On the projection which appears
in the variational treatment of
elasto-plastic torsion problem

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Abstract

In the treatment of variational inequalities, the projection operator $P_K$ from some Hilbert space $V$ onto a certain closed convex subset $K$ plays an important role. But, only for few problems, it is known how to get the explicit form of $P_Ku$ for each given $u \in V$. In this article, we consider $K = \{ f \in H^1_0(\Omega); |\nabla f| \leq 1 \text{ a.e.} \}$, which is related to elasto-plastic torsion problems, and propose an iterative method to approximate $P_Ku$ for 1 dimensional case $\Omega = (a, b)$. We also show an expansion of it for higher dimensional but radial symmetric cases.

1 Problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary and

$$K := \{ f \in H^1_0(\Omega); |\nabla f| \leq 1 \text{ a.e.} \}.$$ 

We will denote by $P_K$ the projection mapping from $H^1_0(\Omega)$ into its convex closed subset $K$, namely, for $u \in H^1_0(\Omega)$ and $v \in K$,

$$P_Ku = v \quad \text{def}, \quad \|u-v\|_{H^1_0(\Omega)} = \inf_{f \in K} \|u-f\|_{H^1_0(\Omega)}.$$ 

For convenience sake, we take

$$\|u\|_{H^1_0(\Omega)} := \|\nabla u\|_{L^2(\Omega)} = \left\{ \int_{\Omega} |\nabla u(x)|^2 \, dx \right\}^{1/2},$$

throughout this article. (Note that $\Omega$ is bounded.) The problem is to find $v = P_Ku \in K$ for each given $u \in H^1_0(\Omega)$.

This projection $P_K$ appears in the variational treatment of elasto-plastic torsion problem. Consider an infinitely long cylindrical elastic-plastic bar of
cross section $\Omega$ to which some torsion momentum ($\tau$ denotes the torsion angle per unit length) is applied (Fig. 1). It is known that the stress vector $\sigma$ in $\Omega$ is determined by the minimizer $u$ of

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \tau \int_\Omega u dx \quad (v \in K),$$

namely, $\sigma = \nabla u$ [2, p.42]. This minimizing problem is equivalent to finding $u \in K$ such that

$$u = P_K(u - \rho(Au - l))$$

for some $\rho > 0$, where $A \in \mathcal{L}(V, V)$ and $l \in V$ are defined by

$$(Af, g) = \frac{1}{2} \int_\Omega \nabla f \cdot \nabla g dx,$$

$$(l, f) = \tau \int_\Omega f dx \quad (\langle \cdot, \cdot \rangle : \text{inner product of } V)$$

for $f, g \in V := H_0^1(\Omega)$, respectively [2, p.3].

The projection $P_K$ also plays an important role in the error estimates of the corresponding penalized elliptic variational inequalities [5].

2 Rewriting the problem

We introduce a functional $J_u : K \to \mathbb{R}$ for each given $u \in H_0^1(\Omega)$:

$$J_u(f) := \|u - f\|_{H_0^1(\Omega)}^2 = \int_\Omega |\nabla u(x) - \nabla f(x)|^2 dx. \quad (1)$$

By using it, the problem can be rewritten such as "To find the minimizer $v$ of $J_u$ on $K"." On this problem, one can easily show:
Proposition 1  If there exists a solution $v \in H^1_0(\Omega)$ to
\[ \nabla v = C(\nabla u) \quad (a.e. \ in \ \Omega), \] (2)
then $v$ is the minimizer of $J_u$ on $K$, where $C(z) := \{ z \ (|z| \leq 1), \ z/|z| \ (|z| > 1) \}$. 

Especially, for 1 dimensional case $\Omega = (a, b) \subset \mathbb{R}$, put
\[ v(x) := \int_a^x C(u'(\xi)) \, d\xi \quad (a \leq x \leq b) \] (3)
for a given function $u \in H^1_0(a, b)$. If this function $v \in H^1(a, b) \cap C([a, b])$ satisfies that $v(b) = 0$, then $v$ belongs to $H^1_0(a, b)$ and hence $v = P_Ku$. An example of this kind: $u(x) = -\frac{3}{10} \cos\left(\frac{3}{2}\pi x\right)$ and $v$ defined by (3) for $\Omega = (-1, 1)$ are shown in Fig. 2. We also plot their derivatives in Fig. 3. In this case, $P_Ku$ and $v$ coincide perfectly (see Fig. 2), and $(P_Ku)'$ is only the "cut-off" of $u'$, namely, $(P_Ku)' = C(u')$ (see Fig. 3).

![Figure 2: the case $v(b) = 0$; $u(x) = -\frac{3}{10} \cos\left(\frac{3}{2}\pi x\right)$](image)

In fact, for 1 dimensional case $\Omega = (a, b)$, one can easily show that if the given function $u$ is symmetric (i.e., $u(a+\xi) = u(b-\xi)$ for any $\xi$), then $v$ defined by (3) satisfies that $v(b) = 0$ and hence $v = P_Ku$. 
But it is rather special. We will show an example for the case $v(b) \neq 0$:

$$u(x) = 4(x + 1)^2(x + \frac{1}{2})(x - \frac{1}{5})(x - \frac{3}{5})(x - \frac{4}{5})(x - 1)$$

for $\Omega = (-1, 1)$. The graphs of $u$, corresponding $v$ and $P_Ku$ are shown in Fig. 4. Also the derivatives $u'$ and $(P_Ku)'$ are plotted in Fig. 5.

Figure 3: $u'$ and $(P_Ku)'$; $u(x) = -\frac{3}{10} \cos \frac{3}{2} \pi x$.

Figure 4: the case $v(b) \neq 0$; $u(x) = 4(x + 1)^2(x + \frac{1}{2})(x - \frac{1}{5})(x - \frac{3}{5})(x - \frac{4}{5})(x - 1)$.
In such a case, it is clear that any primitive function of \( C(u') \) can not belong to \( H_0^1(\Omega) \) since its values at 2 boundary points are not equal. In other words, (2) has no solution in \( H_0^1(\Omega) \), in general.

Then, instead of (2), we consider the following system of equations:

\[
\begin{cases}
\nabla v = C(\nabla u - \nabla w) & \text{(a.e. in } \Omega), \\
\Delta w = 0 & \text{(weak sense).}
\end{cases}
\]

It means that at first, we alter \( u \) by subtracting the appropriate quantity, namely, a function \( w \in H^1(\Omega) \) satisfying \( \Delta w = 0 \). Then we "cut-off" its gradient and get the primitive function. If the obtained function \( v \) belongs to \( H_0^1(\Omega) \), then the next theorem assures that \( v = P_K u \).

**Theorem 1** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with smooth boundary. If there exists a solution \((v, w)\) in \( H_0^1(\Omega) \times H^1(\Omega) \) to the system of equations (4) with a given parameter \( u \in H_0^1(\Omega) \), then \( v \) belongs to \( K \) and minimizes the functional \( J_u \) defined by (1).

**(Proof)** It is clear that \( v \in K \). Hence, it suffices to show that

\[
\forall f \in K, \quad J_u(f) - J_u(v) \geq 0.
\]

Let denote \( \Omega_p := \{x \in \Omega; |\nabla(u - w)| > 1\} \) and \( \Omega_z := \Omega \backslash \Omega_p \). Fix \( f \in K \) and put \( \delta := f - v \in H_0^1(\Omega) \). For this \( \delta \), we can easily show

\[
\nabla\delta \cdot \nabla v = \nabla f \cdot \nabla v - |\nabla v|^2 = \nabla f \cdot \nabla v - 1 \leq 0 \quad \text{(a.e. in } \Omega_p)
\]
since $|\nabla f| \leq 1$ and $|\nabla v| = 1$ (a.e. in $\Omega_p$), and hence
\[
\nabla \delta \cdot (\nabla u - \nabla w) \leq 0 \quad \text{(a.e. in } \Omega_p). \]

On the other hand, since $\triangle w = 0$ (weak sense in $H^1(\Omega)$) and $\delta \in H^1_0(\Omega)$,
\[
\int_{\Omega} \nabla \delta \cdot \nabla w ~dX = \int_{\Omega_p} \nabla \delta \cdot \nabla w ~dX + \int_{\Omega_z} \nabla \delta \cdot \nabla w ~dX = 0.
\]

By using these facts, we get
\[
J_u(f) - J_u(v) = \int_{\Omega} |\nabla u - \nabla (v + \delta)|^2 ~dx - \int_{\Omega} |\nabla u - \nabla v|^2 ~dx
= \int_{\Omega} |\nabla \delta|^2 ~dx - 2 \int_{\Omega} \nabla \delta \cdot (\nabla u - \nabla v) ~dx
= \int_{\Omega} |\nabla \delta|^2 ~dx - 2 \int_{\Omega_p} \nabla \delta \cdot \left( \frac{\nabla u - \nabla v}{|\nabla u - \nabla v|} \right) ~dx - 2 \int_{\Omega_z} \nabla \delta \cdot \nabla w ~dX
= \int_{\Omega} |\nabla \delta|^2 ~dx + 2 \int_{\Omega_p} \nabla \delta \cdot \left( \frac{\nabla u - \nabla w}{|\nabla u - \nabla w|} - \nabla u \right) ~dx + 2 \int_{\Omega_p} \nabla \delta \cdot \nabla w ~dX
\geq \int_{\Omega} |\nabla \delta|^2 ~dx \geq 0.
\]

3 1 dimensional case

Theorem 1 assures that if one could solve the system of equations (4) with a given parameter $u \in H^1_0(\Omega)$, one get the projection $P_K u$. But unfortunately, there may not be any solution to (4) in general, except 1 dimensional case. In fact, when $\Omega = (a, b) \subset \mathbb{R}^1 (-\infty < a < b < \infty)$, the equation $w'' = 0$ can be solved such as $w' \equiv \text{const.}$ a.e. in $(a, b)$. Hence it is sufficient to solve
\[
v' = C(u' - \alpha) \quad \text{(a.e. in } \Omega) \tag{5}
\]
for $v \in H^1_0(a, b)$ and $\alpha \in \mathbb{R}$ instead of (4). And we got an iterative solution to (5), namely, an algorithm to produce the sequences $\{v_k\} \subset H^1(a, b)$ and $\{\alpha_k\} \subset \mathbb{R}$ which approximate $v$ and $\alpha$, respectively.
Algorithm I \hspace{0.5cm} \text{Put } \alpha_0 := 0 \text{ and iterate the followings on } k = 0, 1, 2, \cdots.

1. Define \( v_k \in H^1(a, b) \cap C([a, b]) \) by using \( \alpha_k \) such as
   \[ v_k(x) := \int_a^x C(u'(\xi) - \alpha_k) \, d\xi \quad (a \leq x \leq b). \]

2. Put \( \delta_k := \frac{v_k(b)}{b-a} \) and \( \alpha_{k+1} := \alpha_k + \delta_k \).

When \( v_k \to v \) in \( H^1(a, b) \) and \( \alpha_k \to \alpha \) in \( \mathbb{R} \) as \( k \to \infty \), one can expect \( v(b) = 0 \), i.e., \( v \in H_0^1(a, b) \). If it holds, the pair of \( v \) and \( \alpha \) solves to (4). In fact, these properties are assured by the following theorem.

Theorem 2 \hspace{0.5cm} \text{For any } u \in H_0^1(a, b), \text{ each sequence } \{\alpha_k\} \text{ and } \{v_k\} \text{ in Algorithm I converges. Moreover, the limit function of } v_k \text{ belongs to } H_0^1(a, b). \]

Theorem 2 is the direct result of following 3 lemmas. At first, we will prove the convergence of \( \{\alpha_k\} \) by showing the monotonicity and the boundedness of it.

Lemma 1 (monotonicity) \hspace{0.5cm} \text{In Algorithm I, if}

\[ \alpha_1 = \delta_0 := \frac{1}{b-a} \int_a^b C(u'(\xi)) \, d\xi > 0, \]

then the sequence \( \{\delta_k\} \) satisfies that \( 0 \leq \delta_{k+1} \leq \delta_k \) (\( k = 0, 1, 2, \cdots \)).

\textbf{(Proof)} \hspace{0.5cm} \text{Fix } k \in \{0, 1, 2, \cdots \} \text{ and assume } \delta_k \geq 0. \text{ Let denote}

\[ \Omega_{p}(f) := \{x \in \Omega; \, f(x) > 1\}, \quad \Omega_{n}(f) := \{x \in \Omega; \, f(x) < -1\}, \]

\[ \Omega_{z}(f) := \Omega \setminus (\Omega_{p}(f) \cup \Omega_{n}(f)), \]

where \( \Omega = (a, b) \), and define \( \Omega_{ij} \) by

\[ \Omega_{ij} := \Omega_{i}(u' - \alpha_{k+1}) \cap \Omega_{j}(u' - \alpha_k) \quad (i, j \in \{p, z, n\}). \]

For brevity, we will use the notations

\[ |\Omega_{ij}| := \int_{\Omega_{ij}} dx \quad \text{and} \quad \omega_{ij} := \frac{|\Omega_{ij}|}{|\Omega|} = \frac{1}{|\Omega|} \int_{\Omega_{ij}} dx \quad (i, j \in \{p, z, n\}). \]
Note that

\[ |\Omega| := b - a = \sum_{i,j} |\Omega_{ij}| \quad \text{and} \quad \sum_{i,j} \omega_{ij} = 1 \quad (i, j \in \{p, z, n\}), \]

and \( |\Omega_{pz}| = |\Omega_{pn}| = |\Omega_{zn}| = 0 \) since \( \alpha_{k+1} = \alpha_k + \delta_k \geq \alpha_k \). By using them, we can write

\[
\delta_{k+1} - \delta_k = \frac{1}{|\Omega|} \sum_{i,j} \int_{\Omega_{ij}} \{C(u'-\alpha_{k+1}) - C(u' - \alpha_k)\} \, dx
= \frac{1}{|\Omega|} \left\{ \int_{\Omega_{zp}} (u' - \alpha_{k+1} - 1) \, dx + \int_{\Omega_{zz}} (-\alpha_{k+1} + \alpha_k) \, dx 
+ \int_{\Omega_{np}} (-2) \, dx + \int_{\Omega_{nz}} (-1 - u' + \alpha_k) \, dx \right\}
= \frac{1}{|\Omega|} \int_{\Omega_{zp}} (u' - \alpha_{k+1} - 1) \, dx - \omega_{zp} \delta_k - 2 \omega_{np} + \frac{1}{|\Omega|} \int_{\Omega_{nz}} (-1 - u' + \alpha_k) \, dx.
\]

From the definition of \( \Omega_{zp} \) and \( \Omega_{nz} \), we obtain the following evaluations:

\[-\min\{2, \delta_k\} \leq u'(x) - \alpha_{k+1} - 1 \leq 0 \quad (\text{a.e. in } \Omega_{zp}),
-\min\{2, \delta_k\} \leq -1 - u'(x) + \alpha_k \leq 0 \quad (\text{a.e. in } \Omega_{nz}).\]

By the estimates from above, we get the monotone decreasingness of \( \{\delta_k\} \):

\[ \delta_{k+1} \leq (1 - \omega_{zp}) \delta_k - 2 \omega_{np} \leq \delta_k. \]

Next, we will show the non-negativeness of \( \{\delta_k\} \). By the estimates from below, we get

\[ \delta_{k+1} \geq -\min\{2, \delta_k\} \omega_{zp} + (1 - \omega_{zp}) \delta_k - 2 \omega_{np} - \min\{2, \delta_k\} \omega_{nz}. \]

When \( \delta_k \geq 2 \), we can deduce from this estimate

\[ \delta_{k+1} \geq 2(-\omega_{zp} + 1 - \omega_{zp} - \omega_{np} - \omega_{nz}) \geq 0. \]

In the other hand, when \( \delta_k < 2 \), we can easily show that \( \omega_{np} = 0 \), and hence

\[ \delta_{k+1} \geq \delta_k(-\omega_{zp} + 1 - \omega_{zp} - \omega_{nz}) \geq 0. \]

One can get similar result as Lemma 1 for the case \( \delta_0 < 0 \).
Corollary 2

In Algorithm I, if

$$\alpha_1 = \delta_0 := \frac{1}{b-a} \int_a^b C(u'(\xi)) \, d\xi < 0,$$

then the sequence \( \{\delta_k\} \) satisfies that

$$0 \geq \delta_{k+1} \geq \delta_k \quad (k = 0, 1, 2, \cdots).$$

It is obvious that \( \delta_k = 0 \) implies \( \delta_{k'} = 0 \) for all \( k' \in \{k, k+1, k+2, \cdots\} \). Since \( \alpha_k = \sum_{j=0}^{k-1} \delta_j \), it is easy to look that \( \{\alpha_k\} \) is also monotone and that the sign of \( \alpha_k \) is “same” as that of \( \delta_k \) in the sense considering the sign of 0 to belong to both of plus and minus one. Hence, we get the following.

Corollary 3

For the sequences \( \{\delta_k\} \) and \( \{\alpha_k\} \) generated by Algorithm I, it holds that

$$\alpha_k > 0 \Rightarrow \delta_k \geq 0 \quad \text{and} \quad \alpha_k < 0 \Rightarrow \delta_k \leq 0 \quad (k = 0, 1, 2, \cdots).$$

We use this property in the proof of Lemma 4.

Lemma 4 (boundedness)

In Algorithm I, the sequence \( \{\alpha_k\} \) is bounded such as

$$|\alpha_k| \leq \left( \frac{2}{b-a} \right)^{1/2} \|u\|_{H^1_0(a,b)} + 1 \quad (k = 0, 1, 2, \cdots).$$

Proof

When \( u = 0 \) in \( H^1_0(a,b) \), it is clear that \( \alpha_k = 0 \) for any \( k \in \{0, 1, 2, \cdots\} \). Then, we take \( u \neq 0 \), namely, \( \|u\|_{H^1_0(a,b)} = \|u'\|_{L^2(a,b)} > 0 \). And we will show only for the case \( \alpha_k > 0 \) here. Almost the same proof works for the case \( \alpha_k < 0 \).

For each fixed \( \varepsilon > 0 \), assume that

$$\exists k \in \mathbb{N} \quad \text{s.t.} \quad \alpha_k \geq \left( \frac{2 + \varepsilon}{b-a} \right)^{1/2} \|u\|_{H^1_0(a,b)} + 1. \quad (*)$$

Note that \( \delta_k > 0 \) since \( \alpha_k > 0 \). Putting

$$\Omega_1 := \{x \in \Omega; u'(x) - \alpha_k \geq -1\}, \quad \Omega_2 := (a,b) \setminus \Omega_1,$$

we get the inequality

\[
(b-a)\delta_k = \int_{\Omega_1} C(u'(\xi) - \alpha_k) \, d\xi + \int_{\Omega_2} C(u'(\xi) - \alpha_k) \, d\xi \\
\leq \int_{\Omega_1} |C(u'(\xi) - \alpha_k)| \, d\xi - \int_{\Omega_2} d\xi \leq |\Omega_1| - |\Omega_2|,
\]
where $|\Omega_i| := \int_{\Omega_i} dx$. Since $|\Omega_2| = (b - a) - |\Omega_1|, |\Omega_1| = 0$ implies that $\delta_k < 0$ which contradicts to the assumption $(*)$. Then, we assume $|\Omega_1| > 0$ hereafter. By using $(\ast)$ and the definition of $\Omega_1$, we can easily show that

$$\xi \in \Omega_1 \Rightarrow |u'(\xi)|^2 \geq (\alpha_k - 1)^2 \geq \frac{2 + \varepsilon}{b - a} \|u'\|_{L^2(a,b)}^2.$$ 

Hence, it follows that

$$\|u'\|_{L^2(a,b)}^2 \geq \int_{\Omega_1} |u'(\xi)|^2 d\xi \geq \frac{2 + \varepsilon}{b - a} \|u'\|_{L^2(a,b)}^2 |\Omega_1|,$$

and then,

$$|\Omega_2| - |\Omega_1| \geq \varepsilon |\Omega_1|.$$ 

This and $(\dagger)$ lead that $\delta_k < 0$ which contradicts to $(\ast)$.

Lemma 1 (Corollary 2) and Lemma 4 show the convergence of $\{\alpha_k\}$ generated by Algorithm I. Then, we will show the convergence of $\{v_k\}$ in $H^1(a,b)$.

**Lemma 5** For $\{\alpha_k\}$ and $\{v_k\}$ generated by Algorithm I, denoting

$$\alpha := \lim_{k \to \infty} \alpha_k \quad \text{and} \quad v(x) := \int_a^x C(u'(\xi) - \alpha) d\xi \quad (a \leq x \leq b),$$

it holds that $v_k \to v \ (k \to \infty)$ in $H^1(a,b)$ and $v \in H_0^1(a,b)$.

**(Proof)** It is easy to see that

$$\forall z_1, z_2 \in \mathbb{R}, \quad |C(z_1) - C(z_2)| \leq |z_1 - z_2|.$$ 

By using this property and the definitions of $v$ and $v_k$, we get

$$|v'(x) - v'_k(x)| = |C(u'(x) - \alpha) - C(u'(x) - \alpha_k)| \leq |\alpha - \alpha_k| \quad (\text{a.e. in } \Omega).$$ 

Therefore, we obtain

$$\|v - v_k\|_{H^1(\Omega)}^2 := \int_a^b |v'(x) - v'_k(x)|^2 dx + \int_a^b |v(x) - v_k(x)|^2 dx 
= \int_a^b \left( \int_a^x (v'(\xi) - v'_k(\xi)) d\xi \right)^2 dx + \int_a^b |v'(x) - v'_k(x)|^2 dx 
\leq \int_a^b |\alpha - \alpha_k|^2 (x - a)^2 dx + \int_a^b |\alpha - \alpha_k|^2 dx 
= |\alpha - \alpha_k|^2 \left( \frac{1}{3} (b - a)^3 + (b - a) \right),$$
and then the convergence $v_k \to v$ in $H^1(a, b)$. Furthermore, since
\[
|v(b) - v_k(b)| = \left| \int_a^b (v'(x) - v'_k(x)) \, dx \right| \\
\leq \int_a^b |v'(x) - v'_k(x)| \, dx \leq |\alpha - \alpha_k| (b - a),
\]
it holds that $v_k(b) \to v(b)$ ($k \to \infty$). In the other hand,
\[
v_k(b) = \delta_k(b-a) = (\alpha_{k+1} - \alpha_k)(b-a)
\]
implies $v_k(b) \to 0$, hence we get $v(b) = 0$, namely, $v \in H^1_0(a, b)$. \hfill \square

## 4 Radial symmetric case

For higher dimensional cases, the system of equations (4) may not have any solution, in general. But, when both of domain $\Omega$ and given function $u$ are radial symmetric, the problem is reducible to 1 dimensional one, and can be solved. In this section, we consider that both $\Omega$ and $u$ are radial symmetric.

At first, we mention about the most simple (trivial) case, namely, the domain $\Omega$ is spherical one:
\[
\Omega = \{ x \in \mathbb{R}^N; |x| < a \} \quad \text{with} \quad 0 < a < \infty.
\]
In this case, it is obvious that $v = P_K u$ can be obtained such as
\[
v(x) := - \int_{|x|}^a C(\tilde{u}'(\rho)) \, d\rho \quad (x \in \Omega),
\]
where $\tilde{u} : \mathbb{R} \to \mathbb{R}$ is defined by $\tilde{u}(|x|) := u(x)$.

For more interesting case, we consider a ring domain:
\[
\Omega = \{ x \in \mathbb{R}^N; a < |x| < b \} \quad \text{with} \quad 0 < a < b < \infty.
\]
(6)
In this case, the system of equations (4) can be written as
\[
\begin{cases}
    v_r = C(u_r - w_r) & \text{(a.e. in $\Omega$)}, \\
    w_{rr} + \frac{N-1}{r} w_r = 0 & \text{(weak sense)}
\end{cases}
\]
with $r := |x|$. Since the 2nd equation of this system is solvable such as
\[
w_r(x) = \alpha r^{1-N} \quad \text{(a.e. in $\Omega$),}
\]
with arbitrary constant $\alpha$, it suffices to solve
\[ \tilde{v}'(r) = C \left( \tilde{u}'(r) - \alpha r^{1-N} \right) \quad (\text{a.e. } r \in [a, b]) \quad (7) \]
for $\tilde{v} \in H^1_0(a, b)$ and $\alpha \in \mathbb{R}$. The equation (7) is similar to (5) and we can expand Algorithm I to solve it as followings.

**Algorithm II**  \quad Put $\alpha_0 := 0$ and iterate the followings for $k = 0, 1, 2, \cdots$.

1. Define $v_k(x)$ by using $\alpha_k$ such as
\[ v_k(x) := \int_a^{\|x\|} C \left( \tilde{u}'(\rho) - \frac{\alpha_k}{\rho^{N-1}} \right) d\rho \quad (x \in \Omega). \]

2. Put $\delta_k := \frac{a^{N-1}}{b - a} \lim_{\|x\| \to b} v_k(x)$ and $\alpha_{k+1} := \alpha_k + \delta_k$.

This algorithm is justified by the next theorem.

**Theorem 3**  \quad If $\Omega$ is a ring domain such as (6) and $u \in H^1_0(\Omega)$ is radial symmetric one, then each sequence of $\{\alpha_k\}$ and $\{v_k\}$ in Algorithm II converges.

The sequence $\{\alpha_k\}$ generated by Algorithm II also has the monotonicity and the boundedness, and the convergence of $\{\alpha_k\}$ is direct result of them. Once the convergence of $\{\alpha_k\}$ was shown, one can also show the convergence of $\{v_k\}$. These lemmas written below prove Theorem 3.

**Lemma 6** (monotonicity) \quad In Algorithm II, if
\[ \alpha_1 = \delta_0 := \frac{a^{N-1}}{b - a} \int_a^b C(\tilde{u}'(\rho)) d\rho > 0, \]
then the sequence $\{\delta_k\}$ satisfies that $0 \leq \delta_{k+1} \leq \delta_k \quad (k = 0, 1, 2, \cdots)$.

**Lemma 7** (boundedness) \quad In Algorithm II, the sequence $\{\alpha_k\}$ is bounded such as
\[ |\alpha_k| \leq b^{N-1} \left( \frac{2}{b - a} \right)^{1/2} \|\tilde{u}'\|_{L^2(a, b)} + 1 \quad (k = 0, 1, 2, \cdots). \]

**Lemma 8** \quad For $\{\alpha_k\}$ and $\{v_k\}$ generated by Algorithm II, denoting
\[ \alpha := \lim_{k \to \infty} \alpha_k \quad \text{and} \quad v(x) := \int_a^{\|x\|} C \left( \tilde{u}'(\rho) - \frac{\alpha}{\rho^{N-1}} \right) d\rho \quad (x \in \Omega), \]
then it holds that $v_k \to v \quad (k \to \infty)$ in $H^1(\Omega)$ and $v \in H^1_0(\Omega)$.
The proofs of Lemma 6, 7 and 8 are done by almost same arguments as Lemma 1, 4 and 5, respectively, and we omit them here.

Finally, we will show an example of numerical result of Algorithm II. In Fig. 6, $u$ and $P_K u$ defined in 2 dimensional ring domain $\Omega$ such as

$$
\begin{align*}
  u(x) &= 4(|x| + 1)^2(|x| + \frac{1}{2})(|x| - \frac{1}{5})(|x| - \frac{3}{5})(|x| - \frac{4}{5})(|x| - 1), \\
  \Omega &= \{x \in \mathbb{R}^2; 0.5 \leq |x| \leq 2.5\},
\end{align*}
$$

are plotted as 3D graphs.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{u and $P_K u$ for 2 dimensional ring domain case: $u(r) = 4(r+1)^2(r+\frac{1}{2})(r-\frac{1}{5})(r-\frac{3}{5})(r-\frac{4}{5})(r-1)$.}
\end{figure}

In Fig. 7, the same $u$ and $P_K u$ expressed above but for 1, 2 and 3 dimensional domains are plotted as r–u and r–$P_K u$ graphs. One may notice that the difference between the values of $u$ and those of $P_K u$ is rather uniform in 1 dimensional case. But in a higher dimensional case, the difference between the values of $u$ and those of $P_K u$ near the origin is larger than that of them far from the origin.
Figure 7: $u$ and $P_K u$ for higher dimensional cases:
\[
u(r) = 4(r+1)^2(r+\frac{1}{2})(r-\frac{1}{6})(r-\frac{3}{10})(r-\frac{4}{5})(r-1);
\]
v_n denotes $P_K u$ for $n$ dimensional case.

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References


