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Multiple Solutions for Singularly Perturbed Semilinear Elliptic Problems in Bounded and Unbounded Domains

Michinori ISHIWATA
石渡 通徳
Department of Applied Physics,
School of Science and Engineering,
Waseda University
3-4-1 Okubo, Shinjyuku-ku, Tokyo, 169-8555 Japan.

1 Main Theorem

We are concerned with the multiplicity of solutions for the following singularly perturbed semilinear elliptic equations:

$$(\mathrm{P})_\varepsilon \begin{cases} -\varepsilon^2\Delta u + a(x)u = u|u|^{p-2} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varepsilon > 0$, $p \in (2, 2^*)$ (2^* denotes the critical exponent of the Sobolev embedding $H^1(\Omega) \subset L^p(\Omega)$ given by $2^* = 2N/(N-2)$ if $N \geq 3$, $= +\infty$ if $N = 1, 2$) and $a \in C(\Omega)$ is a function with condition (C) specified below.

As for Ω, we assume that

- Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$), or
- $\Omega = \mathbb{R}^N$ ($N \geq 1$).

Without loss of generality, we also assume that $0 \in \operatorname{int}\Omega$.

When $\Omega = \mathbb{R}^N$, the boundary condition should be understood as

$$u(x) \to 0 \text{ as } |x| \to \infty.$$

In order to characterize the topological feature of $a(x)$, we introduce the following condition $(\mathrm{C})_{\delta_0,\delta}$ for some positive numbers δ_0, δ.

$(\mathrm{C})_{\delta_0,\delta}$: *There exist positive constants r, ρ which satisfy the following:*

(C1): $a(x) \geq 1 + \delta_0$ *in* $B(0,\rho)$,

(C2): $a(x) \geq 1 - \delta$ *in* Ω,

(C3): $\cdot$ $\sup_{x \in (\partial K)_r} a(x) \leq 1$, *if* Ω *is a bounded domain,*

$\cdot$ $\sup_{x \in (\partial K)_r} a(x) < 1$ *and* $\lim_{|x| \to \infty} a(x) = 1$, *if* $\Omega = \mathbb{R}^N$,

where K *is some closed subset of* Ω *which satisfies the following condition* (K):

(K): $B(0,\rho) \subset K$. ∂K *is homotopically equivalent to* S^{N-1} *and*

$$(\partial K)_r = \{x \in \mathbb{R}^N \mid \mathrm{dist}(x, \partial K) \leq r\} \subset \Omega.$$

Roughly speaking, conditions above imply that $a(x)$ has a "peak" in K (condition (C1)), the value of $a(x)$ on ∂K is uniformly less than the level of the "peak"(condition (C3)), and ∂K is the set which is surrounding the "peak" and is homotopically equivalent to S^{N-1} (condition (K)).

Putting $v(x) = u(\varepsilon x)$, we see that (the weak form of) $(\mathrm{P})_\varepsilon$ is equivalent to

$$(\mathrm{P}')_\varepsilon \qquad -\Delta v + a(\varepsilon x) v = v|v|^{p-2} \text{ in } \Omega,\ v \geq 0,\ v \in H_0^1(\Omega/\varepsilon),$$

and the solution of $(\mathrm{P}')_\varepsilon$ corresponds to the critical point of the functional $I_\varepsilon(v) = \int_{\Omega/\varepsilon} |(\nabla v|^2 + a(\varepsilon x)|v|^2)$ in $M_p(\Omega/\varepsilon) = \{v \in H_0^1(\Omega/\varepsilon);\ \|v\|_{L^p(\Omega/\varepsilon)} = 1\}$. Hence it is enough to find critical points of I_ε to prove the existence and multiplicity of solutions of $(\mathrm{P})_\varepsilon$.

It is well-known that for small ε, there is a relation between the multiplicity of critical point of I_ε in M_p and the topological feature of $a(x)$. For example, we refer [1, 2, 3, 4]. Our main theorem reads as follows.

Main Theorem.

For all $\delta_0 > 0$, *there exists* $\delta > 0$ *such that if* $a \in C(\Omega)$ *and* $(\mathrm{C})_{\delta_0,\delta}$ *is satisfied, then there exists* $\bar{\varepsilon}$ *such that for any* $\varepsilon \in (0, \bar{\varepsilon})$, $(\mathrm{P})_\varepsilon$ *admits at least* 2 *solutions. Moreover, if* Ω *is a bounded domain, there exists another solution for* ε *above.*

From now on, we shall only deal with the case of bounded domain in the main theorem. The argument for the case of unbounded Ω is almost similar as below.

One can easily prove that if $a(x) \equiv 1$ (in general $a(x) \equiv$ Const.) then $(\mathrm{P})_\varepsilon$ admits at least one solution u_0 (ground state solution) for all value $\varepsilon \in (0, \infty)$ with the aid of Mountain Pass Theorem. In general, one cannot expect the existence of multiple solutions. Indeed, when $a(x) \equiv 1$ and $\Omega =$ ball, the uniqueness result for sufficiently small ε is known (Dancer [1]). The main theorem above says that as soon as one perturb $a(x)$ to have a "peak", one gets the another (high energy) solutions u_1, u_2 even if the perturbation is very small. This "generation of higher energy solution" is a consequence of the change of topology of some level set of the functional associated to $(\mathrm{P})_\varepsilon$ caused by the nontrivial shape of $a(x)$. It is the purpose of this paper to discuss this change of topology.

2 Known Results and Motivations

The interest in studying $(\mathrm{P})_\varepsilon$ arises from several physical and mathematical contexts.

2.1 Physical Contexts

In the physical context, $(\mathrm{P})_\varepsilon$ can be regarded as a (reduced) nonlinear Schrödinger equation and small parameter ε corresponds to the Dirac constant $\hbar$.

It is well known that when $\hbar$ can be well approximated by 0 (this approximation is called "semiclassical approximation"), quantum mechanical equation may have a solution corresponding to a "semiclassical" state, concentrating around a classical mechanical equilibrium. It is also well known that classical equilibrium is often the point which minimize the potential energy. So it is reasonable to expect that for small ε, $(\mathrm{P})_\varepsilon$ has a "semiclassical" solution concentrating around a point which attains the minimum of the energy potential $a(x)$. Hence the structure of $a_{min} = \{y \in \Omega \mid a(y) = \min_\Omega a(x)\}$, the minimum set of $a(x)$, may play a significant role for the existence, multiplicity of solutions of $(\mathrm{P})_\varepsilon$.

From this point of view, del-Pino and Felmer [2] obtain the following result.

Proposition 2.1 (**Effect of weight function**, del Pino-Felmer [2])
Suppose that Λ is a bounded set compactly contained in Ω and $\min_{\partial\Lambda} a(x) > \inf_\Lambda a(x)$. Then for sufficiently small ε, $(P)_\varepsilon$ admits a solution u_ε, which concentrates to a point in Λ which attains the minimum of $a(x)$ as $\varepsilon \to 0$.

Proposition 2.1 implies that there exist at least as many solutions of $(\mathrm{P})_\varepsilon$ as the number of connected components of a_{min} if ε is small enough.

In our situation, $(\mathrm{C})_{\delta_0,\delta}$, a_{min} may have *only one connected component*, so in this case Proposition 2.1 provides *only one solution*. Our main theorem says that not only the number of connected component of a_{min} but also some topological feature of a_{min} (i.e. the fact that ∂K is homotopically equivalent to S^{N-1}) plays some role on the multiplicity of the solutions of $(\mathrm{P})_\varepsilon$.

2.2 Mathematical Contexts

In the mathematical context, $(\mathrm{P})_\varepsilon$ can be regarded as an example verifying the following feature. In many semilinear elliptic problem including small parameters (e.g.

semilinear elliptic equations with critical or nearly critical exponent [6, 7], stationary Cahn-Hilliard equation [8], Ginzburg-Landau equation [9]), it is commonly observed that if the parameter is small enough, then the existence and multiplicity of solutions are controlled by the finite dimensional object. As for singularly perturbed equations in bounded domains, the following result holds.

Proposition 2.2 (**Effect of topology of the domain**, Benci-Cerami [5])
Assume that $a(x) \equiv$ *Const. Assume also that* $\Omega \subset \mathbb{R}^N$ *is bounded and* Ω *is topologically nontrivial in the sense of category, i.e.,* $\mathrm{cat}\,\Omega > 1$. *Then for small* ε, $(P)_\varepsilon$ *admits at least* $\mathrm{cat}\,\Omega + 1$ *solutions.*

In this case, the finite dimensional object referred above is Ω. The following questions naturally arise:

1. Can one replace the "nontriviality of the topology of the domain" by the "nontriviality of the shape of the weight function $a(x)$"?

2. What is the finite dimensional object which control the existence and multiplicity of solutions when $a(x) \not\equiv$ Const.?

Our main theorem gives an affirmative (partial) answer for the first question and suggests that the finite dimensional object asked in question 2 is not Ω as in Proposition 2.2 but S^{N-1}.

3 Variational and Topological Tools

Our main tool relies on the variational approach, which is based on the following fundamental principle.

Proposition 3.1 (Fundamental principle in Morse theory)
Suppose that M *is a Banach-Finsler manifold and* $I \in C^1(M)$ *satisfies the following assumption:*
1. I *satisfies* $(PS)_c$*-condition for all* $c \in [a, b]$.
2. $[I \leq a]$ *and* $[I \leq b]$ *have a "difference in topology".*
Then there exists a critical value $\bar{c} \in [a, b]$
(Here we mean $\{u \in M \mid I(u) \leq a\}$ *by* $[I \leq a]$.*)*

In order to compare the topology of level sets of I, various kinds of topological invariants are known. We shall here use the notion of the "category" of sets, defined by:

Definition 3.2 (Notion of "category")

Assume X be a topological space and Ω, ω are two closed subsets of X with $\omega \subset \Omega$. Then $n = \mathrm{cat}_\Omega[\omega]$ if and only if n is a smallest number among m such that $(\omega_j)_{j=1}^m$ is a closed contractible covering of ω in Ω,

i.e.,

$$\omega = \cup_{j=1}^m \omega_j,\ \exists h_j \in C([0,1]\times\omega_j; X),\ \exists \bar{x}_j \in X \text{ s.t.}$$
$$h_j(0,x) = x\ \forall x \in \omega_j,\ h_j(1,x) = \bar{x}_j\ \forall x \in \omega_j.$$

We simply denote $\mathrm{cat}_\Omega[\Omega]$ by cat Ω.

In terms of this notion, Lysternik-Schnirelman theorem (category version) reads as follows:

Proposition 3.3 (Lysternik-Schnirelman theorem, category version [10])
Suppose that M is a Banach-Finsler manifold, $I \in C^1(M)$, and $a = \inf_M I > -\infty$. Suppose also that for some $b' > b > a$, I satisfies $(PS)_c$ for all $c \in [a, b']$ and $K \cap [I = b] = \phi$ where $K = \{u \in M \mid (dI)_u = 0\}$.
Then $[I \leq b]$ contains at least $\mathrm{cat}[I \leq b]$ critical points.

4 Sketch of Proof of Main Theorem

4.1 Variational Setting and Notations

We introduce the following notations: for $\omega \subset \mathbb{R}^N$,

- $M_p(\omega) = \{u \in H_0^1(\omega) \mid \|u^+\|_{L^p(\omega)} = 1\}$,
- $I_{\varepsilon,\alpha,\omega}(u) = \displaystyle\int_\omega \left(\varepsilon^2|\nabla u|^2 + \alpha u^2\right) dx$,
- $S_p(\varepsilon,\alpha,\omega) = \inf_{u\in M_p(\omega)} I_{\varepsilon,\alpha,\omega}(u)$.

As for $S_p(\varepsilon,\alpha,\omega)$, it is well known that the following result holds:

Proposition 4.1 (Existence [11] and uniqueness [12] for ground state in $\mathbb{R}^N$)
For any $\varepsilon, \alpha \in \mathbb{R}^+$, there exists a unique minimizer (up to translation) for $S_p(\varepsilon,\alpha,\mathbb{R}^N)$ which is positive and radially symmetric with respect to the origin.

In order to discuss the relation between the level set of I (in function space) and $\partial K/\varepsilon$ (in $\mathbb{R}^N$), we define the "truncated barycenter" $\beta_R(u)$.

Let $\eta \in C_0^\infty(\mathbb{R})$ be a cut off function such that $\eta(t) = 1$ iff $|t| < R$, $\eta(t) = R/t$ iff $|t| \geq R$. Set $\beta_R(u) = \displaystyle\int_{\mathbb{R}^N} x\eta(|x|)|u|^p dx$ for $\forall u \in M_p(\mathbb{R}^N)$.

Then it is obvious that for $\forall u \in M_p(\mathbb{R}^N)$, $|\beta_R(u)| \leq R$ holds. Moreover, if the (intuitive) barycenter of $u \in M_p$ is near "infinity", then $\beta_R(u)$ is located near $\partial B_R = \{x \in \mathbb{R}^N \mid |x| = R\}$. Namely,

Proposition 4.2 (The range of truncated barycenter)
Suppose $\bar{u} \in M_p(\mathbb{R}^N)$ *and* $(y_n) \subset \mathbb{R}^N$ *satisfies* $|y_n| \to \infty$ *as* $n \to \infty$.
Then $|\beta_R(\bar{u}(\cdot - y_n))| \to R$ *as* $n \to \infty$.

Setting $v(x) = \varepsilon^{N/p}u(\varepsilon x)$, problem $(\mathrm{P})_\varepsilon$ can be rewritten as

$$(\mathrm{P}')_\varepsilon \qquad -\Delta v + a(\varepsilon x)v = |v|^{p-2}v, \quad v \geq 0, \quad v \in H_0^1(\Omega/\varepsilon).$$

It is well known that to solve $(\mathrm{P}')_\varepsilon$ is equivalent to:

(V) Find critical points of $I_{1,a_\varepsilon,\Omega/\varepsilon}$ on $M_p(\Omega/\varepsilon)$.

thanks to the Lagrange multiplier rule (in (V) we denote $a(\varepsilon x)$ as $a_\varepsilon(x)$.)
So hereafter we carry out the program (V).
Since it is well known that $I_{1,a_\varepsilon,\Omega/\varepsilon}$ satisfies $(\mathrm{PS})_c$ for all c, in order to prove main theorem it is enough to verify that for some $b > a = S_p(1, a_\varepsilon, \Omega)$,

- cat $[I_{1,a_\varepsilon,\Omega/\varepsilon} \leq b] \geq$ cat $S^{N-1} = 2$ and
- there exists another critical value c greater than b

by virtue of Proposition 3.3.
In order to introduce the "limiting functional" $I_{1,b(x),\mathbb{R}^N}$ associated to $I_{1,a_\varepsilon,\Omega/\varepsilon}$, we define $b(x) \in C(\mathbb{R}^N)$ as follows:

$$(\mathrm{B})_{\delta_0,\rho}: \quad 1 \leq b(x) \leq 1 + \delta_0 \text{ in } \mathbb{R}^N,\ b(x) = 1 + \delta_0 \text{ in } B(0, \rho/2),$$
$$b(x) = 1 \text{ in } B(0, \rho)^c.$$

Note that $a_\varepsilon(x) = a(\varepsilon x) \geq b(x) - \delta$ for $\forall\varepsilon \in (0, 1)$ and $\forall x \in \Omega/\varepsilon$.
The nontriviality of the topology of the level set $[I_{1,a_\varepsilon,\Omega/\varepsilon} \leq b]$ is a consequence of the nontriviality of the level set of $I_\infty = I_{1,b(x),\mathbb{R}^N}$, the limiting functional.
We next investigate the level set of I_∞.

4.2 Limiting Problem

Hereafter we fix positive constants δ_0, ρ. Let $b(x)$ be a function defined by $(\mathrm{B})_{\delta_0,\rho}$ in the last section.
In view of Proposition 4.1 and the fact that $S_p(1, 1, \mathbb{R}^N) = S_p(1, b(x), \mathbb{R}^N)$ we can verify the following:

Proposition 4.3 (Inf is not achieved in the limiting problem.)
$S_p(1, b(x), \mathbb{R}^N) = \inf_{u \in M_p(\mathbb{R}^N)} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + b(x)|u|^2\right) dx$ *is not achieved.*

This result imply that all minimizing sequence possesses no convergent subsequence.

Combining this fact with the compactness of embedding $H^1 \subset L^p_{loc}$, we get

Proposition 4.4 (Behavior of minimizing sequences of the limiting problem [13])

For any minimizing sequence $(v_n) \subset M_p(\mathbb{R}^N)$ of I_∞, $\exists (y_n) \subset \mathbb{R}^N$ s.t. $v_n(\cdot) = \bar{v}(\cdot - y_n) + o(1)$ in $H^1(\mathbb{R}^N)$ where $\bar{v}(x) = \bar{v}_{1,1,\mathbb{R}^N}(x)$ is a (unique) minimizer of $S_p(1,1,\mathbb{R}^N)$ (see Proposition 4.1).

That is, for any $v \in M_p(\mathbb{R}^N)$ such that $I_\infty(v) = I_{1,b(x),\mathbb{R}^N}(v)$ is very close to $S_p(1, b(x), \mathbb{R}^N)$, v is almost concentrated at "infinity". So by Proposition 4.2, $|\beta_R(v)| \simeq R$. Namely,

Proposition 4.5 (Concentration lemma at infinity for the limiting functional)

For all $r \in (0, R)$, there exists α such that for all $v \in M_p(\mathbb{R}^N)$,

$$I_\infty(v) \le S_p(1, b(x), \mathbb{R}^N) + \alpha \Longrightarrow \beta_R(v) \notin B(0,r).$$

This proposition says that $[I_\infty \le S_p + \alpha]$ $(\subset M_p(\mathbb{R}^N))$, infinite dimensional object, can be compared with the $B(0,r)^c (\subset \mathbb{R}^N)$, finite dimensional object, via $\beta_R(v)$.

4.3 Original Problem

Now we turn to the original problem. We regard the original functional $I_\varepsilon(v) = \displaystyle\int_{\Omega/\varepsilon} \left(|\nabla v|^2 + a(\varepsilon x)|v|^2\right) dx$ as a perturbed functional of I_∞. We first note that under $(A)_{\delta_0,\delta,\rho}$, the relation between the level set of functional and $B(0,r)^c$ described in Proposition 4.5 still holds for the perturbed functional I_ε.

Proposition 4.6 (Concentration lemma at infinity for original functional)

For all $r \in (0, R)$, there exist δ, η such that for any a satisfying $(A)_{\delta_0,\delta,\rho}$ the following holds:

there exists $\bar{\varepsilon}$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$ and for all $v \in M_p(\Omega/\varepsilon)$,

$$I_\varepsilon(v) \le S_p(1, b(x), \Omega/\varepsilon) + \eta \Longrightarrow \beta_R(v) \notin B(0,r).$$

So we can construct the mapping $\beta_R : [I_\varepsilon \le S_p + \eta] \to B(0,r)^c$. Next we construct the mapping $\Phi_\varepsilon : \partial K/\varepsilon \to [I_\varepsilon \le S_p + \eta]$.

For any $y_\varepsilon \in \partial K/\varepsilon$ set $v_{\varepsilon,y_\varepsilon}(x) = \varphi(\varepsilon(x - y_\varepsilon)/r)\bar{v}_{1,1,\mathbb{R}^N}(x - y_\varepsilon)$ where $\varphi \in C_0^\infty(\mathbb{R}^n)$ is a cut off function such that φ is radially symmetric with respect to the origin, $\varphi(x) = 1$ iff $|x| < 1/2$, $0 \le \varphi(x) \le 1$ iff $1/2 \le |x| < 1$, $\varphi(x) = 0$ iff $|x| \ge 1$. Let us define $\Phi_\varepsilon(y_\varepsilon) = v_{\varepsilon,y_\varepsilon}$, $\forall y_\varepsilon \in \partial K/\varepsilon$. Then the following holds.

Proposition 4.7 (Construction of an embedding mapping from $\mathbb{R}^N$ to the function space)
$I_\varepsilon \circ \Phi_\varepsilon(y_\varepsilon) \to c \leq S_p(1, \sup_{(\partial K)_r} a(x), \mathbb{R}^N) \leq S_p(1, b(x), \mathbb{R}^N)$ *as* $\varepsilon \to 0$ *uniformly in* $y \in \partial K$ *where* $y_\varepsilon = y/\varepsilon$.

It is well known that I_ε satisfies (PS). For η in Proposition 4.6, we can choose ε, b so that $I_\varepsilon \circ \Phi_\varepsilon(y_\varepsilon) \leq b < b' = S_p(1, b(x), \mathbb{R}^N) + 3\eta/4 \ \forall y_\varepsilon \in \partial K/\varepsilon$ by virtue of Proposition 4.7. It is also obvious that without loss of generality we can choose b so as not to be a critical value of I_ε, since otherwise we already get infinitely many critical values. These facts combined with Proposition 3.3 imply that

$$\text{the number of critical points in } [I_\varepsilon \leq b] \geq \text{cat}[I_\varepsilon \leq b].$$

So in order to prove main theorem, we have to estimate cat$[I_\varepsilon \leq b]$ from below. We can carry out this with the aid of comparison theorem of category (Proposition 4.8) and the fact

$$\beta_R : [I_\varepsilon \leq b] \to B(0,r)^c \text{ and } \Phi_\varepsilon : \partial K/\varepsilon \to [I_\varepsilon \leq b]$$

in view of Proposition 4.6 and 4.7.

4.4 Topological argument

The following comparison theorem can be proved by a standard argument:

Proposition 4.8 (Comparison theorem for category)
Suppose A, B*: topological spaces,* $a \subset A$, $b \subset B$*: closed subsets. Suppose also* $\exists \Phi : a \to b$, $\exists \beta : B \to A$*, continuous mappings such that* $\beta \circ \Phi$ *is homotopically equivalent to injection* $a \to A$*. Then* $cat_B[b] \geq cat_A[a]$.

In our case, with some technical argument, we can verify $\beta_R \circ \Phi_\varepsilon$ is homotopically equivalent to injection from $\partial K/\varepsilon$ to $B(0,r)^c$ for small ε.

Applying Proposition 4.8 with $A = B(0,r)^c$, $a = \partial K/\varepsilon$, $B = b = [I_\varepsilon \leq b]$, we find that cat $[I_\varepsilon \leq b] \geq \text{cat}_{B(0,r)^c}[\partial K/\varepsilon] = \text{cat } S^{N-1} = 2$. Thus we have established the existence of at least two critical points of I_ε with the level below b'.

To get another critical value of I_ε greater than b', we follow the following standard argument.

It is easy to find $v \in M_p(\Omega/\varepsilon)$ such that

$$tv + (1-t)\Phi_\varepsilon(y_\varepsilon) \neq 0 \quad \forall t \in [0,1], \ \forall y_\varepsilon \in \partial K/\varepsilon.$$

Let us define $\eta(t, y_\varepsilon) \in C([0,1] \times \partial K/\varepsilon; M_p(\Omega/\varepsilon))$ by

$$\eta(t, y_\varepsilon) = (tv + (1-t)\Phi_\varepsilon(y_\varepsilon))/\|tv + (1-t)\Phi_\varepsilon(y_\varepsilon)\|_p.$$

Then it is easy to see that $b' < c \equiv \max_{t\in[0,1],y_\varepsilon\in\partial K/\varepsilon} I_\varepsilon \circ \eta(t, y_\varepsilon)$. We shall show that there exists at least one critical value in $[b', c]$.

Suppose on the contrary there is no critical value in $[b', c]$. Then by the well known deformation lemma, there exists $f \in C(M_p; M_p)$ such that

$$f = \text{identity on } [I_\varepsilon \leq b] \text{ and } f([I_\varepsilon \leq c]) \subset [I_\varepsilon \leq b'].$$

Then $\partial K/\varepsilon$ is contractible in $B(0,r)^c$ by the contraction $g(t, y_\varepsilon) \in C([0,1] \times \partial K/\varepsilon; B(0,r)^c)$ defined by $g(t, y_\varepsilon) = \beta_R \circ f \circ \eta(t, y_\varepsilon)$. Since $\partial K/\varepsilon$ and $B(0,r)^c$ is both homotopically equivalent to S^{N-1}, we have the contradiction.

In summary, we find that there exists at least

- two critical points in $[I_\varepsilon \leq b']$ and
- one critical value greater than b'.

Thus the main theorem is proved.

Remark. 1. The same type of multiplicity result also holds for $(\mathrm{P})_\varepsilon$ for the case where $\Omega = \mathbb{R}^N$ and $a(x)$ has a peak, or Ω is an exterior domain with bounded complement and $a(x)$ has a "creek" around the "hole" of Ω. These results together with detailed argument of the proof of facts described above will be the subject of the forthcoming paper [14, 15].

2. Another type of multiplicity result for $-\Delta u + u = a(x)u + f(x)$ in $\mathbb{R}^N$ is discussed in Adachi-K. Tanaka [16].

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