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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1181: 110-122</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64565">http://hdl.handle.net/2433/64565</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Vortex State of d-Wave Superconductors in the Ginzburg-Landau Energy

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Abstract

We find a minimizer of a reduced form of the Ginzburg-Landau free energy for d-wave superconductors having distinct degree-one vortices. For a single vortex in the vortex core, we analytically recover the vortex structure with fourfold symmetry.

1 Introduction

In the 1910’s, low-temperature superconductivity was observed on metals and alloys (cf. [9]). Recently, high-temperature superconductivity has been found on some copper-oxide superconductors (cf. [12]). The vortex state of high-temperature superconductors is different from the vortex state of low-temperature superconductors. When the applied magnetic field is close to the lower critical field $H_c$, the single vortex is expected to be symmetric in low-temperature superconductors but it may be asymmetric (fourfold symmetric) in high-temperature superconductors (cf. [8], [31]). Moreover, as the applied magnetic field is close to the upper critical field $H_{c2}$, Abrikosov type vortex lattices are expected to be triangular in low-temperature superconductors but they may be rectangular in high-temperature superconductors (cf. [1], [8], [27], [30], [31] etc).

To distinguish low-temperature and high-temperature superconductivity, an s-wave and a d-wave order parameter were introduced (cf. [13], [21]). Soininen et al. (cf. [3], [28]) introduced the Ginzburg-Landau free energy with an s-wave and a d-wave order parameter. Ren et al. (cf. [24], [25]) present a microscopic derivation of the Ginzburg-Landau equations from the Gor’kov equations by using the finite temperature Green’s-function approximation method. From [31], we learned the two fields Ginzburg-Landau free energy is given by:

$$G(\Psi_s, \Psi_d, A) = \int_{\mathbb{R}^2} \kappa^2 |\text{curl } A - H|^2 + \alpha_s(T)|\Psi_s|^2$$
$$+ \frac{1}{2} (1 - |\Psi_d|^2)^2 + \frac{1}{3} |\Psi_s|^4 + \frac{2}{3} |\Psi_s|^2 |\Psi_d|^2 + \frac{2}{3} (\Psi_s^2 \Psi_d^2 + \Psi_s^2 \Psi_d^2)$$
$$+ 2 |\Pi \Psi_s|^2 + |\Pi \Psi_d|^2 + \{\Pi_x \Psi_x \Psi_x^* - \Pi_y \Psi_x \Pi_y \Psi_x^* + \text{H.C.}\},$$

(1.1)
where $\Psi_s$ is the $s$-wave order parameter, $\Psi_d$ is the $d$-wave order parameter and $A$ is the vector-valued magnetic potential, $\Pi = i\nabla - A$, $H$ is a constant applied magnetic field, $\kappa$ is the Ginzburg-Landau parameter and

$$\alpha_s(T) = C_s/(1 - T/T_c).$$

(1.2)

Here $C_s$ is a positive constant, $T$ is the current temperature and $T_c$ is the $d$-wave transition temperature.

As the current temperature $T$ is close to $T_c$, Franz et al. [8] observed that in a predominantly $d$-wave superconductor, the $s$-wave component is generically very small. They also provided approximation formulas for the order parameters $\Psi_d$ and $\Psi_s$ as follows:

$$|\Psi_s| \ll |\Psi_d|, \quad |\nabla \Psi_s| \ll |\nabla \Psi_d| \quad \text{as} \quad T \to T_c.$$  

(1.3)

Affleck et al. [1] obtained the leading order in $(1 - T/T_c)$ as

$$\Psi_s = \xi (\prod^2_x - \prod^2_y) \Psi_d,$$

(1.4)

where $\xi$ is a parameter satisfying that $\xi \to 0$ as $T \to T_c$. In [7], Du derived (1.4) by the formal asymptotic analysis.

We learned from [5] and [6] that it is reasonable to ignore the magnetic field in strongly type II superconductors when the applied magnetic field is close to $H_{c1}$ and $T \to T_c$. Hence it is valuable to study the two fields Ginzburg-Landau model (1.1) without the magnetic field (i.e. $A,H \equiv 0$). Moreover, Rosenstein et al. [6] took (1.3) and (1.4) into (1.1) and modified the free energy (1.1) as follows:

$$G(\Psi_d) = \int_{\mathbb{R}^2} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta |\square \Psi_d|^2 dx \, dy,$$

(1.5)

where $\square = \partial^2_x - \partial^2_y$ and $\beta$ is a parameter satisfying that $\beta \to 0$ as $T \to T_c$. Here we have ignored the magnetic field (i.e. $A,H \equiv 0$) for strongly type II superconductors.

It is hard to find the minimizer of (1.5) by the standard direct method. Suppose that $\Psi_d \in H^2(\mathbb{R}^2; \mathbb{C})$ is a minimizer of (1.5) over $H^2(\mathbb{R}^2; \mathbb{C})$. Then it is easy to check that

$$G(\Psi_d + v) = G(\Psi_d) + \int_{\mathbb{R}^2} |\nabla v|^2 - (1 - |\Psi_d|^2)|v|^2 + 2(\Psi_d \cdot v)^2$$

$$+ \int_{\mathbb{R}^2} 2|v|^2(\Psi_d \cdot v) + \frac{1}{2}|v|^4 + \beta |\square v|^2,$$

(1.6)

for any test function $v \in C_0^\infty(\mathbb{R}^2)$. Hereafter, $z_1 \cdot z_2 = \frac{1}{2}(z_1 z_2 + z_1 \bar{z}_2)$ for all $z_1, z_2 \in \mathbb{C}$. Let $v_n(z) = \delta_n v_0(z) \sin[\delta_n^{-2/3}(x+y)]$ for $z = x + iy \in \mathbb{C} \cong \mathbb{R}^2$, where $v_0$ is a test function with a nonempty compact support and $\{\delta_n\}$ is a sequence of positive numbers such that $\delta_n \to 0$ as $n \to \infty$. Here we use the fact that the complex plane $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$. Now, we replace $v$ in (1.6) by $v_n$ and we obtain that $G(\Psi_d + v_n) \to G(\Psi_d)$ but $\|\Psi_d + v_n\|_{H^2} \to \infty$ as $n \to \infty$. Hence $\Psi_d + v_n$'s form a minimizing sequence but $\Psi_d + v_n$'s have no converging subsequence.
even weakly converging subsequences in $H_{loc}^2(\mathbb{R}^2; \mathbb{C})$. Thus the free energy (1.5) has a defect on minimization.

From [30], we learned a Ginzburg-Landau energy functional (without the magnetic field) as follows:

$$E(\Psi_d) = \int_{\mathbb{R}^2} |\nabla \Psi_d|^2 + \frac{1}{2} (1 - |\Psi_d|^2)^2 + \eta (|\partial^2_x \Psi_d|^2 + |\partial^2_y \Psi_d|^2) \, dx \, dy,$$

(1.7)

where $\eta$ is a constant depending on the current temperature $T$. The term $|\partial^2_x \Psi_d|^2 + |\partial^2_y \Psi_d|^2$ breaks the circular symmetry and accounts for the square symmetry. Furthermore, Park and Huse [22] introduced a more generalized Ginzburg-Landau free energy (without the magnetic field) for $d$-wave superconductors as follows:

$$F(\Psi_d) = \int_{\mathbb{R}^2} |\nabla \Psi_d|^2 + \frac{1}{2} (1 - |\Psi_d|^2)^2 + \gamma_1 |\Delta \Psi_d|^2 + \beta_1 (|\Box \Psi_d|^2 - 4|\partial_x \partial_y \Psi_d|^2) \, dx \, dy,$$

(1.8)

where $\Delta = \partial^2_x + \partial^2_y$ and $\beta_1, \gamma_1$ are parameters tending to zero as $T \to T_c$.

Hereafter, we assume that $|\Psi_d| \to 1$ and all the derivatives of $\Psi_d$ decay fast as $(|x, y|) \to \infty$. Such an assumption is consistent with the results in [8] and [31]. Using integration by part, we may transform (1.8) into

$$\tilde{G}(\Psi_d) = \int_{\mathbb{R}^2} |\nabla \Psi_d|^2 + \frac{1}{2} (1 - |\Psi_d|^2)^2 + \beta |\Box \Psi_d|^2 + \gamma |\Delta \Psi_d|^2 \, dx \, dy,$$

(1.9)

where $\beta, \gamma$ are parameters tending to zero as $T \to T_c$. In this paper, we assume that $\beta, \gamma > 0$ and $\beta, \gamma \to 0$ as $T \to T_c$. In particular, such an assumption includes the case that $0 < \gamma \ll \beta$ i.e. (1.9) is a small perturbation of (1.5).

In Section 2, we approximate (1.9) by

$$G_e(\Psi_d) = \int_{\frac{1}{\epsilon} \Omega} |\nabla \Psi_d|^2 + \frac{1}{2} (1 - |\Psi_d|^2)^2 + \beta |\Box \Psi_d|^2 + \gamma |\Delta \Psi_d|^2 \, dx \, dy,$$

(1.10)

where $0 < \epsilon \ll 1$ is a small parameter, $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$ having an interior point at the origin and $\frac{1}{\epsilon} \Omega = \{(x, y) : (x, y) \in \Omega\}$. In the rest of this paper, we prove that the minimizer of (1.10) has distinct degree-one vortices in Section 3. In Section 4, we replace $\frac{1}{\epsilon} \Omega$ in (1.10) by $B_{R_0}$ where $B_{R_0}$ is a disk with radius $R_0$ and center at the origin. Here $R_0 > 0$ is a large constant satisfying $1 \ll R_0 \leq 1/\epsilon$. Then (1.10) becomes

$$\tilde{G}(\Psi_d) = \int_{B_{R_0}} |\nabla \Psi_d|^2 + \frac{1}{2} (1 - |\Psi_d|^2)^2 + \beta |\Box \Psi_d|^2 + \gamma |\Delta \Psi_d|^2 \, dx \, dy,$$

(1.11)

where $\beta > 0$ is a small parameter as $T \to T_c$, $\gamma = C \beta$, and $C$ is a positive constant independent of $\beta$. We study then the critical point of (1.11) and find out its single vortex structure with fourfold symmetry. The single vortex structure of $d$-wave superconductors having fourfold symmetry is well known in physics (cf. [5], [6], [8], [27] and [31]). Here we give a mathematical proof of such a vortex structure.
2 Preliminaries

To investigate vortices in $d$-wave superconductors, we assume that the order parameter $\Psi_d$ satisfies $|\Psi_d| \to 1$ and all the derivatives of $\Psi_d$ decay fast as $|(x, y)| \to \infty$. Such an assumption is consistent with the results in [8] and [31]. Hence we may approximate (1.9) by

$$G_{\epsilon}(\Psi_d) = \int_\Omega |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta|\square \Psi_d|^2 + \gamma |\triangle \Psi_d|^2 \, dx \, dy,$$

(2.1)

where $0 < \epsilon \ll 1$ is a small parameter, $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$ having an interior point at the origin and $\frac{1}{\epsilon} \Omega = \{(x, y) : (x, y) \in \Omega\}$. Rescaling the spatial variables $x, y$ by $\epsilon$, (2.1) becomes

$$\hat{G}_{\epsilon}(\Psi_d) = \int_\Omega |\nabla \Psi_d|^2 + \frac{1}{2\epsilon^2}(1 - |\Psi_d|^2)^2 + \delta_{\epsilon}|\square \Psi_d|^2 + \gamma_{\epsilon} |\triangle \Psi_d|^2 \, dx \, dy,$$

(2.2)

where

$$\delta_{\epsilon} = \beta \epsilon^2 \quad \text{and} \quad \gamma_{\epsilon} = \gamma \epsilon^2.$$  

(2.3)

Of course, (2.3) implies that $0 < \delta_{\epsilon}, \gamma_{\epsilon} = O(\epsilon^2)$ as $\epsilon \to 0+$. In Section 2 and 3, we study (2.2) with an assumption that $0 < \delta_{\epsilon}, \gamma_{\epsilon} = O(\epsilon^2)$ as $\epsilon \to 0+$.

This kind of approximation can also be found in $s$-wave superconductors. The conventional $s$-wave Ginzburg-Landau free energy (cf. [9]) without the magnetic field is

$$\int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4}(1 - |u|^2)^2,$$

where $u \in \mathbb{C}$ is the $s$-wave order parameter. Under the hypothesis that $|u| \to 1$ and all the derivatives of $u$ decay fast as $|(x, y)| \to \infty$, we may approximate the $s$-wave Ginzburg-Landau free energy by

$$\int_{\frac{1}{\epsilon} \Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{4}(1 - |u|^2)^2,$$

where $0 < \epsilon \ll 1$ is a small parameter and $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$ having an interior point at the origin. Then we rescale the spatial variables by $\epsilon$ and obtain the energy functional as follows:

$$E_{\epsilon}(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 + \frac{1}{4\epsilon^2}(1 - |u|^2)^2,$$

(2.4)

where $u : \Omega \to \mathbb{C}$ is the $s$-wave order parameter. There are many investigations on the free energy (2.4). For the readers who are interested in these works, please refer to [2], [15], [17], [23] and [29] etc.

In [2] and [29], we learn the minimizer of $E_{\epsilon}$ over $H^1_s(\Omega)$ having $n$ degree-one vortices in $\Omega$, where

$$H^1_s(\Omega) = \{ u \in H^1(\Omega; \mathbb{C}) : u = g \quad \text{on} \quad \partial \Omega \},$$

and $g : \partial \Omega \to S^1$ is smooth with degree $n \geq 1$. Furthermore, the minimizer $u_{\epsilon}$ of (2.4) satisfies
(1) \( E_\epsilon(u_\epsilon) = n\pi \log \frac{1}{\epsilon} + W_\epsilon(a_1, \ldots, a_n) + o_\epsilon(1) \) as \( \epsilon \to 0^+ \),

(2) \( u_\epsilon \) converges to \( u_* \) (up to a subsequence) in \( C^2_{loc}(\Omega \setminus \{a_1, \ldots, a_n\}) \) as \( \epsilon \to 0^+ \),

(3) \( (a_1, \ldots, a_n) \in \Omega^n \) is a global minimizer of the renormalized energy \( W_\epsilon \) defined in [2],

where \( o_\epsilon(1) \) is a small quantity which tends to zero as \( \epsilon \to 0^+ \),

\[
u_\epsilon(z) = \prod_{j=1}^{n} \frac{z - a_j}{|z - a_j|} e^{ih(z)}, \quad \forall z \in \Omega,
\]

and \( h \) is a real-valued harmonic function. Since \( \mathbb{R}^2 \) is isomorphic to \( \mathbb{C} \), we may consider \( \Omega \subset \mathbb{R}^2 \cong \mathbb{C} \). Note that the domain \( \Omega \) is assumed star-shaped in [2]. However, Struwe [29] generalized results of [2] for all bounded smooth domains.

For the minimizer of (2.2), we prove:

**Theorem I.** Suppose \( 0 < \delta_\epsilon, \gamma_\epsilon = O(\epsilon^2) \) as \( \epsilon \to 0^+ \). Then there exists a minimizer \( u_\epsilon \) of (2.2) over \( H^1_g(\Omega) \) such that

(i) \( u_\epsilon \in H^2(\Omega) \) has \( n \) degree-one vortices in \( \Omega \),

(ii) \( \hat{G}_\epsilon(u_\epsilon) = 2n\pi \log \frac{1}{\epsilon} + O(1) \) as \( \epsilon \to 0^+ \),

(iii) \( u_\epsilon \) converges to \( u_* \) (up to a subsequence) strongly in \( L^2(\Omega) \) and weakly in \( H^1_{loc}(\Omega \setminus \{a_1, \ldots, a_n\}) \),

(iv) \( (a_1, \ldots, a_n) \in \Omega^n \) is a global minimizer of the renormalized energy \( W_\epsilon \) in [2].

**Remark.** We may consider the energy functional (2.2) with \( 0 < \delta_\epsilon, \gamma_\epsilon = O(\epsilon^2) \) as a small perturbation of (2.4). However, the perturbation terms are of higher order derivatives. Hence the Euler-Lagrange equation of (2.2) is a singular perturbation problem and the perturbation terms are of the 4-th order derivatives. Until now, there is no general theorem on such a singular perturbation problem.

### 3 Proof of Theorem I

To prove the existence of a minimizer, we define a comparison map as follows:

\[
u_\epsilon(z) = \prod_{j=1}^{n} U_0\left(\frac{z - b_j}{\epsilon}\right) e^{iH_\epsilon(z)},
\]

for \( z \in \Omega \subset \mathbb{C} \), where \( b_j \)'s are \( n \) distinct points in \( \Omega \) and \( H_\epsilon \) is a real-valued smooth function in \( \Omega \) such that

\[
u_\epsilon = g \quad \text{on } \partial \Omega, \quad \|H_\epsilon\|_{C^2(\Omega)} = O(1).
\]
Hereafter, $U_0$ is the symmetric vortex solution (cf. [4], [10], [11]) defined by

$$U_0(z) = f(R) e^{i\theta} \quad \text{for } z \in \mathbb{C},$$

where $R = |z|$ and $(R, \theta)$ is the polar coordinate in $\mathbb{C}$. Moreover, $f(R)$ satisfies

$$\left\{ \begin{array}{l}
  f'' + \frac{1}{R} f' - \frac{1}{R^2} f + (1 - f^2) f = 0 \quad \text{for } R > 0, \\
  f(0) = 0, f(\infty) = 1.
\end{array} \right. \quad (3.3)$$

From [4] and [11], the symmetric vortex solution $U_0$ satisfies

Lemma I.

(i) $f(R) = \alpha_0 R + \alpha_1 R^3 + O(R^5)$ as $R \to 0+$, where $\alpha_0 > 0, \alpha_1 \in \mathbb{R}$ are constants,

(ii) $f(R) = 1 - \frac{1}{2R^2} + O(R^{-4})$ as $R \to +\infty$,

(iii) $U_0 = f(R) e^{i\theta}$ is analytic in $\mathbb{C}$.

Hence it is easy to check that

$$\hat{G}_\epsilon(U_\epsilon) = 2\pi n \log \frac{1}{\epsilon} + O(1) \quad \text{as } \epsilon \to 0+. \quad (3.4)$$

Now, fix $0 < \epsilon \ll 1$. We claim that $\inf_{u \in H_0^1(\Omega)} \hat{G}_\epsilon(u)$ attains a minimizer $u_\epsilon \in H^2(\Omega)$. Let \{u_k\} be a minimizing sequence such that

$$\hat{G}_\epsilon(u_k) \to \inf_{u \in H_0^1(\Omega)} \hat{G}_\epsilon(u). \quad (3.5)$$

Then by (2.2), (3.4) and (3.5), we have

$$\lim_{k \to \infty} \inf \int_{\Omega} |\nabla u_k|^2 + |\Box u_k|^2 + |\Delta u_k|^2 dx dy < +\infty.$$  

Hence there exists a subsequence \{u_{kj}\} such that

$$\|u_{kj}\|_{H^2} \leq K_\epsilon, \quad \forall j \geq 1,$$

where $K_\epsilon > 0$ is a constant independent of $j$. Thus (3.6) implies

$$u_{kj} \to u_\epsilon \quad \text{weakly in } H^2(\Omega) \quad \text{as } j \to \infty. \quad (3.7)$$

Therefore by Fatou's lemma, $u_\epsilon$ is a minimizer of $\hat{G}_\epsilon$ over $H_0^1(\Omega)$.

From (2.2), (2.4), (3.4) and $u_\epsilon$ is a minimizer of $\inf_{u \in H_0^1(\Omega)} \hat{G}_\epsilon(u)$, we obtain

$$E_\epsilon(u_\epsilon) \leq \pi n \log \frac{1}{\epsilon} + O(1). \quad (3.8)$$
Moreover, by (3.8) and [29], we have
\[ E_\epsilon(u_\epsilon) = \pi n \log \frac{1}{\epsilon} + O(1). \] (3.9)

Hence (3.4) and (3.9) imply that
\[ \delta_\epsilon \int_\Omega |\square u_\epsilon|^2 \, dx \, dy = O(1), \] (3.10)
and
\[ \gamma_\epsilon \int_\Omega |\triangle u_\epsilon|^2 \, dx \, dy = O(1). \] (3.11)

Thus we complete the proof of (ii).

By (3.9), Proposition 1.1 and 1.2 in [16], we complete the proof of (i). Furthermore, we obtain that $u_\epsilon$ converges to $u_*$ (up to a subsequence) strongly in $L^2(\Omega)$ and weakly in $H^1_{loc}(\Omega \setminus \{a_1, \cdots, a_n\})$, where $a_1, \cdots, a_n \in \Omega$, $u_*(z) = \prod_{j=1}^{n} \frac{z - a_j}{|z - a_j|} e^{i h(z)}, \forall z \in \Omega \subset \mathbb{C}$ and $h$ is a real-valued function. Now we show that $h$ is a harmonic function as follows: Consider the Euler-Lagrange equation of $\hat{G}_\epsilon$ with respect to the minimizer $u_\epsilon$. Then $u_\epsilon$ satisfies
\[ \triangle u_\epsilon + \frac{1}{\epsilon^2} (1 - |u_\epsilon|^2) u_\epsilon - \delta_\epsilon \square^2 u_\epsilon - \gamma_\epsilon \triangle^2 u_\epsilon = 0 \quad \text{in} \ \Omega. \] (3.12)

Perform the wedge product with $u_\epsilon$ and (3.12). This is a standard trick to erase the cubic nonlinear term in (3.12) (cf. [26] and [29]). Then we have
\[ u_\epsilon \wedge \Delta u_\epsilon - \delta_\epsilon u_\epsilon \wedge \square^2 u_\epsilon - \gamma_\epsilon u_\epsilon \wedge \triangle^2 u_\epsilon = 0 \quad \text{in} \ \Omega. \] (3.13)

Let $p \in C_0^\infty(\Omega)$ be a test function. Multiply (3.13) by $p$ and integrate it on $\Omega$. Then using integration by parts, we obtain
\begin{align*}
- \int_{\Omega} (u_\epsilon \wedge \partial_x u_\epsilon) p_x + (u_\epsilon \wedge \partial_y u_\epsilon) p_y = \\
\delta_\epsilon \int_{\Omega} (u_\epsilon \wedge \square u_\epsilon) \square p + 2(\partial_x u_\epsilon \wedge \square u_\epsilon) p_x - 2(\partial_y u_\epsilon \wedge \square u_\epsilon) p_y \\
+ \gamma_\epsilon \int_{\Omega} (u_\epsilon \wedge \triangle u_\epsilon) \triangle p + 2(\partial_x u_\epsilon \wedge \triangle u_\epsilon) p_x + 2(\partial_y u_\epsilon \wedge \triangle u_\epsilon) p_y
\end{align*}
(3.14)

Here we have used the following formulas:
\begin{align*}
u \wedge \Delta u &= \partial_x (u \wedge \partial_x u) + \partial_y (u \wedge \partial_y u), \\
u \wedge \square^2 u &= \square (u \wedge \square u) - 2(\partial_x u \wedge \square u_x - u_y \wedge \square u_y), \\
u \wedge \Delta^2 u &= \Delta (u \wedge \Delta u) - 2(u_x \wedge \Delta u_x + u_y \wedge \Delta u_y).
\end{align*}

Hence by $0 < \gamma_\epsilon, \delta_\epsilon = O(\epsilon^2)$, (3.9)-(3.11), (3.14) and Holder inequality, the limit map $u_*$ satisfies
\[ u_* \wedge \Delta u_* = 0 \quad \text{in distribution sense}. \] (3.15)
Thus \( u \) is a canonical harmonic map i.e. \( h \) is a harmonic function. Therefore we complete the proof of (iii).

Now we prove (iv) as follows: Let \((\tilde{a}_1, \cdots, \tilde{a}_n) \in \Omega^n\) be a global minimizer of the renormalized energy \( W_g \). The definition of \( W_g \) can be found in [2]. Then we define another comparison map as follows:

\[
V_\epsilon(z) = \begin{cases} 
    u_\epsilon(z - \tilde{a}_j + a_j) & \text{if } z \in B_\epsilon(\tilde{a}_j), j = 1, \ldots, n, \\
    \tilde{U}_\epsilon(z) & \text{if } z \in \Omega_{\epsilon^\alpha} \equiv \Omega \setminus \bigcup_{j=1}^{n} B_\epsilon(\tilde{a}_j),
\end{cases}
\]

(3.16)

where \( 0 < \alpha < 1 \) is a constant and \( \tilde{U}_\epsilon \) is a minimizer of \( E_\epsilon \) over \( H^1_\mathcal{J}^\alpha(\Omega_{\epsilon^\alpha}) \). Here the boundary condition \( \tilde{g} \) is defined by

\[
\tilde{g} = \begin{cases} 
    g & \text{on } \partial \Omega, \\
    u_\epsilon(\cdot - \tilde{a}_j + a_j) & \text{on } \partial B_\epsilon(\tilde{a}_j), j = 1, \cdots, n.
\end{cases}
\]

(3.17)

Hence by (iii), [2] and [29], \( \tilde{U}_\epsilon \) satisfies

\[
\tilde{U}_\epsilon \rightarrow \prod_{j=1}^{n} \frac{z - \tilde{a}_j}{|z - \tilde{a}_j|} e^{\overline{h}(z)} \quad \text{in } C^2(\Omega_{\epsilon^\alpha}) \quad \text{as } \epsilon \rightarrow 0^+, \quad (3.18)
\]

where \( \overline{h} \) is a harmonic function. The convergence of (3.18) may be up to a subsequence. However, this does not affect the following argument. Thus by (3.18) and [2], it is easy to check that

\[
\hat{G}_\epsilon(V_\epsilon) = \sum_{j=1}^{n} \int_{B_\epsilon(a_j)} \hat{g}_\epsilon(u_\epsilon) + 2\pi n \alpha \log \frac{1}{\epsilon} + 2W_g(\tilde{a}_1, \cdots, \tilde{a}_n) + o_\epsilon(1),
\]

(3.19)

where \( \hat{g}_\epsilon(u) = |\nabla u|^2 + \frac{1}{2} (1 - |u|^2)^2 + \delta_\epsilon \nabla u^2 + \gamma_\epsilon \Delta u^2 \) is the energy density of \( \hat{G}_\epsilon \) and \( o_\epsilon(1) \) is a small quantity which tends to zero as \( \epsilon \rightarrow 0^+ \). On the other hand, by (iii) and [2], we have

\[
\int_{\hat{\Omega}_{\epsilon^\alpha}} \frac{1}{2} |\nabla u_\epsilon|^2 + \frac{1}{4\epsilon^2} (1 - |u_\epsilon|^2)^2 \geq \pi n \alpha \log \frac{1}{\epsilon} + W_g(a_1, \cdots, a_n) + o_\epsilon(1),
\]

(3.20)

where \( \hat{\Omega}_{\epsilon^\alpha} = \Omega \setminus \bigcup_{j=1}^{n} B_\epsilon(a_j) \). Hence (3.20) implies that

\[
\hat{G}_\epsilon(u_\epsilon) \geq \sum_{j=1}^{n} \int_{B_\epsilon(a_j)} \hat{g}_\epsilon(u_\epsilon) + 2\pi n \alpha \log \frac{1}{\epsilon} + 2W_g(a_1, \cdots, a_n) + o_\epsilon(1),
\]

(3.21)

Thus by (3.19) and (3.21), we obtain

\[
W_g(a_1, \cdots, a_n) \leq W_g(\tilde{a}_1, \cdots, \tilde{a}_n) + o_\epsilon(1)
\]

(3.22)

Since \((\tilde{a}_1, \cdots, \tilde{a}_n)\) is a global minimizer of \( W_g \), then we complete the proof of (iv) by (3.22).
4 Single Vortex Structure in the Vortex Core

In this section, we assume that the single vortex structure is in the vortex core $B_{R_0}$, where $R_0 > 0$ is a large constant satisfying $1 \ll R_0 \leq \frac{1}{\epsilon}$. Hereafter, we denote $B_{R_0}$ as a disk in $\mathbb{R}^2$ with radius $R_0$ and center at the origin. To study the vortex structure in the vortex core, we restrict (1.9) in the vortex core $B_{R_0}$ as follows:

$$\hat{G}(\Psi_d) = \int_{B_{R_0}} |\nabla \Psi_d|^2 + \frac{1}{2} (1 - |\Psi_d|^2)^2 + \beta |\Box \Psi_d|^2 + \gamma |\Delta \Psi_d|^2 \, dx \, dy,$$  \hspace{1cm} (4.1)

where $\gamma = C \beta, C > 0$ is a constant independent of $\beta$, and $\beta > 0$ is a small parameter tending to zero as $T \to T_c$. We investigate (4.1) with $\beta > 0$ a small parameter to see the phase transition of d-wave superconductors.

The Euler-Lagrange equation of (4.1) is

$$\Delta \Psi_d + (1 - |\Psi_d|^2) \Psi_d - \beta (\Box^2 + C \Delta^2) \Psi_d = 0 \quad \text{in} \quad B_{R_0}. \hspace{1cm} (4.2)$$

Note that $E \equiv \Box^2 + C \Delta^2$ is an elliptic operator as $C > 0$. Moreover, by the Lax-Milgram theorem, $E : H^2(B_{R_0}; \mathbb{C}) \to H^{-2}(B_{R_0}; \mathbb{C})$ is invertible and we denote $E^{-1}$ as its inverse. Hence the standard elliptic regularity theorem (cf. [20]) can be applied in (4.2).

We state the main result on (4.2) as follows:

Theorem II. There exists a solution $\Psi_d$ of (4.2) satisfying

$$\Psi_d(z, \beta) = f(R) e^{i\theta} + \beta (a(R) e^{-4i\theta} + b(R) e^{4i\theta} + c(R)) e^{i\theta} + O(\beta^2) \quad \text{as} \quad \beta \to 0, \hspace{1cm} (4.3)$$

where $a, b$ and $c$ are smooth real-valued functions.

The equation (4.3) implies that the $d$-wave order parameter $\Psi_d$ is fourfold symmetric in the vortex core. In [27], we learn a well approximated solution of (4.2) with fourfold symmetry. Here we find an exact solution of (4.2) with the fourfold symmetry.

Proof of Theorem II.

To solve (4.2), we set

$$\Psi_d(z, \beta) = U_0(z) + \beta w_1(z) + \beta^2 w_2(z) + \beta^3 w(z, \beta), \hspace{1cm} (4.4)$$

where $U_0$ is the symmetric vortex solution defined in (3.2) and (3.3). Here $w_1$ satisfies

$$L w_1 - E U_0 = 0 \quad \text{in} \quad B_{R_0}, \quad w_1 = 0 \quad \text{on} \quad \partial B_{R_0}, \hspace{1cm} (4.5)$$

where $Lv = \Delta v + (1 - |U_0|^2)v - 2(U_0 \cdot v)U_0$ is the linearized operator of the equation (4.2) with respect to a trivial solution $(\Psi_d, \beta) = (U_0, 0)$. In addition, $w_2$ satisfies that

$$L w_2 = 2(U_0 \cdot w_1) w_1 + |w_1|^2 U_0 + E w_1 \quad \text{in} \quad B_{R_0}, \quad w_2 = 0 \quad \text{on} \quad \partial B_{R_0}. \hspace{1cm} (4.6)$$
It is easy to check that
\[ EU_0 = h_{-3}(R) e^{-3i\theta} + h_1(R) e^{i\theta} + h_5(R) e^{5i\theta}, \]  
(4.7)
where \( h_{-3}, h_1 \) and \( h_5 \) are real-valued smooth functions. By [14], [18], [19] and [23], \( L \) is a bijection from \( H^1_0(B_{R_0}; \mathbb{C}) \) onto \( H^{-1}(B_{R_0}; \mathbb{C}) \). Hence by (4.5)-(4.7), we have
\[ w_1 = a(R) e^{-3i\theta} + b(R) e^{5i\theta} + c(R) e^{i\theta}, \]  
(4.8)
\[ w_2 = \sum_{k=0}^{2} a_{1-4k}(R) e^{i(1-4k)\theta} + a_{1+4k}(R) e^{i(1+4k)\theta}, \]  
(4.9)
where \( a, b, c \) and \( a_{1\pm 4k} \)'s are smooth real-valued functions.

Taking (4.4) into (4.2), we obtain that
\[ Lw = 2[(U_0 \cdot (w_2 + \beta w))w_1 + (U_0 \cdot w_1)(w_2 + \beta w)] + \beta w_2 + \beta w|w_0|^2U_0 + \beta(U_0 \cdot (w_2 + \beta w))(w_2 + \beta w) + 2(w_1 \cdot (w_2 + \beta w))U_0 + |w_1 + \beta(w_2 + \beta w)|^2(w_1 + \beta(w_2 + \beta w)) + Ew_2 + \beta Ew \quad \text{in} \quad B_{R_0}.
\]  
(4.10)
Hence (4.10) is equivalent to
\[ E^{-1}Lw = E^{-1}\{2[(U_0 \cdot (w_2 + \beta w))w_1 + (U_0 \cdot w_1)(w_2 + \beta w)] + \beta w_2 + \beta w|w_0|^2U_0 + \beta(U_0 \cdot (w_2 + \beta w))(w_2 + \beta w) + 2(w_1 \cdot (w_2 + \beta w))U_0 + |w_1 + \beta(w_2 + \beta w)|^2(w_1 + \beta(w_2 + \beta w))\} + w_2 + \beta w \quad \text{in} \quad B_{R_0}.
\]  
(4.11)
Note that (4.11) has a trivial solution \((w, \beta) = (w_3, 0)\), where \( w_3 \) satisfies that
\[ Lw_3 = 2[(U_0 \cdot w_2)w_1 + (U_0 \cdot w_1)w_2] + 2(w_1 \cdot w_2)U_0 + |w_1|^2 w_1 + E w_2 \quad \text{in} \quad B_{R_0},
\]  
(4.12)
\[ w_3 = 0 \quad \text{on} \quad \partial B_{R_0}.
\]
Since \( U_0, w_1, w_2 \) are smooth functions and \( L \) is bijective from \( H^1_0(B_{R_0}; \mathbb{C}) \) onto \( H^{-1}(B_{R_0}; \mathbb{C}) \), then the standard elliptic regularity theorem implies that \( w_3 \) is also a smooth function. Furthermore, since \( E \) is bijective from \( H^1_0(B_{R_0}; \mathbb{C}) \) onto \( H^{-2}(B_{R_0}; \mathbb{C}) \) and \( H^{-1}(B_{R_0}; \mathbb{C}) \) is embedded in \( H^{-2}(B_{R_0}; \mathbb{C}) \), then \( E \) is a bijection from \( H^1_0(B_{R_0}; \mathbb{C}) \cap H^3(B_{R_0}; \mathbb{C}) \) onto \( H^{-1}(B_{R_0}; \mathbb{C}) \). We denote \( E^{-1} \) as the inverse of \( E \). Hence \( E^{-1}L \) is a bijection from \( H^1_0(B_{R_0}; \mathbb{C}) \) onto \( H^2_0(B_{R_0}; \mathbb{C}) \cap H^3(B_{R_0}; \mathbb{C}) \). Thus by the implicit function theorem, (4.11) has a unique solution \( w \in H^1_0(B_{R_0}; \mathbb{C}) \) as \(|\beta|\) is sufficiently small. Moreover, the standard elliptic regularity theorem may imply the smoothness of \( w \). Therefore (4.2) has a solution \( \Psi_d \) satisfying (4.4) as \(|\beta|\) is sufficiently small. By (4.4), (4.8) and (4.9), we obtain (4.3) and complete the proof of Theorem II.

Final Remark: By (1.4) with \( A \equiv 0 \) and (4.3), we have
\[ \Psi_s(z) = \xi \Box [U_0 + \beta(a(R)e^{-4i\theta} + b(R)e^{4i\theta} + c(R))e^{i\theta} + O(\beta^2)] \quad \text{as} \quad \beta \to 0.
\]  
(4.13)
Since \( U_0(z) = f(R)e^{i\theta} \), then
\[ \Box U_0(z) = \frac{1}{2}(f' + \frac{1}{R}f)e^{-i\theta} + \frac{1}{2}[(f' - \frac{1}{R}f)' - \frac{2}{R}(f' - \frac{1}{R}f)]e^{3i\theta}.
\]  
(4.14)
Hence by (i), (ii) of Lemma I and (4.14), \( \Box U_0 \) satisfies
\[
\Box U_0(z) = 4\alpha_1 R e^{-i\theta} + O(R^3) \quad \text{as } R \to 0^+,
\]
and
\[
\Box U_0(z) = -\frac{1}{2R^2} e^{-i\theta} + \frac{3}{2R^2} e^{3i\theta} + O(R^{-4}) \quad \text{as } R \to +\infty.
\]
By (4.15) and (4.16), the degree of \( \Box U_0 \) is minus one in \( B_{r_1} \) and three in \( B_{R_1} \) as \( 0 < r_1 \ll 1 \) and \( R_1 \gg 1 \). Moreover, by [4] and [11], it is easy to check that
\[
\frac{d}{dz} \Box U_0(z) \neq 0 \quad \text{if } \Box U_0(z) = 0.
\]
Hence (iii) of Lemma I and (4.17) imply that \( \Box U_0 \) has only simple zeros in \( \mathbb{C} \). Thus \( \Box U_0 \) has a single zero with degree minus one at the origin and another four zeros with degree one away from the origin. Therefore as \( |\beta| \) is sufficiently small, \( \Psi_s \) has a single zero with degree minus one near the origin and another four zeros with degree one away from the origin. This indicates the four-lobe structure of \( \Psi_s \) in the vortex core. The numerical simulation can be found in [7], [8] and [31].

**Acknowledgement.**

The second author wishes to express his sincere thanks to B. Rosenstein for helpful discussions. He also sincerely thanks the referees for their suggestions.

**References**


