Title
Multiple solutions of inhomogeneous $\mathbf{H}$-systems with zero Dirichlet boundary conditions (Variational Problems and Related Topics)

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Citation
数理解析研究所講究録 (2001), 1181: 199-209

Issue Date
2001-01

URL
http://hdl.handle.net/2433/64558

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Multiple solutions of inhomogeneous H-systems
with zero Dirichlet boundary conditions

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1. Introduction

This article is an abbreviated version of [TF1].

In this paper, we study the existence of multiple solutions to the Dirichlet problem of the inhomogeneous H-system:

\[
\begin{align*}
\Delta u &= 2Hu_x \wedge u_y + f \quad \text{in } \Omega, \\
u|_{\partial \Omega} &= 0,
\end{align*}
\]

where $$\Omega \subset \mathbb{R}^2$$ is a bounded smooth domain, $$H > 0$$ is a given constant, and $$f \in H^{-1}(\Omega; \mathbb{R}^3)$$ is a given function. $$a \wedge b$$ denotes the usual vector product of $$a, b \in \mathbb{R}^3$$.

Solutions of (1.1) in $$H_0^1(\Omega; \mathbb{R}^3)$$ correspond to critical points of the energy functional:

\[
E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{2H}{3} Q(u) + \int_{\Omega} f \cdot u,
\]

where

\[
Q(u) = \int_{\Omega} u \cdot u_x \wedge u_y
\]

is the oriented volume functional.

This problem is interesting from the variational view point because the functional $$E$$ does not satisfy the Palais-Smale(PS) compactness condition globally on $$H_0^1(\Omega; \mathbb{R}^3)$$. In the case $$f \equiv 0$$, it is known that the existence or the non-existence of multiple solutions of (1.1) depends on the topology of the domain. More precisely, it is known that when $$f \equiv 0$$ and $$\Omega$$ is simply-connected, then $$u \equiv 0$$ is the only solution of (1.1); on the other hand, when $$\Omega$$ is doubly-connected, there exists at least one non-trivial solution [W].

In [Ta1], G.Tarantello treated the following Dirichlet problem of semilinear elliptic equations involving critical Sobolev exponent:

\[
\begin{align*}
-\Delta u &= u|u|^{2^*-2} + f \quad \text{in } \Omega, \\
u|_{\partial \Omega} &= 0,
\end{align*}
\]

where $$\Omega \subset \mathbb{R}^N (N \geq 3)$$ is a bounded smooth domain, $$2^* = \frac{2N}{N-2}$$ is the critical Sobolev exponent for the embedding $$H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$$ and $$f \in H^{-1}(\Omega)$$. It is well known that when $$f \equiv 0$$ and $$\Omega$$ is star-shaped, (1.2) has the only solution $$u \equiv 0$$ [P]. On the other hand, there is a vast literature on the effect of the domain topology...
or geometry on the existence of multiple positive solutions of (1.2) when $f \equiv 0$; see [BaC], [Co], [Pa] and references therein. In spite of a possible lack of compactness, she obtained the existence of at least two non-trivial weak solutions of (1.2) for $f \neq 0$ satisfying some suitable smallness condition.

Here, following her methods, we pursue the analogous results for the problem (1.1).

Before stating our results, we introduce a set of assumptions on the function $f$:

\begin{itemize}
  \item[(f.1)] $f \in H^{-1} \cap L^1(\Omega; \mathbb{R}^3)$,
  \item[(f.2)] $-\int_{\Omega} f \cdot u < \frac{(\int_{\Omega} |\nabla u|^2)^2}{-8HQ(u)}$ for all $u \in H_0^1(\Omega; \mathbb{R}^3)$ with $Q(u) < 0$,
  \item[(f.3)] $\|f\|_{H^{-1}} < \frac{\sqrt{3\pi}}{12H}$.
\end{itemize}

We remark that by the isoperimetric inequality for $H_0^1$-mappings [BC]:

$$S|Q(u)|^{\frac{2}{3}} \leq \int_{\Omega} |\nabla u|^2 \quad \text{for all} \quad u \in H_0^1(\Omega; \mathbb{R}^3),$$

where $S = (32\pi)^{1/3}$, it is easy to see that the assumption (f.2) always holds if $f \in H^{-1}(\Omega; \mathbb{R}^3)$ satisfies

$$\|f\|_{H^{-1}} < \frac{S^{3/2}}{8H} = \frac{\sqrt{2\pi}}{2H},$$

so, the assumption (f.2) appears essentially the smallness condition of $f$.

Our main results are the following:

**Theorem 1.** Let $f \neq 0$ satisfy the assumptions (f.1) and (f.2), then the problem (1.1) admits at least one solution $u$ in $H_0^1(\Omega; \mathbb{R}^3)$.

**Theorem 2.** Let $f \neq 0$ satisfy the assumptions (f.1) and (f.3), then, $u$ obtained in Theorem 1 is a strict local minimum for the functional $E$ in $H_0^1(\Omega; \mathbb{R}^3)$, and the problem (1.1) admits at least one more solution $\bar{u}$ in $H_0^1(\Omega; \mathbb{R}^3)$.

This paper is organized as follows. In section 1, we prove Theorem 1 by using Ekeland’s variational principle and Nehari variational method.

In section 2, we prove Theorem 2 by utilizing the strict local minimality of the first solution and the Mountain Pass Theorem.

### 2. Existence of the first solution

In this section, we prove Theorem 1 by considering a suitable minimization problem for the functional $E$. To this end, let us denote

$$\Lambda = \{u \in H_0^1(\Omega; \mathbb{R}^3) : \langle E'(u), u \rangle = 0\}$$

(2.1)

$$= \{u \in H_0^1(\Omega; \mathbb{R}^3) : \int_{\Omega} |\nabla u|^2 + 2HQ(u) + \int_{\Omega} f \cdot u = 0\},$$

(2.2)
where $\langle , \rangle$ denotes the usual dual pairing of $H^{-1}$ and $H^1_0$, and

\[
\Lambda_0 = \{u \in \Lambda : \int_{\Omega} |\nabla u|^2 + 4HQ(u) = 0\}, \tag{2.3}
\]
\[
\Lambda_+ = \{u \in \Lambda : \int_{\Omega} |\nabla u|^2 + 4HQ(u) > 0\}, \tag{2.4}
\]
\[
\Lambda_- = \{u \in \Lambda : \int_{\Omega} |\nabla u|^2 + 4HQ(u) < 0\}. \tag{2.5}
\]

Recall that $Q$ is analytic on $H^1_0(\Omega; \mathbb{R}^3)$ and $\langle Q'(u), u \rangle = 3Q(u)$. A is called the "Nehari manifold" and it contains all critical points for $E$ in $H^1_0(\Omega; \mathbb{R}^3)$. Therefore, to obtain the solution of the problem (1.1), it is natural to consider the minimization problem:

\[
c_0 = \inf_{u \in \Lambda} E(u). \tag{2.6}
\]

We shall prove that under the assumptions (f.1) and (f.2), the infimum in (2.6) is achieved by some $u \in \Lambda$ and $u$ defines a critical point for $E$ in $H^1_0(\Omega; \mathbb{R}^3)$.

We note that if we set

\[
K(u) = \int_{\Omega} |\nabla u|^2 + 2HQ(u) + \int_{\Omega} f \cdot u, \quad u \in H^1_0(\Omega; \mathbb{R}^3),
\]

then $\Lambda = \{u \in H^1_0(\Omega; \mathbb{R}^3) : K(u) = 0\}$ and $\Lambda$ is in fact a smooth submanifold of $H^1_0(\Omega; \mathbb{R}^3)$ if $K'(u) \neq 0$ for any $u \in \Lambda$. Now we calculate

\[
\langle K'(u), u \rangle = \int_{\Omega} |\nabla u|^2 + 4HQ(u), \quad \text{for } u \in \Lambda,
\]

so, for the minimizer $u$ for (2.6) (if it exists) to be a critical point of $E$ in $H^1_0(\Omega; \mathbb{R}^3)$, we must ensure that $\Lambda_0 = \{0\}$.

We start with a lemma which shows the assumption (f.2) is indeed a sufficient condition for $\Lambda_0 = \{0\}$.

**Lemma 2.1.** Suppose the assumption (f.2) holds, then for any $u \in \Lambda$, $u \neq 0$, we have

\[
\int_{\Omega} |\nabla u|^2 + 4HQ(u) \neq 0.
\]

*Proof:* Assume

\[
\int_{\Omega} |\nabla u|^2 + 4HQ(u) = 0 \tag{2.7}
\]

holds for some $u \in \Lambda, u \neq 0$. Then $Q(u) < 0$ and, because $u$ also satisfies

\[
\int_{\Omega} |\nabla u|^2 + 2HQ(u) + \int_{\Omega} f \cdot u = 0, \tag{2.8}
\]

we have

\[
\int_{\Omega} f \cdot u = 2HQ(u) \tag{2.9}
\]
by (2.7),(2.8).
Now from (f.2),(2.9) and (2.7) we derive:
\[
0 < \int_{\Omega} f \cdot u + \frac{\left(\int_{\Omega} |\nabla u|^{2}\right)^{2}}{-8HQ(u)} = 2HQ(u) + \frac{\left(\int_{\Omega} |\nabla u|^{2}\right)^{2}}{-8HQ(u)}
\]
\[
= |Q(u)| \cdot \{-2H + \frac{(-4HQ(u))^{2}}{8H|Q(u)|^{2}}\} = 0,
\]
which is a contradiction. \(\square\)

**Lemma 2.2.** Suppose the assumption (f.2) holds. Then for any \(u \in \Lambda, u \neq 0\), there exist an \(\varepsilon > 0\) and a smooth function
\[
t : \{w \in H_{0}^{1}(\Omega; \mathbb{R}^{3}) : \|w\|_{H_{0}^{1}} < \varepsilon\} \to \mathbb{R}
\]
such that
\[
t(0) = 1, \quad t(w) \cdot (u - w) \in \Lambda \quad \text{for} \quad \|w\|_{H_{0}^{1}} < \varepsilon,
\]
and
\[
\langle t'(0), w \rangle = \frac{2\int_{\Omega} \nabla u \cdot \nabla w + 6H \int_{\Omega} w \cdot u_{x} \wedge u_{y} + \int_{\Omega} f \cdot w}{\int_{\Omega} |\nabla u|^{2} + 4HQ(u)}.
\]

**Proof:** Define a smooth map \(F : \mathbb{R} \times H_{0}^{1}(\Omega; \mathbb{R}^{3}) \to \mathbb{R}\) as
\[
F(t, w) = t \int_{\Omega} |\nabla(u - w)|^{2} + 2Ht^{2}Q(u - w) + \int_{\Omega} f \cdot (u - w).
\]
Since \(F(1, 0) = 0\) for \(u \in \Lambda\) and
\[
F_{t}(1, 0) = \int_{\Omega} |\nabla u|^{2} + 4HQ(u) \neq 0
\]
by Lemma 2.1, we can apply the Implicit Function Theorem at the point \((1, 0) \in \mathbb{R} \times H_{0}^{1}(\Omega; \mathbb{R}^{3})\) and the result follows. \(\square\)

**Lemma 2.3.** Let \(f \neq 0\) satisfy the assumption (f.1), then
\[
\mu_{0} := \inf_{\substack{u \in H_{0}^{1}(\Omega; \mathbb{R}^{3}) \\text{such that} \\|Q(u)\|_{H_{0}^{1}} = 1}} \frac{1}{8H} \left(\int_{\Omega} |\nabla u|^{2}\right)^{2} + \int_{\Omega} f \cdot u
\]
is achieved. In addition if \(f\) satisfies (f.2), then \(\mu_{0} > 0\).

The proof of Lemma 2.3 is a modification of that for the minimization problem treated in [TF2], so we omit it. (However, different from [TF2], the extra assumption that \(f \in L^{1}(\Omega; \mathbb{R}^{3})\) is needed in the current case.)
In the following, we proceed to the proof of Theorem 1 assuming that \( f \neq 0 \) satisfies (f.1) and (f.2) simultaneously.

First, we give an upper and lower bound for \( c_0 \) in (2.6).

**Proposition 2.1.** There exists a \( t_0 > 0 \) such that
\[
-\frac{2}{3}\|f\|_{H^{-1}}^2 \leq c_0 < -\frac{t_0^2}{6}\|f\|_{H^{-1}}^2
\]  

holds.

**Proof:** First, we show that \( E \) is bounded from below on \( \Lambda \). Indeed, by definition (2.2),
\[
\int_\Omega |\nabla u|^2 + 2HQ(u) + \int_\Omega f \cdot u = 0
\]
for \( u \in \Lambda \). Thus we have
\[
E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2H}{3}Q(u) + \int_\Omega f \cdot u
\]
\[
= \left( \frac{1}{2} - \frac{1}{3} \right)\int_\Omega |\nabla u|^2 + \left( 1 - \frac{1}{3} \right)\int_\Omega f \cdot u
\]
\[
\geq \frac{1}{6}\|\nabla u\|_{L^2}^2 - \frac{2}{3}\|f\|_{H^{-1}}\|\nabla u\|_{L^2} \geq -\frac{2}{3}\|f\|_{H^{-1}}^2
\]
for any \( u \in \Lambda \). In particular,
\[
c_0 \geq -\frac{2}{3}\|f\|_{H^{-1}}^2.
\]

In order to obtain an upper bound for \( c_0 \), let \( v \in H^1_0(\Omega; \mathbb{R}^3) \) be the unique solution of \( \triangle v = f \) in \( \Omega \).

Then for \( f \neq 0 \), we have
\[
\int_\Omega f \cdot v = -\int_\Omega |\nabla v|^2 < 0.
\]
Now we divide the proof according to the sign of \( Q(v) \).

If \( Q(v) > 0 \), then
\[
\varphi(t) = \varphi^v(t) := t\int_\Omega |\nabla v|^2 + 2Ht^2Q(v)
\]
is a convex quadratic function in \( t \in \mathbb{R} \) with \( \varphi(0) = \varphi \left( \frac{\int_\Omega |\nabla v|^2}{-2HQ(v)} \right) = 0 \). Note that, if \( \varphi'(t) > 0 \) at some \( t \neq 0 \) satisfying \( \varphi(t) = -\int_\Omega f \cdot v \), then \( tv \in \Lambda_+ \).

Now we have \( -\int_\Omega f \cdot v > 0 \), so easy observation shows there exists a unique \( t_0 > 0 \) such that \( t_0v \in \Lambda_+ \). Thus, by definition of \( \Lambda \) and \( \Lambda_+ \), we have
\[
E(t_0v) = -\frac{1}{2}\int_\Omega |\nabla(t_0v)|^2 - \frac{4H}{3}Q(t_0v)
\]
\[
< -\frac{1}{2}\int_\Omega |\nabla(t_0v)|^2 + \frac{1}{3}\int_\Omega |\nabla(t_0v)|^2
\]
\[
= -\frac{t_0^2}{6}\int_\Omega |\nabla v|^2 = -\frac{t_0^2}{6}\|f\|_{H^{-1}}^2,
\]
which yields an upper bound of $c_0$ in this case.

Next if $Q(v) < 0$, then $\varphi(t)$ in (2.13) is a concave quadratic function in $t$ and

$$\max_{t \in \mathbb{R}} \varphi(t) = \varphi\left(\frac{\int_{\Omega} |\nabla v|^2}{-4HQ(v)}\right) = \frac{\left(\int_{\Omega} |\nabla v|^2\right)^2}{-8HQ(v)}.$$ 

Now, by the assumption (f.2) we again obtain unique $t_0 > 0$ with $t_0v \in \Lambda_+$, so the rest is the same as in the former case.

Finally if $Q(v) = 0$, then $v \in \Lambda_+$ and we can choose $t_0 = 1$. □

At this point, we are ready to apply the Ekeland’s variational principle \cite{AE} to the minimization problem (2.6).

**Ekeland’s variational principle.** Let $M$ be a complete metric space with metric $d$, and let $E : M \to \mathbb{R} \cup +\infty$ be lower semicontinuous, bounded from below, and $\not\equiv \infty$.

Then for any $\varepsilon, \delta > 0$, for any $u \in M$ with

$$E(u) \leq \inf_M E + \varepsilon,$$

there exists an element $v \in M$ such that

1. $E(v) \leq E(u)$,
2. $d(u, v) \leq \delta$,
3. $E(v) < E(w) + \frac{\varepsilon}{\delta}d(v, w)$, for all $w \neq v$.

**Proposition 2.2.** There exists a minimizing sequence $\{u^n\} \subset \Lambda$ for (2.6) with the following properties:

(a) $E(u^n) < c_0 + \frac{1}{n}$,
(b) $E(w) \geq E(u^n) - \frac{1}{n}\|\nabla(u^n - w)\|_{L^2}$, for any $w \in \Lambda$,
(c) $\frac{2}{3}\|f\|_{H^{-1}} < \|\nabla u^n\|_{L^2} < 4\|f\|_{H^{-1}}$, where $t_0$ is given by Proposition 2.1, and
(d) $\|E'(u^n)\|_{H^{-1}} \to 0$ as $n \to \infty$.

**Sketch of Proof:** $\Lambda$ is closed with respect to the strong $H^1_0$-topology and $E$ is bounded from below, continuous, and $\not\equiv \infty$ on $\Lambda$. Therefore we can apply Ekeland’s variational principle to (2.6), and the statements (a),(b) are the direct consequences of this.

By taking $n$ large enough, from (a) and (2.12) we have

$$E(u^n) = \frac{1}{6} \int_{\Omega} |\nabla u^n|^2 + \frac{2}{3} \int_{\Omega} f \cdot u^n < c_0 + \frac{1}{n} < -\frac{t_0^2}{6}\|f\|_{H^{-1}}^2 < 0. \quad (2.14)$$
This implies $u^n \not\equiv 0$ and
\[ \frac{1}{6} \int_{\Omega} |\nabla u^n|^2 < -\frac{2}{3} \int_{\Omega} f \cdot u^n \leq \frac{2}{3} \|f\|_{H^{-1}} \|u^n\|_{H_0^1}. \]
Consequently, we have
\[ \|u^n\|_{H_0^1} < 4 \|f\|_{H^{-1}}. \]
On the other hand, (2.14) implies
\[ 0 < \frac{t_0^2}{6} \|f\|_{H^{-1}}^2 < -\frac{2}{3} \int_{\Omega} f \cdot u^n \]
for $n$ large, which gives
\[ \frac{t_0^2}{4} \|f\|_{H^{-1}} < \|\nabla u^n\|_{L^2}. \]
This proves (c).

Finally, to obtain (d), we shall argue by contradiction and assume $\|E'(u^n)\|_{H^{-1}} > 0$ for $n$ large. Then we can get the contradiction, using Lemma 2.2 and Lemma 2.3. 
\[ \square \]

Proof of Theorem 1: From Proposition 2.2 we have obtained a minimizing Palais-Smale sequence $\{u^n\}$ for $E$, with a uniform $H_0^1$-bound. Let $u \in H_0^1(\Omega; \mathbb{R}^3)$ be the weak limit of (a subsequence of) $\{u^n\}$. From (2.15) we note that $-\int_{\Omega} f \cdot u > 0$.

By Proposition 2.2(d) and the fact that
\[ \langle E'(u^n), w \rangle \to \langle E'(u), w \rangle, \quad \forall w \in H_0^1(\Omega; \mathbb{R}^3) \]
(this follows from the weak continuity of $u^n_x \wedge u^n_y$ : 
\[ u^n_x \wedge u^n_y \rightharpoonup u_x \wedge u_y \quad \text{in} \ D'(\Omega; \mathbb{R}^3), \]
See [BC:Lemma A.9]), we have 
\[ \langle E'(u), w \rangle = 0 \quad \text{for any} \ w \in H_0^1(\Omega; \mathbb{R}^3). \]
That is, $u$ is a weak solution of (1.1) and in particular $u \in \Lambda$.

Therefore 
\[ c_0 \leq E(u) = \frac{1}{6} \int_{\Omega} |\nabla u|^2 + \frac{2}{3} \int_{\Omega} f \cdot u \leq \liminf_{n \to \infty} E(u^n) = c_0. \]
Consequently $u^n \to u$ strongly in $H_0^1$ and $E(u) = c_0 = \inf_{\Lambda} E$.

This proves Theorem 1. 
\[ \square \]
3. Existence of the second solution

In this section, we shall prove the existence of the second solution of problem (1.1) by the Mountain Pass Theorem of Ambrosetti-Rabinowitz [AR].

To this end, we first derive that \( \underline{u} \) is a strict local minimum for \( E \) in \( H^1_0(\Omega; \mathbb{R}^3) \), if \( f \) satisfies the assumption (f.3).

**Proposition 3.1.** Let \( f \not\equiv 0 \) satisfy (f.1) and (f.3), then \( \underline{u} \) obtained in Theorem 1.1 is a strict local minimum for \( E \) in \( H^1_0(\Omega; \mathbb{R}^3) \).

**Proof:** For any \( v \in H^1_0(\Omega; \mathbb{R}^3) \) we expand:

\[
E(\underline{u} + v) = \frac{1}{2} \int_{\Omega} |\nabla(\underline{u} + v)|^2 + \frac{2H}{3} Q(\underline{u} + v) + \int_{\Omega} f \cdot (\underline{u} + v)
\]

\[
= \frac{1}{2} \int_{\Omega} |\nabla \underline{u}|^2 + \frac{2H}{3} Q(\underline{u}) + \int_{\Omega} f \cdot \underline{u} + \left[ \int_{\Omega} \nabla \underline{u} \cdot \nabla v + 2H \int_{\Omega} \underline{u}_x \wedge \underline{u}_y \cdot v + \int_{\Omega} f \cdot v \right]
\]

\[
+ \frac{1}{2} \int_{\Omega} |\nabla v|^2 + 2H \int_{\Omega} u \cdot v_x \wedge v_y + \frac{2H}{3} Q(v)
\]

\[
= E(\underline{u}) + \frac{1}{2} \int_{\Omega} |\nabla v|^2 + 2H \int_{\Omega} u \cdot v_x \wedge v_y + \frac{2H}{3} Q(v)
\]

Now, by Wente's \( L^2 \)-estimate and the isoperimetric inequality, we have

\[
\frac{1}{2} \int_{\Omega} |\nabla v|^2 + 2H \int_{\Omega} u \cdot v_x \wedge v_y + \frac{2H}{3} Q(v)
\]

\[
\geq \frac{1}{2} ||\nabla v||_{L^2}^2 - 2HC_{L^2} \cdot ||\nabla \underline{u}||_{L^2} ||\nabla v||_{L^2}^2 - \left( \frac{2H}{3} \right) \left( \frac{1}{S} \right)^{3/2} ||\nabla v||_{L^2}^3,
\]

where \( C_{L^2} = \sqrt{\frac{3}{16\pi}} \) is the best constant for Wente's \( L^2 \)-estimate [Ge] and \( S = (32\pi)^{1/3} \).

We denote

\[
h(x) = \left( \frac{1}{2} - 2HC_{L^2} ||\nabla \underline{u}||_{L^2} \right) x^2 - \left( \frac{2H}{3} \right) \left( \frac{1}{S} \right)^{3/2} x^3, \quad x \geq 0,
\]

then it is easy to see that \( h(x) > 0 \) for sufficiently small \( x > 0 \) if \( \frac{1}{2} - 2HC_{L^2} ||\nabla \underline{u}||_{L^2} > 0 \), that is,

\[
||\nabla \underline{u}||_{L^2} < \frac{1}{4HC_{L^2}}.
\]

Recall that \( \underline{u} \) satisfies the estimate \( ||\nabla \underline{u}||_{L^2} \leq 4||f||_{H^{-1}} \) (by Proposition 2.2(c)), therefore if

\[
||f||_{H^{-1}} < \frac{1}{16HC_{L^2}},
\]

that is, under the assumption (f.3), we certainly have (3.2).

In conclusion, (f.3) implies that

\[
E(\underline{u} + v) = E(\underline{u}) + h(||v||_{H^1_0}) > E(\underline{u})
\]
for every sufficiently small $v \in H^1_0(\Omega; \mathbb{R}^3)$, so $\underline{u}$ is a strict local minimum for $E$. □

Next, we study the compactness properties of the functional $E$. The following proposition is now more or less a standard result in this direction.

**Proposition 3.2 (local compactness).** $E$ satisfies the $(PS)_c$ condition for all $c < c_0 + \frac{4\pi}{3H^2}$. That is, every sequence $\{u^n\} \subset H^1_0(\Omega; \mathbb{R}^3)$ satisfying:

(a) $E(u^n) \to c < c_0 + \frac{4\pi}{3H^2}$,

(b) $||E'(u^n)||_{H^{-1}} \to 0$,

has a strong convergent subsequence.

To proceed further, we need some definition. Let

$$\varphi^\varepsilon(x, y) = \frac{2\varepsilon}{\varepsilon^2 + x^2 + y^2} \begin{pmatrix} x \\ y \\ \varepsilon \end{pmatrix}, \quad \varepsilon > 0$$

be an extremal function for the isoperimetric inequality in $\mathbb{R}^2$.

For $a = (x_0, y_0) \in \Omega$, denote $\varphi^{\varepsilon,a}(x, y) = \varphi^\varepsilon(x - x_0, y - y_0)$, and let $\zeta_a \in C_0^\infty(\Omega)$ be the cut-off function with $0 \leq \zeta_a \leq 1$, $\zeta_a = 1$ near $a$. We set

$$v^{\varepsilon,a}(x, y) = \zeta_a(x, y)\varphi^{\varepsilon,a}(x, y).$$

Now, by calculating directly along the explicit path, we get the following proposition.

**Proposition 3.3.** For every $R > 0$ and almost everywhere $a = (x_0, y_0) \in \{(x, y) \in \Omega : \nabla \underline{u}(x, y) \neq 0\}$, there exist an $\varepsilon_0 = \varepsilon_0(R, a) > 0$ and an orthonormal basis $(\vec{i}, \vec{j}, \vec{k})$ in $\mathbb{R}^3$ having the same orientation as the canonical basis of $\mathbb{R}^3$, such that

$$E(\underline{u} - Rv^{\varepsilon,a}) < c_0 + \frac{4\pi}{3H^2}$$

holds for every $0 < \varepsilon < \varepsilon_0$. Here we assume that $\varphi^{\varepsilon,a}$ is written with respect to $(\vec{i}, \vec{j}, \vec{k})$.

At this point, we recall the famous Mountain Pass Theorem of Ambrosetti-Rabinowitz [AR] in its standard form.

**Mountain Pass Theorem.** Let $F$ be a $C^1$-functional on a Banach space $V$. Suppose

1. $F(0) = 0$;
2. $\exists \rho, \alpha > 0$ such that $||v||_V = \rho \implies F(v) \geq \alpha$;
3. $\exists v^* \in V$ such that $||v^*||_V \geq \rho$ and $F(v^*) < 0$. 
Define
\[ \Gamma = \{ \gamma \in C^0([0,1]; V) : \gamma(0) = 0, \gamma(1) = v^* \} \]
and
\[ c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} F(\gamma(t)) (\geq \alpha). \]
Then, there exists a sequence \( \{v^n\} \subset V \) such that
\[ F(v^n) \to c \]
and
\[ F'(v^n) \to 0 \quad \text{in} \ V^*. \]
Further if \( F \) satisfies the \((PS)_c\) condition, then there exists a critical point at the level \( c \).

Proof of Theorem 2:
We only need to apply the Mountain Pass Theorem to the functional \( F(v) = E(\underline{u} + v) - E(\underline{u}) \) on \( V = H^1_0(\Omega; \mathbb{R}^3) \). (1) is trivially satisfied and Proposition 3.1 implies (2). (3) is also verified because \( E(\underline{u} - Rv^{\epsilon,a}) \to -\infty \) as \( R \to \infty \); we set \( v^* = R_0(-v^{\epsilon,a}) \) for some \( R_0 > 0 \) large enough.
Proposition 3.2 and 3.3 implies the \((PS)_c\) condition for \( F \). Therefore we have a critical point \( v^0 \) of \( F, F(v^0) = c \geq \alpha > 0 \), that is, we conclude there exists a critical point \( \bar{u} := \underline{u} + v^0 \) of \( E, \bar{u} \not\equiv \underline{u} \).
The proof is completed. \qed

References


