数理解析研究所講究録
1181 巻 2001 年 123-134

ON A HÖLDER REGULARITY OF GRADIENTS FOR EVOLUTIONAL $P-$LAPLACIAN SYSTEMS

Masashi Misawa† (三沢 正史)

Dept. of Computer Science, Fac. of Electro-Communications,
The Univ. of Electro-Communications

Dedicated to Professor Norio Kikuchi on his sixtieth birthday

1. INTRODUCTION

Let Ω be a domain in an Euclidean space R^m for $m \geq 2$ and T be a positive number. Suppose that $2 < p < \infty$. We consider the evolutional p-Laplacian system

$$\partial_t u^i - D_\alpha \left(|Du|_{gh}^{p-2} g^{\alpha\beta} D_\beta u^j h_{i,j} \right) = \operatorname{div} \left(|F|^{p-2} F^i \right), \quad i = 1, \cdots, n, \tag{1.1}$$

where the function F is given and defined on $Q = (0,T) \times \Omega$ with values into R^{mn}, $\left(g^{\alpha\beta}(z)\right)$ and $(h_{ij}(z))$ are symmetric matrices with measurable coefficients satisfying the uniform ellipticity and boundedness condition with positive constants λ, Λ

$$\lambda |\xi|^2 \leq g^{\alpha\beta}(z) \xi_\alpha^i \xi_\beta^j h_{ij}(z) \leq \Lambda |\xi|^2 \quad \text{for any } \xi = (\xi_\alpha^i) \in R^{mn} \text{ and almost every } z \in Q \tag{1.2}$$

and the notation $|\xi|_{gh}^2 = g^{\alpha\beta} \xi_\alpha^i \xi_\beta^j h_{ij}$ and $|\xi|^2 = \xi \cdot \xi = \xi_\alpha^i \xi_\alpha^i$ is used. Here and in what follows, the summation notation over repeated indices is adopted.

Such evolution systems as (1.1) describe the gradient flow of the $p-$energy functional with variable coefficients and lower order terms.

We are interested in how the regularity of the function F is reflected to the solutions under some assumption on the coefficients. Let us consider a Hölder regularity of the gradient of a solution for a given Hölder continuous function F. Such Hölder regularity is known to hold for elliptic and parabolic systems of divergence form (see [9, pp. 87–89], [15] and the references in them). The $C^{1,\alpha}-$regularity for evolutional $p-$Laplacian systems with only principal term was established in [5, 6, 7, 2] and the results become fundamental to the regularity theory for evolutional $p-$Laplacian systems. Concerning $p-$Laplacian systems with differentiable coefficients and lower order terms, we have the corresponding results in [4, 3, 16, 17, 10]. For stationary $p-$Laplacian systems with non-differentiable lower order terms, a Hölder regularity of the gradient is studied in [8] for the degenerate case $p > 2$. In [8, 11, 12], L^q-estimates for the gradient for $p-$Laplacian systems are also obtained and these are of interest itself (also see [13]). In the results above, a interior regularity of a "local" solution is investigated. On the other hand, a interior regularity for

†This research was partially supported by the Grant-in-Aid for Scientific Research (C) No.12740102 at the Ministry of Educations, Science, Sports and Culture.

evolutional $p-$Laplacian system with non-differentiable coefficients and lower order terms seems to have not been successfully investigated (refer to [4, 1]). In this paper, we study a interior Hölder regularity for the initial boundary value problem for (1.1).

Let u_0 be a smooth function defined on Q with values into R^n. For $F \in L^p(Q, R^{mn})$, consider a weak solution of the initial boundary value problem for (1.1): $u \in L^\infty((0,T); L^2(\Omega, R^n)) \cap L^p((0,T); W^{1,p}(\Omega, R^n))$ satisfying

$$\int_Q \left\{-u \cdot \partial_t \phi + |Du|^{p-2} g^{\alpha\beta} D_\beta u^j D_\alpha \phi^i h_{ij} + |F|^{p-2} F \cdot D\phi\right\} dz = 0 \text{ for all } \phi \in C_0^\infty(Q, R^n),$$

$$u = u_0 \qquad \text{on } \partial_p Q = [0,T] \times \partial\Omega \cup \{t=0\} \times \Omega, \tag{1.3}$$

where the initial and boundary conditions are understood to hold in the following sense

$$\begin{aligned} &u(t) = u_0(t) \quad \text{in the trace sense of } W^{1,p}(\Omega, R^n) \text{ for almost every } t, 0 \le t \le T, \\ &\lim_{t \searrow 0} |u(t) - u_0(0)|_{2,\Omega} = 0. \end{aligned} \tag{1.4}$$

Our main result is the following.

Theorem 1 *Suppose that the coefficients and F are Hölder continuous functions in Q with an exponent β, $0 < \beta < 1$, on the usual parabolic metric. Let u be a weak solution of the initial boundary problem for (1.1) in Q with smooth initial and boundary value u_0. Then there exist an exponent α, $0 < \alpha < 1$, depending only on m, p and β, and a positive constant γ depending only on $m, p, \lambda, \Lambda, \beta, |\partial_t u_0, Du_0|_{\infty,Q}$ and $|Du_0, Du|_{p,Q}$ such that the gradient of the solution is locally Hölder continuous in Q with an exponent α on the usual parabolic metric and the Hölder constant is bounded by the constant γ.*

Because of a difficulty due to a property of the evolutional $p-$Laplace operator, we assume that our solution under consideration is not a "local" solution in Q, but a solution of the initial boundary value problem. It seems more natural to study a interior regularity of a "local" solution, defined on a local parabolic cylinder. However, the evolutional $p-$Laplace operator has no homogeneity, that is, the solution multiplied by a constant is not a solution. This property gives us a technical difficulty for a interior regularity of a "local" solution. We do not also know whether the Hölder exponent α can be chosen to equal to the Hölder exponent β of the function F.

2. GROWTH OF LOCAL $P-$ENERGY

In this section, we study the growth on the radius for the local $p-$energy. For any $z = (t,x) \in R^{m+1}$, $\rho > 0$ and $\theta > 0$, put $Q_\rho^\theta(z) = (t - \rho^\theta, t) \times B_\rho(x)$ and $Q_\rho^2(z)$ is abbreviated to $Q_\rho(z)$. To reduce (1.1) to a equation with homogeneous data on $\partial_p Q$, put $w = u - u_0$ in Q. Then w is a weak solution in Q with zero initial and boundary value of

$$\begin{aligned} &\partial_t w - D_\alpha\left(|Dw + Du_0|_{gh}^{p-2} g^{\alpha\beta} D_\beta w^j h_{i,j}\right) \\ &\quad = \operatorname{div}\left(|F|^{p-2}F^i\right) - \partial_t u_0 + D_\alpha\left(|Dw + Du_0|_{gh}^{p-2} g^{\alpha\beta} D_\beta (u_0)^j h_{i,j}\right) \quad \text{in } Q. \end{aligned} \tag{2.1}$$

Let $\tilde{w}$ be the extension of w into $(-\infty,T]\times\Omega$ such that $\tilde{w}=w$ in Q and $\tilde{w}=0$ in $(-\infty,0]\times\Omega$. We can also define the extension $\tilde{g}^{\alpha\beta}$, $\tilde{h}_{ij}$ and $\tilde{F}$ of the coefficients and the function F to be Hölder continuous functions with compact support in $(-\infty,T]\times\Omega$. Then we see that $\tilde{w}\in L^\infty((-\infty,T],L^2(\Omega,R^n))\cap L^p((-\infty,T];W_0^{1,p}(\Omega,R^n))$ is a weak solution of (2.1). By this reason, let $u\in L^\infty((-\infty,T],L^2(\Omega,R^n))\cap L^p((-\infty,T];W_0^{1,p}(\Omega,R^n))$ is a weak solution of (2.1) with $u=0$ in $(-\infty,0]\times\Omega$. Let $z_0=(t_0,x_0)\in Q$ be taken arbitrarily and $R_0=\frac{1}{2}\mathrm{dist}(x_0,\partial\Omega)$. By translation and a scaling transformation, we assume that our solution u is defined on $\tilde{Q}_2=(-\infty,0)\times B_2(0)$. Note that the support of the solution u is contained in the closure of $(-\frac{t_0}{R_0^p},0)\times B_2(0)$.

The main lemma in this section is the following.

Lemma 2 *For any α, $0<\alpha<1$, there exists a positive constant γ depending only on m,p,λ,Λ, $|\partial_t u_0|_{\infty,\tilde{Q}_2}$, $|Du_0|_{\infty,\tilde{Q}_2}$, $[gh]_{\beta,\tilde{Q}_2}$, $[|F|^{p-2}F]_{\beta,\tilde{Q}_2}$, $|Du|_{p,\tilde{Q}_2}$ and α such that*

$$\int_{Q_\rho(\tilde{z})}(1+|Du|^p)dz\le\gamma\rho^{m+2-\alpha p} \tag{2.2}$$

holds for any $\tilde{z}\in\tilde{Q}_1$ and all ρ, $0<\rho\le 1$.

For the proof of Lemma 2, let $\theta\ge 2$, R, $0<R\le 1$ and $\tilde{z}=(\tilde{t},\tilde{x})\in\tilde{Q}_1$. By translation, we assume that $\tilde{z}$ is the origin. To prove Lemma 2, we use the perturbation argument with the homogeneous $p-$Laplacian system, similarly as in [1] and [4, pp. 292–315]. Let $v\in L^\infty(-R^\theta,0;L^2(B_R,R^n))\cap L^p(-R^\theta,0;W^{1,p}(B_R,R^n))$ be a solution to the homogeneous $p-$Laplacian system with constant coefficients (for the existence of a weak solution, refer to [14, Theorem 6.7, pp 466–475]).

$$\begin{aligned}\partial_t v^i&=D_\alpha\left(|Dv|^{p-2}_{g(0)h(0)}g^{\alpha\beta}(0)D_\beta v^j h_{ij}(0)\right)\quad\text{in } Q_R^\theta,\\ v&=u\quad\text{on } \partial_p Q_R^\theta.\end{aligned} \tag{2.3}$$

We know that the $L^\infty-$estimate holds for the gradient of solutions (see [4, Theorem 5.1, pp. 238]).

Lemma 3 *There exists a positive constant γ depending only on m and p such that*

$$\sup_{Q^\theta_{\frac{R}{2}}}|Dv|^p\le\gamma\left(\frac{R^{\theta-2}}{|Q_R^\theta|}\int_{Q_R^\theta}|Dv|^p dz\right)^{\frac{p}{2}}+\gamma R^{\frac{p(2-\theta)}{p-2}}. \tag{2.4}$$

We now estimate the difference of u from v in the local L^p-norm.

Lemma 4 *There exists a positive constant γ depending only on m,p,λ,Λ, $|\partial_t u_0|_{\infty,\tilde{Q}_2}$, $|Du_0|_{\infty,\tilde{Q}_2}$, $[gh]_{\beta,\tilde{Q}_2}$ and $[|F|^{p-2}F]_{\beta,\tilde{Q}_2}$ such that*

$$\int_{Q_R^\theta}|Du-Dv|^p dz\le\gamma R^{\frac{\beta p}{p-1}}\int_{Q_R^\theta}(1+|Du|^p)dz+\gamma|Q_R^\theta|^{\frac{1}{p-1}}\left(\int_{Q_R^\theta}(1+|Du|^p)dz\right)^{\frac{p-2}{p-1}}, \tag{2.5}$$

where $[f]_{\beta,\tilde{Q}_2}$ denote the Hölder semi-norm of a function f in $\tilde{Q}_2$ with exponent β.

Proof. Subtract (2.1) from (2.3) and use a test function $v-u$, which is shown to be admissible by the usual approximation argument, in the weak form of the resulting equation. By Young's inequality and some algebraic inequalities, we make routine calculation to have, for any $\epsilon > 0$,

$$
\begin{aligned}
&\gamma \int_{Q_R^\theta} |Dv - Du|^p dz \\
&\leq \int_{Q_R^\theta} |Dv - Du| \Big((|Du| + |Du_0|)^{p-2}|Du_0| + |gh - (gh)(0)||Du + Du_0|^{p-1} \\
&\qquad + \Big||F|^{p-2}F - (|F|^{p-2}F)_{Q_R^\theta}\Big|\Big) + |\partial_t u_0||v - w| dz \\
&\leq \epsilon \int_{Q_R^\theta} |Dv - Du|^p dz + \int_{Q_R^\theta} (1 + |Du|^p)^{\frac{p-2}{p-1}} dz \\
&\quad + \gamma \int_{Q_R^\theta} |gh - (gh)(0)|^{\frac{p}{p-1}} |Du + Du_0|^p + \Big||F|^{p-2}F - (|F|^{p-2}F)_{Q_R^\theta}\Big|^{\frac{p}{p-1}} dz \\
&\qquad + \left(\int_{Q_R^\theta} |\partial_t u_0|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_{Q_R^\theta} |v - w|^p \right)^{\frac{1}{p}} \qquad (2.6) \\
&\leq \epsilon \int_{Q_R^\theta} |Dv - Du|^p dz + \gamma |Q_R^\theta|^{\frac{1}{p-1}} \left(\int_{Q_R^\theta} (1 + |Du|^p) dz \right)^{\frac{p-2}{p-1}} \\
&\quad + \gamma R^{\frac{p\beta}{p-1}} \left([gh]_{\beta,\tilde{Q}_2}^{\frac{p}{p-1}} \int_{Q_R^\theta} (1 + |Du|^p) dz + |Q_R^\theta| \left([|F|^{p-2}F]_{\beta,\tilde{Q}_2}^{\frac{p}{p-1}} + |\partial_t u_0|_{\infty,Q_R^\theta}^{\frac{p}{p-1}} \right) \right),
\end{aligned}
$$

where the positive constant γ depends only on ϵ^{-1}, p, and $|Du_0|_{\infty,\tilde{Q}_2}$ and we used (1.2) and, in the last inequality, Poincaré inequality available for functions in $W_0^{1,p}(B_R)$ and the Hölder continuity of the coefficients g, h and the function $|F|^{p-2}F$ in $\tilde{Q}_2$ with exponent β on the parabolic metric.

Combining (2.5) with (2.4), we arrived at the following estimation.

Lemma 5 *Set $\theta = 2 + \alpha(p-2)$ for any positive number α. Then there exists a positive constant γ having the same dependence as the one in Lemma 4 such that*

$$
\begin{aligned}
\int_{Q_\rho^\theta} (1 + |Du|^p) dz &\leq \gamma \left\{ \left(\frac{R^{\alpha p}}{|Q_R^\theta|} \int_{Q_R^\theta} (1 + |Du|^p) dz \right)^{\frac{p-2}{2}} + 2^{m+\theta} \right\} \left(\frac{\rho}{R} \right)^{m+\theta} \int_{Q_R^\theta} |Du|^p dz \\
&\quad + \gamma \rho^{m+\theta} R^{-\alpha p} \qquad (2.7) \\
&\quad + \gamma R^{m+\theta-\alpha p+\frac{p\beta}{p-1}} \left(\frac{R^{\alpha p}}{|Q_R^\theta|} \int_{Q_R^\theta} (1 + |Du|^p) dz \right) \\
&\quad + R^{m+\theta-\alpha p\left(\frac{p-2}{p-1}\right)} \left(\frac{R^{\alpha p}}{|Q_R^\theta|} \int_{Q_R^\theta} (1 + |Du|^p) dz \right)^{\frac{p-2}{p-1}}
\end{aligned}
$$

holds for any $\theta \geq 2$ and all ρ, R, $0 < \rho < R \leq 1$.

Proof. Noting that $0 < R \le 1$, by (2.6), we have

$$\begin{aligned}\int_{Q_R^\theta} |Dv|^p dz &\le \gamma \int_{Q_R^\theta} |Dv - Du|^p + |Du|^p dz \\ &\le \gamma \int_{Q_R^\theta} (1 + |Du|^p) dz + \gamma R^{m+2+\frac{p\beta}{p-1}}. \qquad (2.8)\end{aligned}$$

We substitute (2.8) into (2.4) and combine (2.5) with the resulting inequality. Then it follows that, for any ρ, $0 < \rho \le \frac{R}{2}$,

$$\begin{aligned}\int_{Q_\rho^\theta} (1 + |Du|^p) dz &\le \gamma \sup_{Q^\theta_{\frac{R}{2}}} |Dv|^p + \gamma \int_{Q_R^\theta} |Du - Dv|^p dz + |Q_\rho^\theta| \\ &\le \gamma |Q_\rho^\theta| \left\{ \left(\frac{R^{\theta-2}}{|Q_R^\theta|} \int_{Q_R^\theta} |Du|^p dz \right)^{\frac{p}{2}} + R^{\frac{p^2\beta}{2(p-1)}} + R^{\frac{p(2-\theta)}{p-2}} \right\} \qquad (2.9) \\ &\quad + \gamma R^{\frac{\beta p}{p-1}} \int_{Q_R^\theta} (1 + |Du|^p) dz + \gamma |Q_R^\theta|^{\frac{1}{p-1}} \left(\int_{Q_R^\theta} (1 + |Du|^p) dz \right)^{\frac{p-2}{p-1}}.\end{aligned}$$

The first term in the right hand side of (2.9) is bounded by

$$\gamma \left(\frac{R^{\frac{p(\theta-2)}{p-2}}}{|Q_R^\theta|} \int_{Q_R^\theta} |Du|^p dz \right)^{\frac{p-2}{2}} (\tfrac{\rho}{R})^{m+\theta} \int_{Q_R^\theta} |Du|^p dz. \qquad (2.10)$$

For $\rho > \frac{R}{2}$, we trivially have

$$\int_{Q_\rho^\theta} (1 + |Du|^p) dz \le 2^{m+\theta} (\tfrac{\rho}{R})^{m+\theta} \int_{Q_R^\theta} (1 + |Du|^p) dz. \qquad (2.11)$$

Proof of Lemma 2. We use the nonlinear iteration introduced in [1, Lemma 3.1, pp. 297-299].

Lemma 6 *Let ϕ be a non-negative non-decreasing function defined on* $[0, 1]$. *Suppose that*

$$\phi(\rho) \le \gamma_0 (\tfrac{\rho}{R})^l \phi(R) + \gamma_0 (R^{l-\sigma\kappa} + \rho^l R^{-\kappa}) \qquad (2.12)$$

holds for all ρ, R, $0 < \rho < R \le 1$, where γ_0, l, κ and σ are given positive constants with $l > \kappa$ and $0 < \sigma < 1$. Then, for any positive number δ satisfying

$$0 \le \delta < \kappa \left(\tfrac{\kappa(1-\sigma)}{\kappa(1-\sigma)+l} \right), \qquad (2.13)$$

and any R, $0 < R \le 1$, there exist a positive constant γ_1, depending only on $\gamma_0, l, \kappa, \sigma$ and δ, and a positive integer n_0, depending on the same constants as γ_1 and also on R, such that

$$\phi(\rho) \le \gamma_1 \rho^{l-\kappa+\delta} \left(\tfrac{1}{R^{lq^{n_0+2}}} \phi(R) + 1 \right) \qquad (2.14)$$

holds for all ρ, R, $0 < \rho < R \le 1$, where $q = 1 + \frac{\kappa(1-\sigma)}{l}$.

Define the sequences $\{\alpha_k\}$ and $\{\theta_k\}$ by

$$\begin{aligned} \alpha_0 = \tfrac{m+2}{2}, \quad \alpha_{k+1} = \alpha_k \tfrac{\alpha_k(p-2)+m+2}{\alpha_k(p-2+\frac{\beta}{\alpha_0})+m+2}; \\ \theta_k = 2 + \alpha_k(p-2), \quad k = 1, \cdots. \end{aligned} \tag{2.15}$$

Then we find from induction that $\{\alpha_k\}$ and $\{\theta_k\}$ are positive decreasing sequences with $\alpha_k \searrow 0$ and $\theta_k \searrow 2$ as $k \nearrow \infty$. Now we claim that

Lemma 7 *For each $k = 0, 1, \cdots$, there exists a positive constant γ_k depending only on $\alpha_k, \alpha_0, \beta$, $|Du|_{p,\tilde{Q}_1}$ and the same quantities as in Lemma 4 such that*

$$\tfrac{1}{|Q_\rho^{\theta_k}|} \int_{Q_\rho^{\theta_k}(z_0)} (1 + |Du|^p) dz \leq \gamma_k \rho^{-\alpha_k p} \tag{2.16}$$

holds for any $z_0 \in \tilde{Q}_1$ and all ρ, $0 < \rho \leq 1$.

Proof of Lemma 7. We prove the validity of Lemma 7 by induction on $k = 0, 1, \cdots$. From $u \in L^p((-\infty, 0); W^{1,p}(B_2(0), R^n))$, we see that

$$\tfrac{1}{|Q_\rho^{\theta_0}|} \int_{Q_\rho^{\theta_0}(z_0)} (1 + |Du|^p) dz \leq \rho^{-\alpha_0 p} \left(1 + \gamma \int_{\tilde{Q}_2} |Du|^p dz\right) \tag{2.17}$$

holds for any $z_0 \in \tilde{Q}_1$ and all ρ, $0 < \rho \leq 1$, where we used that $m + \theta_0 = \alpha_0 p$ by $\alpha_0 = \frac{m+2}{2}$. Suppose by induction that (2.16) holds for some $k = 1, \cdots$. Let us show that (2.16) holds for $k+1$. For all ρ, $\frac{1}{2} < \rho \leq 1$, we trivially have

$$\tfrac{1}{|Q_\rho^{\theta_{k+1}}|} \int_{Q_\rho^{\theta_{k+1}}(z_0)} (1 + |Du|^p) dz \leq \rho^{-\alpha_{k+1} p} \left(1 + \gamma 2^{m+\theta_0} \int_{\tilde{Q}_2} |Du|^p dz\right). \tag{2.18}$$

We now proceed our estimation for each $z_0 \in \tilde{Q}_1$. Fix $z_0 \in \tilde{Q}_1$ and put $\alpha = \alpha_k, \theta = \theta_k$. Then we obtain from (2.7) that, for all ρ, R, $0 < \rho < R \leq 1$,

$$\begin{aligned} \int_{Q_\rho^\theta(z_0)} (1 + |Du|^p) dz \ \leq \ & \gamma \left\{ (\gamma_k)^{\frac{p-2}{2}} + 2^{m+\theta_0} \right\} \left(\frac{\rho}{R}\right)^{m+\theta} \int_{Q_R^\theta(z_0)} (1 + |Du|^p) dz \\ & + \gamma \rho^{m+\theta} R^{-\alpha p} \\ & + \gamma(\gamma_k + (\gamma_k)^{\frac{p-2}{p-1}}) R^{m+\theta-\alpha p\left(1-\frac{\beta}{\alpha_0(p-1)}\right)}. \end{aligned} \tag{2.19}$$

In Lemma 6, choose $R = \frac{1}{2}$, $\phi(\rho) = \int_{Q_\rho^{\theta_k}(z_0)} (1 + |Du|^p) dz$ and

$$l = m + \theta_k, \quad \kappa = \alpha_k p, \quad \sigma = 1 - \tfrac{\beta}{\alpha_0(p-1)} \tag{2.20}$$

and then apply Lemma 6 for (2.19). Noting that

$$0 < \tilde{\delta} = \tfrac{\frac{\beta}{\alpha_0}\alpha_k}{\frac{\beta}{\alpha_0}\alpha_k + m + \theta_k} < \tfrac{\alpha_k \frac{p\beta}{\alpha_0(p-1)}}{\alpha_k \frac{p\beta}{\alpha_0(p-1)} + m + \theta_k} = \tfrac{\kappa(1-\sigma)}{\kappa(1-\sigma)+l}, \tag{2.21}$$

we obtain from Lemma 6 that, for all ρ, $0 < \rho \leq \frac{1}{2}$,

$$\int_{Q_\rho^{\theta_k}(z_0)} (1+|Du|^p)dz \leq \gamma_1 \rho^{l-\kappa+\tilde{\delta}\kappa}\Big(2^{lq^{n_0+1}} \int_{Q_{\frac{1}{2}}^{\theta_k}(z_0)} (1+|Du|^p)dz + 1\Big), \tag{2.22}$$

where we note that the positive constants n_0 and γ_1 depend only on $\gamma_k, \alpha_k, \alpha_0$ and β and that

$$\begin{aligned} q &= 1 + \tfrac{\alpha_k p\beta}{\alpha_0(p-1)(m+\theta_k)}, \\ l-\kappa+\tilde{\delta}\kappa &= m+\theta_k - \alpha_k p + \alpha_k p \frac{\frac{\beta}{\alpha_0}\alpha_k}{\frac{\beta}{\alpha_0}\alpha_k + m+\theta_k} = m+\theta_k - \alpha_{k+1}p. \end{aligned} \tag{2.23}$$

Thus we can choose a positive constant γ_{k+1} depending only on $\gamma_k, \alpha_k, \alpha_0, \beta$ and $|Du|_{p,\tilde{Q}_2}$ such that

$$\int_{Q_\rho^{\theta_k}(z_0)} (1+|Du|^p)dz \leq \gamma_{k+1}\rho^{m+\theta_k-\alpha_{k+1}p} \tag{2.24}$$

holds for all ρ, $0 < \rho \leq 1$.

To show that (2.16) holds for $Q_\rho^{\theta_{k+1}}(z_0)$, divide $Q_\rho^{\theta_{k+1}}(z_0)$, in the time direction, into $s_0 = [\rho^{\theta_{k+1}}/\rho^{\theta_k}] + 1$ cylinders $Q_\rho^{\theta_k}(t_i, x_0)$, $i = 0, 1, \cdots, s_0 - 1$. Adopt (2.24) in each region $Q_\rho^{\theta_k}(t_i, x_0)$, $i = 0, 1, \cdots, s_0 - 1$. Here note that the support of the solution u is in the closure of $(-\frac{t_0}{R_0^p}, 0) \times B_2(0)$. Sum up the inequalities to have

$$\begin{aligned} \int_{Q_\rho^{\theta_{k+1}}(z_0)} (1+|Du|^p)dz &\leq \sum_{i=0}^{s_0-1} \int_{Q_\rho^{\theta_k}(t_i,x_0)} (1+|Du|^p)dz \\ &\leq \gamma_{k+1}\, s_0\, \rho^{m+\theta_k-\alpha_{k+1}p} \leq 2\gamma_{k+1}\rho^{m+\theta_{k+1}-\alpha_{k+1}p}. \end{aligned} \tag{2.25}$$

Hence, Lemma 7 follows from (2.17), (2.18) and (2.25).

Finally, we derive the assertion in Lemma 2 from Lamma 7. Take a positive number ρ, $0 < \rho \leq 1$, arbitrarily and fix it. For any positive number α, $0 < \alpha < 1$, let k be a positive integer such that $\alpha_k \leq \alpha < \alpha_{k-1}$. Then adopt (2.16) with k and ρ and use the decomposition argument above to conclude (2.2) in Lemma 2.

3. GROWTH OF LOCAL MEAN OSCILLATION

In this section, we study the growth on the radius of the local mean oscillation. Let $z_0 = (t_0, x_0) \in Q$ be taken arbitrarily and $R_0 = \frac{1}{2}\mathrm{dist}_p(z_0, \partial_p Q)$, where $\mathrm{dist}_p(z_1, z_2) = \min\{|t_2-t_1|^{\frac{1}{p}}, |x_2-x_1|\}$ for any $z_i = (t_i, x_i) \in R^{m+1}, i = 1, 2$. By translation and a scaling transformation, let $u \in L^\infty((-2^p, 0], L^2(B_2, R^n)) \cap L^p((-2^p, 0]; W_0^{1,p}(B_2, R^n))$ is a weak solution of (1.1) in $Q_2^p = (-2^p, 0] \times B_2(0)$. Using the notation in Sect.2, we recall that $u = u_0 + w = u_0 + \tilde{w}$ in $Q_2^p \subset P_2 = (-\frac{t_0}{R_0^p}, 0) \times B_2(0)$. Then we see from Lemma 2 that, for any α, $0<\alpha<1$, there exists a positive constant γ depending only on α, $|Du|_{p,P_2}$, $|Du_0|_{p,P_2}$ and the same quantities as in Lemma 4 such that

$$\begin{aligned}\int_{Q_\rho(z_0)} |Du|^p dz &\le \gamma \int_{Q_\rho(z_0)} (|Dw|^p + |Du_0|^p) dz \\ &\le \gamma(1+|Du_0|^p_{\infty,Q_2^p})\rho^{m+2-\alpha p} \end{aligned} \tag{3.1}$$

holds for any $z_0 \in Q_1^p$ and all ρ, $0<\rho\le 1$.

Our main lemma in this section is the following.

Lemma 8 *There exist a positive number β_1, $0<\beta_1<1$, depending only on m, p and β, and a positive constant γ, depending only on m, p, β, $|Du|_{p,P_2}$, $|Du_0|_{p,P_2}$ and the same quantities as in Lemma 4, such that*

$$\begin{aligned}&\int_{Q_\rho(z_0)} |Du-(Du)_\rho|^p dz \\ &\le \gamma\rho^{m+2+p\beta_1}\left(\int_{Q_{\frac{1}{2}}(z_0)} |Du-(Du)_R|^p dz + 1\right)\end{aligned} \tag{3.2}$$

holds for any $z_0 \in Q_1^p$ and all ρ, $0<\rho\le \frac{1}{2}$.

We apply the isomorphism theorem due to Campanato (see [9, Theorem 1.2, p. 70; Theorem 1.3, p. 72]) to see that Du is Hölder continuous in Q_1^p with an exponent β_1. Hence, we conclude the assertion in Theorem 1.

Similarly as in Sect.2, we use the perturbation argument with the homogeneous $p-$Laplac -ian systems. However, here we make a little device to change the power of radius of a local parabolic cylinder, on which we solve the homogeneous $p-$Laplacian systems. By this device, we can appropriately adopt the Hölder estimate for the gradient for the homogeneous $p-$Laplacian systems to make estimation of the mean oscillation of the gradient of a solution in the L^p-norm.

Let $z_0 \in Q_1^p \subset P_1 = (-\frac{t_0}{R_0}, 0)\times B_1(0)$ be arbitrarily taken and, for any R, $0<R\le\frac{1}{2}$, and any δ, $0<\delta<1$, set $r=R^{1-\delta}$. For brevity, we assume that z_0 is the origin. Let $v\in L^\infty(-r,0:L^1(B_r,R^n))\cap L^p(-r,0:W^{1,p}(B_r,R^n))$ be a weak solution of (2.3), in which θ, R and w are replaced by $2, r$ and u, respectively. Similarly as in (2.6) in the proof of Lemma 4, we subtract (1.1) from (2.3) replaced θ, R and w by $2, r$ and u, respectively, and use a test function $v-u$ in the weak form of the resulting equation to make estimation for the difference $v-u$ in the L^p-norm

$$\int_{Q_r} |Dv-Du|^p dz \le \gamma R^{(1-\delta)\frac{p\beta}{p-1}} \int_{Q_r} (1+|Du|^p) dz, \tag{3.3}$$

where a positive constant γ depends only on m, p, λ, Λ, $[gh]_{\beta,Q_2^p}$ and $[|F|^{p-2}F]_{\beta,Q_2^p}$.

Now we observe that the $L^\infty-$estimate holds in the following form.

Lemma 9 *For any α, $0 < \alpha < 1$, there exists a positive constant $\gamma(\alpha)$ having the same dependence as the one in (3.1) such that*

$$|Dv|_{\infty,Q_{\frac{R^{1-\delta}}{2}}} \leq \gamma(\alpha) R^{-\alpha p(1-\delta)}. \tag{3.4}$$

Proof. We choose $\theta = 2$ and $R = r$ in Lemma 3 to have

$$|Dv|_{\infty,Q_{\frac{r}{2}}} \leq \gamma \left(\frac{1}{|Q_r|} \int_{Q_r} |Dv|^p dz \right)^{\frac{1}{2}} + \gamma. \tag{3.5}$$

Noting that $0 < r < 1$, we find from (3.3) that

$$\begin{aligned} \int_{Q_r} |Dv|^p dz &\leq \int_{Q_r} |Dv - Du|^p dz + \int_{Q_r} |Du|^p dz \\ &\leq \gamma \int_{Q_r} (1 + |Du|^p) dz. \end{aligned} \tag{3.6}$$

Substitute (3.6) into (3.5) to have

$$|Dv|_{\infty,Q_{\frac{r}{2}}} \leq \gamma \left(\frac{1}{|Q_r|} \int_{Q_r} |Du|^p dz \right)^{\frac{1}{2}} + \gamma. \tag{3.7}$$

Adopt (3.1) with $\rho = r = R^{1-\delta}$ in (3.7) to arrived at (3.4).

We need the estimation for a oscillation of the gradient of a solution v.

Lemma 10 *For any positive number δ, $0 < \delta < 1$, there exist a positive number α_1, $0 < \alpha_1 < 1$, depending only on m, p and δ, and a positive constant $\gamma(\delta)$, depending only on $\delta, |Du|_{p,P_2}, |Du_0|_{p,P_2}$ and the same quantities as in Lemma 4, such that*

$$\operatorname*{osc}_{Q_R}(Dv) \leq \gamma(\delta) R^{\alpha_1}. \tag{3.8}$$

Proof. We know that the Hölder estimate holds in the following form (see [4, Theorem 1.1', pp. 256]). There exist positive constants γ and α_0 depending only on m and p such that

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(Dv) \leq \gamma |Dv|_{\infty,Q_{\frac{R^{1-\delta}}{2}}} \left(\frac{\frac{R}{2} + \frac{R}{2} \max\{1, |Dv|^{\frac{p-2}{2}}_{\infty,Q_{\frac{R^{1-\delta}}{2}}}\}}{\operatorname{dist}_2(Q_{\frac{R}{2}}, \partial_p Q_{\frac{R^{1-\delta}}{2}})} \right)^{\alpha_0}. \tag{3.9}$$

If $R^\delta \geq \frac{1}{4}$, then we have

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(Dv) \leq 8R^\delta |Dv|_{\infty,Q_{\frac{R^{1-\delta}}{2}}}. \tag{3.10}$$

Since, if $0 < R^\delta \le \frac{1}{4}$, then

$$\begin{aligned}\mathrm{dist}_2(Q_{\frac{R}{2}}, \partial_p Q_{\frac{R^{1-\delta}}{2}}) &= \tfrac{1}{2}\min\{R^{1-\delta} - R, \sqrt{R^{1-\delta} - R}\} \\ &\ge \tfrac{3}{8}R^{1-\delta},\end{aligned}$$

we obtain from (3.4) and (3.9)

$$\begin{aligned}\operatorname*{osc}_{Q^2_{\frac{R}{2}}}(Dv) &\le \gamma |Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}} \max\{1, |Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}}^{\frac{\alpha_0(p-2)}{2}}\} R^{\delta\alpha_0} \\ &\le \gamma(\alpha) R^{\delta\alpha_0 - \alpha p(1-\delta)\left(1 + \frac{\alpha_0(p-2)}{2}\right)}. \qquad (3.11)\end{aligned}$$

We choose a positive number α to satisfy

$$0 < \alpha < \frac{\delta\alpha_0}{p(1-\delta)\left(1 + \frac{\alpha_0(p-2)}{2}\right)} \qquad (3.12)$$

and then we put

$$\alpha_1 = \delta\alpha_0 - \alpha p(1-\delta)\left(1 + \frac{\alpha_0(p-2)}{2}\right) > 0 \qquad (3.13)$$

to have the conclusion of Lemma 10.

Now let us finish the proof of Lemma 8.

Proof of Lemma 8. For all R, $0 < R \le 1$, and all ρ, $\frac{R}{4} \le \rho < \frac{R}{2}$, we trivially have

$$\int_{Q_\rho} |Dv - (Dv)_\rho|^p dz \le 4^{m+4}\left(\frac{\rho}{R}\right)^{m+4} \int_{Q_{\frac{R}{2}}} |Dv - (Dv)_{\frac{R}{2}}|^p dz. \qquad (3.14)$$

From Lemma 10, we find that

$$\int_{Q_\rho} |Dv - (Dv)_\rho|^p dz \le \tilde{\gamma} R^{m+2+p\alpha_1} \qquad (3.15)$$

holds for all R, $0 < R \le 1$, and all ρ, $0 < \rho < \frac{R}{4}$, where positive constants $\tilde{\gamma}$ and α_1, $0 < \alpha_1 < 1$, depend only on the same quantities as in Lemma 10 for each positive number δ, $0 < \delta < 1$. Gathering (3.14) and (3.15) gives that

$$\int_{Q_\rho} |Dv - (Dv)_\rho|^p dz \le 4^{m+4}\left(\frac{\rho}{R}\right)^{m+4} \int_{Q_{\frac{R}{2}}} |Dv - (Dv)_{\frac{R}{2}}|^p dz + \tilde{\gamma} R^{m+2+p\alpha_1} \qquad (3.16)$$

holds for all ρ, R $0 < \rho < \frac{R}{2} \le \frac{1}{2}$, where positive constants $\tilde{\gamma}$ and α_1, $0 < \alpha_1 < 1$, depend only on the same quantities as in (3.15). From (3.1), (3.3) and (3.16), it follows that, for all ρ, R, $0 < \rho < \frac{R}{2} \leq \frac{1}{2}$,

$$\begin{aligned}
&\int_{Q_\rho} |Du - (Du)_\rho|^p dz \\
&\leq \gamma \int_{Q_R} |Dv - (Dv)_\rho|^p dz + \gamma \int_{Q_R} |Dv - Du|^p dz \\
&\leq \gamma \left(\frac{\rho}{R}\right)^{m+4} \int_{Q_R} |Du - (Du)_R|^p dz + \tilde{\gamma} R^{m+2+p\alpha_1} + \gamma R^{(1-\delta)\left(\frac{\beta p}{p-1}\right)} \int_{Q_R} (1 + |Du|^p) dz \\
&\leq \gamma \left(\frac{\rho}{R}\right)^{m+4} \int_{Q_R} |Du - (Du)_R|^p dz + \tilde{\gamma} R^{m+2+p\alpha_1} \qquad (3.17) \\
&\qquad + \gamma \left(R^{(1-\delta)\left(m+2+\frac{\beta p}{p-1}\right)} + \gamma R^{(1-\delta)\left(m+2-p\alpha+\frac{\beta p}{p-1}\right)} \right).
\end{aligned}$$

Fix a positive number δ to satisfy

$$0 < \delta < \frac{\frac{\beta p}{p-1}}{m + 2 + \frac{\beta p}{p-1}}. \tag{3.18}$$

Next, choose a positive number α satisfying (3.12) and

$$0 < \alpha < \frac{(1-\delta)\frac{\beta p}{p-1} - \delta(m+2)}{p(1-\delta)}. \tag{3.19}$$

Then we can choose a positive number α_2 to be

$$m + 2 + p\alpha_2 = (1-\delta)\left(m + 2 - p\alpha + \frac{\beta p}{p-1}\right) > m + 2. \tag{3.20}$$

Finally, set $\beta_1 = \min\{\alpha_1, \alpha_2\}$, where α_1 is in (3.13), and apply the iteration argument in [9, Lemma 2.1, p. 86] to arrive at (3.2).

Acknowledgements

The author would like to thank Professor Norio Kikuchi, Keio University, for his constant encouragement.

References

[1] Y.-Z. Chen, E. DiBenedetto, Boundary estimates for solutions of nonlinear degenerate parabolic systems, J. reine angew. Math. **395** (1989) 102-131 .

[2] H.J. Choe, Hölder regularity for the gradient of solutions of certain singular parabolic systems, Commun. Partial Differential Equations **16(11)** (1991) 1709-1732.

[3] H.J. Choe, Hölder continuity for solutions of certain degenerate parabolic systems, IMA preprint series **712** (1990).

[4] E. DiBenedetto, Degenerate Parabolic Equations, Universitext, Springer-Verlag (1994).

[5] E. DiBenedetto, A. Friedman, Regularity of solutions of nonlinear degenerate parabolic systems, J. reine angew. Math. **349** (1984) 83-128.

[6] E. DiBenedetto, A. Friedman, Hölder estimates for nonlinear degenerate parabolic systems, J. reine angew. Math. **357** (1985) 1-22.

[7] E. DiBenedetto, A. Friedman, Addendum to "Hölder estimates for nonlinear degenerate parabolic systems", J. reine angew. Math. **363** (1985) 217-220.

[8] E. DiBenedetto, J. Manfredi, On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems, Amer. J. Math. **115** (1993) 1107-1134.

[9] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems Ann. of Math. Stud. **105**, Princeton University Press, Princeton (1983).

[10] M. Giaquinta, G. Modica, Remarks on the regularity of the minimizers of certain degenerate functionals, Manusc. Math. **57**, (1986) 55-99

[11] J. Kinuunen, S. Zhou, A local estimates for nonlinear equations with discontinuous coefficients, Commun. Partial Differential Equations **24(11&12)** (1999) 2043-2068.

[12] J. Kinuunen, J. L. Lewis, Higher integrability for parabolic systems of $p-$Laplacian type, Duke Math. J. **102(2)** (2000) 253-271.

[13] J. L. Lewis, On very weak solutions of certain elliptic systems, Commun. Partial Differential Equations **18(9&10)** (1993) 1515-1537.

[14] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tzeva, Linear and quasilinear equations of parabolic type, Transl. Math. Monogr. **23** AMS Providence R-I (1968).

[15] W. Schlag, Schauder and L^p-estimates for parabolic systems via Campanato spaces, Commun. Partial Differential Equations **21(7&8)** (1996) 1141-1175.

[16] P. Tolksdorf, Everywhere-regularity for some quasilinear systems with a lack of ellipticity, Annali. Mat. pura appl. **134** (1983) 241-266.

[17] K. Uhlenbeck Regularity for a class of nonlinear elliptic systems, Acta Math. **138** (1970) 219-240 .