_1\overline{\zeta}_2) \\
& - \frac{1}{6}(f_2^2\zeta_1\zeta_2\overline{\zeta}_1\overline{\zeta}_2 + f_3^2\zeta_1\zeta_2\overline{\zeta}_1\overline{\zeta}_2) - \frac{5}{48}(f_2^2\zeta_1^2\overline{\zeta}_1^2 + f_3^2\zeta_2^2\overline{\zeta}_2^2) \\
& - \frac{1}{8}(f_2^2\zeta_1^2\overline{\zeta}_2^2 + f_2^2\zeta_2^2\overline{\zeta}_1^2 + f_3^2\zeta_1^2\overline{\zeta}_2^2 + f_3^2\zeta_2^2\overline{\zeta}_1^2) \\
& + \frac{1}{16}(f_1 f_3\zeta_1^2\overline{\zeta}_2^2 + f_1 f_3\zeta_2^2\overline{\zeta}_1^2 + f_2 f_4\zeta_1^2\overline{\zeta}_2^2 + f_2 f_4\zeta_2^2\overline{\zeta}_1^2).
\end{aligned} \tag{75}$$

We solve the degree-4 inverse problem for the BG-normal form $\mathcal{G}$ in turn: By ANFER, we have the following polynomial of degree-4 as the solution [1];

$$\mathcal{H}(q,p) = \frac{1}{2}\sum_{j=1}^{2}\left(p_j^2 + q_j^2\right) + \mathcal{H}_3(q,p) + \mathcal{H}_4(q,p), \tag{76}$$

with

$$\begin{aligned}
\mathcal{H}_3(q,p) = {} & a_1 z_1^3 + a_2 z_1^2 z_2 + a_3 z_1 z_2^2 + a_4 z_2^3 + a_5 z_1^2\overline{z}_1 + a_6 z_1^2\overline{z}_2 \\
& + a_7 z_1 z_2\overline{z}_1 + a_8 z_1 z_2\overline{z}_2 + a_9 z_2^2\overline{z}_1 + a_{10} z_2^2\overline{z}_2 \\
& + \overline{a}_1\overline{z}_1^3 + \overline{a}_2\overline{z}_1^2\overline{z}_2 + \overline{a}_3\overline{z}_1\overline{z}_2^2 + \overline{a}_4\overline{z}_2^3 + \overline{a}_5 z_1\overline{z}_1^2 + \overline{a}_6 z_2\overline{z}_1^2 \\
& + \overline{a}_7 z_1\overline{z}_1\overline{z}_2 + \overline{a}_8 z_2\overline{z}_1\overline{z}_2 + \overline{a}_9 z_1\overline{z}_2^2 + \overline{a}_{10} z_2\overline{z}_2^2,
\end{aligned} \tag{77}$$

and

$$\mathcal{H}_4(q,p) = c_1 z_1^4 + c_2 z_1^3 z_2 + c_3 z_1^2 z_2^2 + c_4 z_1 z_2^3$$

$$
\begin{aligned}
&+c_5 z_2^4 + c_6 z_1^3 \overline{z}_1 + c_7 z_1^3 \overline{z}_2 + c_8 z_1^2 z_2 \overline{z}_1 \\
&+c_9 z_1^2 z_2 \overline{z}_2 + c_{10} z_1 z_2^2 \overline{z}_1 + c_{11} z_1 z_2^2 \overline{z}_2 + c_{12} z_2^3 \overline{z}_1 + c_{13} z_2^3 \overline{z}_2 \\
&+\overline{c}_1 \overline{z}_1^4 + \overline{c}_2 \overline{z}_1^3 \overline{z}_2 + \overline{c}_3 \overline{z}_1^2 \overline{z}_2^2 + \overline{c}_4 \overline{z}_1 \overline{z}_2^3 \\
&+\overline{c}_5 \overline{z}_2^4 + \overline{c}_6 z_1 \overline{z}_1^3 + \overline{c}_7 z_2 \overline{z}_1^3 + \overline{c}_8 z_1 \overline{z}_1^2 \overline{z}_2 \\
&+\overline{c}_9 z_2 \overline{z}_1^2 \overline{z}_2 + \overline{c}_{10} z_1 \overline{z}_1 \overline{z}_2^2 + \overline{c}_{11} z_2 \overline{z}_1 \overline{z}_2^2 + \overline{c}_{12} z_1 \overline{z}_2^3 + \overline{c}_{13} z_2 \overline{z}_2^3 \\
&+8(a_6 \overline{a}_6 z_1 z_2 \overline{z}_1 \overline{z}_2 + a_9 \overline{a}_9 z_1 z_2 \overline{z}_1 \overline{z}_2 + a_5 \overline{a}_6 z_1 z_2 \overline{z}_1^2 \\
&+a_6 \overline{a}_5 z_1^2 \overline{z}_1 \overline{z}_2 + a_9 \overline{a}_{10} z_2^2 \overline{z}_1 \overline{z}_2 + a_{10} \overline{a}_9 z_1 z_2 \overline{z}_2^2) \\
&+6(a_1 \overline{a}_1 z_1^2 \overline{z}_1^2 + a_4 \overline{a}_4 z_2^2 \overline{z}_2^2 + a_5 \overline{a}_5 z_1^2 \overline{z}_1^2 + a_{10} \overline{a}_{10} z_2^2 \overline{z}_2^2) \\
&+4(a_1 \overline{a}_2 z_1^2 \overline{z}_1 \overline{z}_2 + a_8 \overline{a}_9 z_1^2 \overline{z}_2^2 + a_3 \overline{a}_4 z_1 z_2 \overline{z}_2^2 + a_5 \overline{a}_8 z_1 z_2 \overline{z}_1 \overline{z}_2 \\
&+a_6 \overline{a}_7 z_1^2 \overline{z}_2^2 + a_6 \overline{a}_8 z_1 z_2 \overline{z}_2^2 + a_7 \overline{a}_9 z_1^2 \overline{z}_1 \overline{z}_2 + a_7 \overline{a}_{10} z_1 z_2 \overline{z}_1 \overline{z}_2) \\
&+4(a_2 \overline{a}_1 z_1 z_2 \overline{z}_1^2 + a_4 \overline{a}_3 z_2^2 \overline{z}_1 \overline{z}_2 + a_8 \overline{a}_5 z_1 z_2 \overline{z}_1 \overline{z}_2 + a_7 \overline{a}_6 z_2^2 \overline{z}_1^2 \\
&+a_8 \overline{a}_6 z_2^2 \overline{z}_1 \overline{z}_2 + a_9 \overline{a}_7 z_1 z_2 \overline{z}_1^2 + a_{10} \overline{a}_7 z_1 z_2 \overline{z}_1 \overline{z}_2 + a_9 \overline{a}_8 z_2^2 \overline{z}_1^2) \\
&+\frac{8}{3}(a_2 \overline{a}_2 z_1 z_2 \overline{z}_1 \overline{z}_2 + a_3 \overline{a}_3 z_1 z_2 \overline{z}_1 \overline{z}_2) \\
&+2(-a_6 \overline{a}_6 z_1^2 \overline{z}_1^2 + a_7 \overline{a}_7 z_1^2 \overline{z}_1^2 + a_8 \overline{a}_8 z_2^2 \overline{z}_2^2 - a_9 \overline{a}_9 z_2^2 \overline{z}_2^2) \\
&+2(a_1 \overline{a}_3 z_1^2 \overline{z}_2^2 + a_2 \overline{a}_4 z_1^2 \overline{z}_2^2 + a_5 \overline{a}_7 z_1^2 \overline{z}_1 \overline{z}_2 - a_5 \overline{a}_9 z_1^2 \overline{z}_2^2 \\
&-a_6 \overline{a}_8 z_1^2 \overline{z}_1 \overline{z}_2 - a_6 \overline{a}_{10} z_1^2 \overline{z}_2^2 + a_7 \overline{a}_8 z_1 z_2 \overline{z}_1^2 \\
&+a_7 \overline{a}_8 z_2^2 \overline{z}_1 \overline{z}_2 - a_7 \overline{a}_9 z_1 z_2 \overline{z}_2^2 + a_8 \overline{a}_{10} z_1 z_2 \overline{z}_2^2) \\
&+2(a_3 \overline{a}_1 z_2^2 \overline{z}_1^2 + a_4 \overline{a}_2 z_2^2 \overline{z}_1^2 + a_7 \overline{a}_5 z_1 z_2 \overline{z}_1^2 - a_9 \overline{a}_5 z_2^2 \overline{z}_1^2 \\
&-a_8 \overline{a}_6 z_1 z_2 \overline{z}_1^2 - a_{10} \overline{a}_6 z_2^2 \overline{z}_1^2 + a_8 \overline{a}_7 z_1^2 \overline{z}_1 \overline{z}_2 \\
&+a_8 \overline{a}_7 z_1 z_2 \overline{z}_2^2 - a_9 \overline{a}_7 z_2^2 \overline{z}_1 \overline{z}_2 + a_{10} \overline{a}_8 z_2^2 \overline{z}_1 \overline{z}_2) \\
&+\frac{4}{3}(a_2 \overline{a}_3 z_1^2 \overline{z}_1 \overline{z}_2 + a_2 \overline{a}_3 z_1 z_2 \overline{z}_2^2 + a_3 \overline{a}_2 z_1 z_2 \overline{z}_1^2 + a_3 \overline{a}_2 z_2^2 \overline{z}_1 \overline{z}_2) \\
&+\frac{2}{3}(a_2 \overline{a}_2 z_1^2 \overline{z}_1^2 + a_3 \overline{a}_3 z_2^2 \overline{z}_2^2) \\
&-\frac{15}{16}(f_1^2 z_1^2 \overline{z}_1^2 + f_4^2 z_2^2 \overline{z}_2^2) - \frac{3}{4}(f_1 f_3 z_1 z_2 \overline{z}_1 \overline{z}_2 + f_2 f_4 z_1 z_2 \overline{z}_1 \overline{z}_2) \\
&-\frac{5}{8}(f_1 f_2 z_1^2 \overline{z}_1 \overline{z}_2 + f_1 f_2 z_1 z_2 \overline{z}_1^2 + f_3 f_4 z_1 z_2 \overline{z}_2^2 + f_3 f_4 z_2^2 \overline{z}_1 \overline{z}_2) \\
&-\frac{5 f_2 f_3}{24}(z_1^2 \overline{z}_1 \overline{z}_2 + z_1 z_2 \overline{z}_1^2 + z_1 z_2 \overline{z}_2^2 + z_2^2 \overline{z}_1 \overline{z}_2) \\
&-\frac{1}{6}(f_2^2 z_1 z_2 \overline{z}_1 \overline{z}_2 + f_3^2 z_1 z_2 \overline{z}_1 \overline{z}_2) \\
&-\frac{1}{8}(f_2^2 z_1^2 \overline{z}_2^2 + f_2^2 z_2^2 \overline{z}_1^2 + f_3^2 z_1^2 \overline{z}_2^2 + f_3^2 z_2^2 \overline{z}_1^2) - \frac{5}{48}(f_2^2 z_1^2 \overline{z}_1^2 + f_3^2 z_2^2 \overline{z}_2^2) \\
&+\frac{1}{16}(f_1 f_3 z_2^2 \overline{z}_1^2 + f_1 f_3 z_1^2 \overline{z}_2^2 + f_2 f_4 z_1^2 \overline{z}_2^2 + f_2 f_4 z_2^2 \overline{z}_1^2),
\end{aligned}
\tag{78}
$$

where a_h $(h = 1, \cdots, 10)$ and c_ℓ $(\ell = 1, \cdots, 13)$ are the complex-valued parameters chosen arbitrarily, and f_k $(k = 1, \cdots, 4)$ the real-valued parameters in $\mathcal{F}^{(3)}(q)$ (see

(73)). Namely, we have 46-degree-of freedom in the solution, $\mathcal{H}$, of the inverse problem of the PHOCP if (f_k) fixed. Note that if (a_h), (c_ℓ) are chosen to be

$$\begin{aligned} &a_1 = a_3 = a_5 = a_8 = a_9 = 0, \\ &2a_2 = 2a_6 = a_7 = \frac{1}{4}, \quad 3a_4 = a_{10} = \frac{3\mu}{8}, \end{aligned} \tag{79}$$

and

$$c_\ell = 0 \quad (\ell = 1, \cdots, 13), \tag{80}$$

respectively, and (f_k) to be (74), $\mathcal{H}$ becomes equal to the one-parameter Hénon-Heiles Hamiltonian. After (76)-(78), one might understand the necessity of computer algebra in the inverse problem.

5.2 The BDIC for PHOCPs and PHOQPs

We wish to find the condition for (a_h), (c_ℓ) and (f_k) to bring $\mathcal{H}$ into the PHOQP-Hamiltonian

$$\mathcal{F}^{(4)}(q,p) = \frac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) + (g_1 q_1^4 + g_2 q_1^3 q_2 + g_3 q_1^2 q_2^2 + g_4 q_1 q_2^3 + g_5 q_2^4), \tag{81}$$

where (g_ℓ) are real-values parameters. A straightforward calculation shows the following (see [1] for detail) .

Theorem 5.1 *A PHOCP-Hamiltonian $\mathcal{F}^{(3)}$ shares its BG-normal form with a PHOQP-Hamiltonian $\mathcal{F}^{(4)}$ up to degree-4 if and only if the PHOCP-Hamiltonian $\mathcal{F}^{(3)}$ satisfies the BDIC,*

$$3(f_1 f_3 + f_2 f_4) - (f_2^2 + f_3^2) = 0, \tag{82}$$

for PHOCPs. Under (82), the PHOQP-Hamiltonian $\mathcal{F}^{(4)}$ sharing its BG-normal form with the PHOCP-Hamiltonian $\mathcal{F}^{(3)}$ is equal to

$$\begin{aligned} \mathcal{Q} \;=\; & \frac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) - \frac{5}{18}(9f_1^2 + f_2^2)q_1^4 - \frac{10}{9}(3f_1 + f_3)f_2 q_1^3 q_2 - \frac{5}{3}(f_2^2 + f_3^2)q_1^2 q_2^2 \\ & - \frac{10}{9}(3f_4 + f_2)f_3 q_1 q_2^3 - \frac{5}{18}(9f_4^2 + f_3^2)q_2^4, \end{aligned} \tag{83}$$

where (f_h) are subject to the BDIC (82).

We remark here that the BDIC (82) appears again in Appendix.

We are now in a position to show the integrability of the PHOQP with $\mathcal{Q}$ with (f_h) subject to (82). It is easily seen that that $\mathcal{F}^{(4)}$ becomes $\mathcal{Q}$ under the substitution,

$$\begin{aligned} &g_1 = -\frac{5}{18}(9f_1^2 + f_2^2), \quad g_2 = -\frac{10}{9}(3f_1 + f_3)f_2, \quad g_3 = -\frac{5}{3}(f_2^2 + f_3^2), \\ &g_4 = -\frac{10}{9}(3f_4 + f_2)f_3, \quad g_5 = -\frac{5}{18}(9f_4^2 + f_3^2). \end{aligned} \tag{84}$$

A long but straightforward calculation shows that (g_ℓ) given by (84) with (82) satisfy the BDIC

$$\begin{aligned} &9g_2^2 + 4g_3^2 - 24g_1g_3 - 9g_2g_4 = 0, \\ &9g_4^2 + 4g_3^2 - 24g_3g_5 - 9g_2g_4 = 0, \\ &(g_2 + g_4)g_3 - 6(g_1g_4 + g_2g_5) = 0, \end{aligned} \tag{85}$$

for PHOQPs (see Appendix). To summarize, we have the following.

Theorem 5.2 *If a PHOCP and a PHOCP share the same BG-normal form up to degree-4, then both oscillators are integrable in the sense that they satisfy the BDIC.*

6 Concluding Remarks

We have described the procedure in ANFER and the application of ANFER to the BDIC for the PHOCPs. In the following, Several remarks are made on ANFER itself and its application

(1) (Improving the program): As pointed out at the end of section 4, the procedure in ANFER based on the composition of canonical transformation would save save the required-memory comparing with the procedure given in section 2.3. There would be much room for improvement on this direction. It would be done in writing-up process, so that the author, not so familiar with writing programs, is trying to improve ANFER with several collaborators [11]. Recently ANFER has been updated to be able to reproduce the BDIC for the PHOCPs inside the program. Even if restricted to for PHOCPs, the inverse problem is expected to have rich related subjects listed below. Hence,those who are interested on improving ANFER will be welcome writing-up, because the improvement on ANFER is expected to advance various studies related with the inverse problem of the BG-normalization.

(2) (the separability): By a simple calculation, we see that the PHOCP-Hamiltonian with $f_1 = f_3/3 = a + b$, $f_2/3 = f_4 = a - b$ and the PHOQP-Hamiltonian with $g_1 = g_3/6 = g_5 = -5(a^2 + b^2)$, $g_2 = g_4 = 20(a^2 + b^2)$ share the same BG-normal form up to degree-4, where a and b are real-valued parameters [1]. These oscillator Hamiltonians are well known to be separable in $q_1 \pm q_2$ (see [17]), Theorem *** is hence understood to provide a significant relation between the separability of the PHOCP-Hamiltonians and that of PHOQP. Since $\mathcal{F}^{(3)}$ with (88) and $\mathcal{Q}$ are thought to include several classes of Hamiltonians separable in several coordinate systems other than $q_1 \pm q_2$, the separability will be worth studying extensively from the BG-normalization viewpoint in future.

(3) (quantum bifurcation): Since the perturbed oscillators referred to in theorems 5.1 and 5.2 are integrable, their quantum spectra are expected to be obtained exactly. These oscillators are hence expected to provide good examples of the quantum bifurcation in the BG-normalized Hamiltonian systems [20, 21, 22, 23] to study whether or not the quantum bifurcation in the BG-normalized Hamiltonian system for those oscillators approximates the bifurcation in these oscillators in a good extent.

(4) (integrability): As is easily seen, the solution (76, 77, 78) of the inverse problem for $\mathcal{G}$ admits fifty real-valued parameters. We may hence expect to obtain other integrable systems, so-called the electromagnetic type [18], for example.

On closing this section, we wish to mention again of the role of computer algebra in the inverse problem: Without computer algebra, for example, in section 5, it would have been very difficult to find theorems 5.1 and 5.2.

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7 Appendix: the BDIC

In this Appendix, we review the BDIC briefly. We will restrict our attention here to the two-degree-of-freedom natural Hamiltonian systems associated with the Euclidean metric because the PHOCPs and the PHOQPs dealt with in section 5 are of such type.

The theorem due to Bertrand and Darboux is stated as follows (see [13, 14], for example).

Theorem 7.1 (Bertrand-Darboux) *Let F be a natural Hamiltonian of the form,*

$$F(q,p) = \frac{1}{2}\sum_{j=1}^{2} p_j^2 + V(q), \tag{86}$$

where $V(q)$ a differentiable function in q. Then, the following three statements are equivalent for the Hamiltonian system with F.

(1) *There exists a set of real-valued constants, $(\alpha, \beta, \beta', \gamma, \gamma') \neq (0,0,0,0,0)$, for which $V(q)$ satisfies*

$$\begin{aligned}\left(\frac{\partial^2 V}{\partial q_2^2} - \frac{\partial^2 V}{\partial q_1^2}\right)&(-2\alpha q_1 q_2 - \beta' q_2 - \beta q_1 + \gamma)\\ &+ 2\frac{\partial^2 V}{\partial q_1 \partial q_2}(\alpha q_2^2 - \gamma q_1^2 + \beta q_2 - \beta' q_1 + \gamma')\\ &\quad+ \frac{\partial V}{\partial q_1}(6\alpha q_2 + 3\beta) - \frac{\partial V}{\partial q_2}(6\alpha q_1 + 3\beta') = 0.\end{aligned} \tag{87}$$

(2) *The Hamiltonian system with F admits an integral of motion quadratic in momenta.*

(3) *The Hamiltonian F is separable in either Cartesian, polar, parabolic or elliptic coordinates.*

Due to the statement (2) in theorem 7.1, a natural Hamiltonian system with F is always integrable if (87) holds true. In this regard, we refer to (87) as the Bertrand-Darboux integrability condition (BDIC).

For the PHOCPs and the PHOQPs, Yamaguchi and Nambu [14] have given a more explicit expression of the BDIC (87) convenient for section 5:

Lemma 7.2 *Let $\mathcal{F}^{(k)}(q,p)$ $(k=3,4)$ be the Hamiltonians of the form (73) and (81).*

(1) *For the PHOCP with $\mathcal{F}^{(3)}(q,p)$, the BDIC (87) is equivalent to either of the following conditions, (88), (89), or (90);*

$$3(f_1f_3+f_2f_4)-(f_2^2+f_3^2)=0, \tag{88}$$
$$f_1=2f_3, \quad f_2=f_4=0, \tag{89}$$
$$f_4=2f_2, \quad f_1=f_3=0. \tag{90}$$

(2) *For the PHOQP with $\mathcal{F}^{(4)}(q,p)$, the BDIC (87) is equivalent to either of the following conditions, (91) or (92);*

$$\begin{cases} g_3=2g_1=2g_5, \\ g_2=g_4=0, \end{cases} \tag{91}$$
$$\begin{cases} 9g_2^2+4g_3^2-24g_1g_3-9g_2g_4=0, \\ 9g_4^2+4g_3^2-24g_3g_5-9g_2g_4=0, \\ (g_2+g_4)g_3-6(g_1g_4+g_2g_5)=0. \end{cases} \tag{92}$$

The equations, (88) and (92), are referred to as the BDIC (82) for PHOCPs and (85) for PHOQPs, respectively, in section 5.

In closing Appendix, we wish to mention little more of the BDIC: As stated in Theorem 7.1, the BDIC (87) provide not only a necessary and sufficient condition for an existence of first integrals quadratic in momenta [15, 19] but also for the separability in either Cartesian, polar, parabolic or elliptic coordinates [13]. Indeed, the BDIC has been studied repeatedly from various viewpoints; the separation of variables [13, 16], the complete integrability [17], so-called the direct method [18] and the renormalization of Hamiltonian equation [14], for example (see also [19] as an older reference and the references in the above-cited literature).

References

[1] Y. Uwano, J. Phys. **A33**, 6635-6653 (2000).

[2] Y. Uwano, N. A. Chekanov, V. A. Rostovtsev and S. I. Vinitsky, *Computer Algebra in Scientific Computing* ed Ganzha V G et al (Berlin: Springer-Verlag), p 441 (1999).

[3] N. A. Chekanov, V. A. Rostovtsev, Y. Uwano and S. I. Vinitsky, *Comput. Phys. Commun.* **126** 47 (2000).

[4] N. A. Chekanov, M. Hongo, V. A. Rostovtsev, Y. Uwano and S. I. Vinitsky, *Physics of Atomic Nuclei* **61** 2029 (1998).

[5] F. G. Gustavson, *Astronomical Journal* **71**, 670 (1966).

[6] M. Kummer, *Commun. Math. Phys.* **48**, 53 (1976).

[7] R. Cushman, *Proc. R. Soc.* **A382** 361 (1982).

[8] J. K. Moser, *Lectures on Hamiltonian Systems* Memoirs of A.M.S. **81** (Providence: A.M.S.) p 10 (1968).

[9] H. Goldstein, *Classical Mechanics* 2nd ed., p 241 (Reading: Addison-Wesley, 1950).

[10] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, p 215 (New York: Springer-Verlag, 1980).

[11] Y. Uwano, http://yang.amp.i.kyoto-u.ac.jp/~uwano/ (2000).

[12] M. Spivak, *Calculus on Manifolds* (New York: Benjamin), p 41 (1965).

[13] I. Marshall and S. Wojciechowski, *J. Math. Phys.* **29**, 1338 (1988).

[14] Y. Y. Yamaguchi and Y. Nambu, *Prog. Theor. Phys.* **100**, 199 (1998).

[15] G. Darboux, *Archives Neerlandaises* (ii) **6** 371 (1901).

[16] C. Grosche, G. S. Pogosyan and A. N. Sissakian, *Fortschr. Phys.* **43**, 453 (1995).

[17] A. M. Perelomov, *Integrable Systems of Classical Mechanics and lie Algebras* vol.1 (Basel: Birkhäuser-Verlag), p 81 (1990).

[18] J. Hietarinta, *Phys. Rep.* **147**, 87 (1987).

[19] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* 4th ed. (Cambridge: Cambridge U.P.), p 332 (1937).

[20] Y. Uwano, *J. Phys. Soc. Jpn* **63 Suppl. A**, 31 (1994).

[21] Y. Uwano, *J. Phys.* **A28**, 2041 (1995).

[22] Y. Uwano, *Int. J. of Bifurcation and Chaos* **8**, 941 (1998).

[23] Y. Uwano, *Rep. Math. Phys.* **44**, 267 (1999).