On the projection which appears in the variational treatment of elasto-plastic torsion problem (Variational Problems and Related Topics)

Author(s)
Idogawa, Tomoyuki

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On the projection which appears in the variational treatment of elasto-plastic torsion problem

Tomoyuki Idogawa

Abstract

In the treatment of variational inequalities, the projection operator $P_K$ from some Hilbert space $V$ onto a certain closed convex subset $K$ plays an important role. But, only for few problems, it is known how to get the explicit form of $P_Ku$ for each given $u \in V$. In this article, we consider $K = \{ f \in H_0^1(\Omega); |\nabla f| \leq 1 \text{ a.e.} \}$, which is related to elasto-plastic torsion problems, and propose an iterative method to approximate $P_Ku$ for 1 dimensional case $\Omega = (a, b)$. We also show an expansion of it for higher dimensional but radial symmetric cases.

1 Problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary and

$$K := \{ f \in H_0^1(\Omega); |\nabla f| \leq 1 \text{ a.e.} \}.$$ 

We will denote by $P_K$ the projection mapping from $H_0^1(\Omega)$ into its convex closed subset $K$, namely, for $u \in H_0^1(\Omega)$ and $v \in K$,

$$P_Ku = v \iff \|u - v\|_{H_0^1(\Omega)} = \inf_{f \in K} \|u - f\|_{H_0^1(\Omega)}.$$ 

For convenience sake, we take

$$\|u\|_{H_0^1(\Omega)} := \|
abla u\|_{L^2(\Omega)} = \left\{ \int_{\Omega} |\nabla u(x)|^2 \, dx \right\}^{1/2},$$

throughout this article. (Note that $\Omega$ is bounded.) The problem is to find $v = P_Ku \in K$ for each given $u \in H_0^1(\Omega)$.

This projection $P_K$ appears in the variational treatment of elasto-plastic torsion problem. Consider an infinitely long cylindrical elastic-plastic bar of
cross section $\Omega$ to which some torsion momentum ($\tau$ denotes the torsion angle per unit length) is applied (Fig. 1). It is known that the stress vector $\sigma$ in $\Omega$ is determined by the minimizer $u$ of

$$J(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \tau \int_{\Omega} v \, dx \quad (v \in K),$$

namely, $\sigma = \nabla u$ [2, p.42]. This minimizing problem is equivalent to finding $u \in K$ such that

$$u = P_K(u - \rho(Au - l))$$

for some $\rho > 0$, where $A \in \mathcal{L}(V, V)$ and $l \in V$ are defined by

$$(Af, g) = \frac{1}{2} \int_{\Omega} \nabla f \cdot \nabla g \, dx,$$

$$(l, f) = \tau \int_{\Omega} f \, dx \quad \left( (\cdot, \cdot) : \text{inner product of } V \right)$$

for $f, g \in V := H_0^1(\Omega)$, respectively [2, p.3].

The projection $P_K$ also plays an important role in the error estimates of the corresponding penalized elliptic variational inequalities [5].

## 2 Rewriting the problem

We introduce a functional $J_u : K \to \mathbb{R}$ for each given $u \in H_0^1(\Omega)$:

$$J_u(f) := ||u - f||_{H_0^1(\Omega)}^2 = \int_{\Omega} |\nabla u(x) - \nabla f(x)|^2 \, dx. \quad (1)$$

By using it, the problem can be rewritten such as "To find the minimizer $v$ of $J_u$ on $K."$ On this problem, one can easily show:
Proposition 1  If there exists a solution \( v \in H_0^1(\Omega) \) to

\[
\nabla v = C(\nabla u) \quad (\text{a.e. in } \Omega),
\]

(2)

then \( v \) is the minimizer of \( J_u \) on \( K \), where \( C(z) := \begin{cases} 
z & (|z| \leq 1), \\
z/|z| & (|z| > 1). \end{cases} \)

Especially, for 1 dimensional case \( \Omega = (a, b) \subset \mathbb{R} \), put

\[
v(x) := \int_a^x C(u'(\xi)) \, d\xi \quad (a \leq x \leq b)
\]

(3)

for a given function \( u \in H_0^1(a, b) \). If this function \( v \in H^1(a, b) \cap C([a, b]) \) satisfies that \( v(b) = 0 \), then \( v \) belongs to \( H_0^1(a, b) \) and hence \( v = P_K u \). An example of this kind: \( u(x) = -\frac{3}{10} \cos \frac{3}{2} \pi x \) and \( v \) defined by (3) for \( \Omega = (-1, 1) \) are shown in Fig. 2. We also plot their derivatives in Fig. 3. In this case, \( P_K u \) and \( v \) coincide perfectly (see Fig. 2), and \( (P_K u)' \) is only the "cut-off" of \( u' \), namely, \( (P_K u)' = C(u') \) (see Fig. 3).

![Figure 2: the case \( v(b) = 0; u(x) = -\frac{3}{10} \cos \frac{3}{2} \pi x \).](image)

In fact, for 1 dimensional case \( \Omega = (a, b) \), one can easily show that if the given function \( u \) is symmetric (i.e., \( u(a + \xi) = u(b - \xi) \) for any \( \xi \)), then \( v \) defined by (3) satisfies that \( v(b) = 0 \) and hence \( v = P_K u \).
But it is rather special. We will show an example for the case $v(b) \neq 0$: $u(x) = 4(x+1)^2(x+\frac{1}{2})(x-\frac{1}{5})(x-\frac{3}{5})(x-\frac{4}{5})(x-1)$ for $\Omega = (-1,1)$. The graphs of $u$, corresponding $v$ and $P_K u$ are shown in Fig. 4. Also the derivatives $u'$ and $(P_K u)'$ are plotted in Fig. 5.

Figure 4: the case $v(b) \neq 0$; $u(x) = 4(x+1)^2(x+\frac{1}{2})(x-\frac{1}{5})(x-\frac{3}{5})(x-\frac{4}{5})(x-1)$. 
Figure 5: $u'$ and $(P_K u)'$; $u(x) = 4(x+1)^2(x+\frac{1}{2})(x-\frac{1}{5})(x-\frac{3}{5})(x-\frac{4}{5})(x-1)$.

In such a case, it is clear that any primitive function of $C(u')$ can not belong to $H_0^1(\Omega)$ since its values at 2 boundary points are not equal. In other words, (2) has no solution in $H_0^1(\Omega)$, in general.

Then, instead of (2), we consider the following system of equations:

\[
\begin{aligned}
\nabla v &= C(\nabla u - \nabla w) \quad \text{(a.e. in } \Omega), \\
\Delta w &= 0 \quad \text{(weak sense)}.
\end{aligned}
\]

It means that at first, we alter $u$ by subtracting the appropriate quantity, namely, a function $w \in H^1(\Omega)$ satisfying $\Delta w = 0$. Then we “cut-off” its gradient and get the primitive function. If the obtained function $v$ belongs to $H_0^1(\Omega)$, then the next theorem assures that $v = P_K u$.

**Theorem 1** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. If there exists a solution $(v, w)$ in $H_0^1(\Omega) \times H^1(\Omega)$ to the system of equations (4) with a given parameter $u \in H_0^1(\Omega)$, then $v$ belongs to $K$ and minimizes the functional $J_u$ defined by (1).

**Proof** It is clear that $v \in K$. Hence, it suffices to show that

$$\forall f \in K, \quad J_u(f) - J_u(v) \geq 0.$$ 

Let denote $\Omega_p := \{x \in \Omega; \ |\nabla(u - w)| > 1\}$ and $\Omega_z := \Omega \setminus \Omega_p$. Fix $f \in K$ and put $\delta := f - v \in H_0^1(\Omega)$. For this $\delta$, we can easily show

$$\nabla \delta \cdot \nabla v = \nabla f \cdot \nabla v - |\nabla v|^2 = \nabla f \cdot \nabla v - 1 \leq 0 \quad \text{(a.e. in } \Omega_p)$$
since $|\nabla f| \leq 1$ and $|\nabla v| = 1$ (a.e. in $\Omega_{p}$), and hence
\[
\nabla \delta \cdot (\nabla u - \nabla w) \leq 0 \quad \text{(a.e. in } \Omega_{p}).
\]
On the other hand, since $\triangle w = 0$ (weak sense in $H^{1}(\Omega)$) and $\delta \in H^{1}_{0}(\Omega)$,
\[
\int_{\Omega} \nabla \delta \cdot \nabla w \, dx = \int_{\Omega_{p}} \nabla \delta \cdot \nabla w \, dx + \int_{\Omega_{z}} \nabla \delta \cdot \nabla w \, dx = 0.
\]
By using these facts, we get
\[
J_{u}(f) - J_{u}(v) = \int_{\Omega} |\nabla u - (v + \delta)|^{2} \, dx - \int_{\Omega} |\nabla u - \nabla v|^{2} \, dx
\]
\[
= \int_{\Omega} |\nabla \delta|^{2} \, dx - 2 \int_{\Omega_{p}} \nabla \delta \cdot (\nabla u - \nabla v) \, dx
\]
\[
= \int_{\Omega} |\nabla \delta|^{2} \, dx - 2 \int_{\Omega_{p}} \nabla \delta \cdot \left( \frac{\nabla u - \nabla v}{|\nabla u - \nabla v|} \right) \, dx - 2 \int_{\Omega_{z}} \nabla \delta \cdot \nabla w \, dx
\]
\[
= \int_{\Omega} |\nabla \delta|^{2} \, dx + 2 \int_{\Omega_{p}} \nabla \delta \cdot \left( \frac{\nabla u - \nabla w}{|\nabla u - \nabla w|} - \nabla u \right) \, dx + 2 \int_{\Omega_{p}} \nabla \delta \cdot \nabla w \, dx
\]
\[
\geq \int_{\Omega} |\nabla \delta|^{2} \, dx \geq 0.
\]
\[\square\]

3 1 dimensional case

Theorem 1 assures that if one could solve the system of equations (4) with a given parameter $u \in H^{1}_{0}(\Omega)$, one get the projection $P_{K}u$. But unfortunately, there may not be any solution to (4) in general, except 1 dimensional case. In fact, when $\Omega = (a, b) \subset \mathbb{R}^{1}$ ($-\infty < a < b < \infty$), the equation $w'' = 0$ can be solved such as $w' \equiv \text{const. a.e. in } (a, b)$. Hence it is sufficient to solve
\[
v' = C(v' - \alpha) \quad \text{(a.e. in } \Omega) \quad (5)
\]
for $v \in H^{1}_{0}(a, b)$ and $\alpha \in \mathbb{R}$ instead of (4). And we got an iterative solution to (5), namely, an algorithm to produce the sequences $\{v_{k}\} \subset H^{1}(a, b)$ and $\{\alpha_{k}\} \subset \mathbb{R}$ which approximate $v$ and $\alpha$, respectively.
Algorithm I. Put $\alpha_0 := 0$ and iterate the followings on $k = 0, 1, 2, \cdots$.

1. Define $v_k \in H^1(a, b) \cap C([a, b])$ by using $\alpha_k$ such as
   \[
   v_k(x) := \int_a^x C(u'(\xi) - \alpha_k) \, d\xi \quad (a \leq x \leq b).
   \]

2. Put $\delta_k := \frac{v_k(b)}{b-a}$ and $\alpha_{k+1} := \alpha_k + \delta_k$.

When $v_k \rightarrow v$ in $H^1(a, b)$ and $\alpha_k \rightarrow \alpha$ in $\mathbb{R}$ as $k \rightarrow \infty$, one can expect $v(b) = 0$, i.e., $v \in H^1_0(a, b)$. If it holds, the pair of $v$ and $\alpha$ solves to (4). In fact, these properties are assured by the following theorem.

Theorem 2. For any $u \in H^1_0(a, b)$, each sequence $\{\alpha_k\}$ and $\{v_k\}$ in Algorithm I converges. Moreover, the limit function of $v_k$ belongs to $H^1_0(a, b)$.

Theorem 2 is the direct result of following 3 lemmas. At first, we will prove the convergence of $\{\alpha_k\}$ by showing the monotonicity and the boundedness of it.

Lemma 1 (monotonicity). In Algorithm I, if

\[
\alpha_1 = \delta_0 := \frac{1}{b-a} \int_a^b C(u'(\xi)) \, d\xi > 0,
\]

then the sequence $\{\delta_k\}$ satisfies that $0 \leq \delta_{k+1} \leq \delta_k$ ($k = 0, 1, 2, \cdots$).

(Proof) Fix $k \in \{0, 1, 2, \cdots\}$ and assume $\delta_k \geq 0$. Let denote

\[
\Omega_p(f) := \{x \in \Omega; f(x) > 1\}, \quad \Omega_n(f) := \{x \in \Omega; f(x) < -1\},
\]

\[
\Omega_z(f) := \Omega \setminus (\Omega_p(f) \cup \Omega_n(f)),
\]

where $\Omega = (a, b)$, and define $\Omega_{ij}$ by

\[
\Omega_{ij} := \Omega_z(u' - \alpha_{k+1}) \cap \Omega_j(u' - \alpha_k) \quad (i, j \in \{p, z, n\}).
\]

For brevity, we will use the notations

\[
|\Omega_{ij}| := \int_{\Omega_{ij}} dx \quad \text{and} \quad \omega_{ij} := \frac{|\Omega_{ij}|}{|\Omega|} = \frac{1}{|\Omega|} \int_{\Omega_{ij}} dx \quad (i, j \in \{p, z, n\}).
\]
Note that

$$|\Omega| := b - a = \sum_{i,j} |\Omega_{ij}| \quad \text{and} \quad \sum_{i,j} \omega_{ij} = 1 \quad (i, j \in \{p, z, n\}),$$

and $|\Omega_{pz}| = |\Omega_{pn}| = |\Omega_{zn}| = 0$ since $\alpha_{k+1} = \alpha_k + \delta_k \geq \alpha_k$. By using them, we can write

$$\delta_{k+1} - \delta_k = \frac{1}{|\Omega|} \sum_{i,j} \int_{\Omega_{ij}} \{C(u' - \alpha_{k+1}) - C(u' - \alpha_k)\} \, dx$$

$$= \frac{1}{|\Omega|} \left\{ \int_{\Omega_{zp}} (u' - \alpha_{k+1} - 1) \, dx + \int_{\Omega_{zz}} (-\alpha_{k+1} + \alpha_k) \, dx \right. \right.$$

$$\left. + \int_{\Omega_{np}} (-2) \, dx + \int_{\Omega_{nz}} (-1 - u' + \alpha_k) \, dx \right\}$$

$$= \frac{1}{|\Omega|} \int_{\Omega_{zp}} (u' - \alpha_{k+1} - 1) \, dx - \omega_{zz} \delta_k - 2\omega_{np} + \frac{1}{|\Omega|} \int_{\Omega_{nz}} (-1 - u' + \alpha_k) \, dx.$$

From the definition of $\Omega_{zp}$ and $\Omega_{nz}$, we obtain the following evaluations:

$$-\min\{2, \delta_k\} \leq u'(x) - \alpha_{k+1} - 1 \leq 0 \quad (\text{a.e. } x \in \Omega_{zp}),$$

$$-\min\{2, \delta_k\} \leq -1 - u'(x) + \alpha_k \leq 0 \quad (\text{a.e. } x \in \Omega_{nz}).$$

By the estimates from above, we get the monotone decreasingness of $\{\delta_k\}$:

$$\delta_{k+1} \leq (1 - \omega_{zz}) \delta_k - 2\omega_{np} \leq \delta_k.$$

Next, we will show the non-negativeness of $\{\delta_k\}$. By the estimates from below, we get

$$\delta_{k+1} \geq -\min\{2, \delta_k\} \omega_{zp} + (1 - \omega_{zz}) \delta_k - 2\omega_{np} - \min\{2, \delta_k\} \omega_{nz}.$$

When $\delta_k \geq 2$, we can deduce from this estimate

$$\delta_{k+1} \geq 2(-\omega_{zp} + 1 - \omega_{zz} - \omega_{np} - \omega_{nz}) \geq 0.$$

In the other hand, when $\delta_k < 2$, we can easily show that $\omega_{np} = 0$, and hence

$$\delta_{k+1} \geq \delta_k(-\omega_{zp} + 1 - \omega_{zz} - \omega_{nz}) \geq 0.$$

One can get similar result as Lemma 1 for the case $\delta_0 < 0$. $\square$
Corollary 2. In Algorithm I, if

\[ \alpha_1 = \delta_0 := \frac{1}{b-a} \int_a^b C(u'(\xi)) d\xi < 0, \]

then the sequence \( \{\delta_k\} \) satisfies that \( 0 \geq \delta_{k+1} \geq \delta_k \) \((k = 0, 1, 2, \cdots)\).

It is obvious that \( \delta_k = 0 \) implies \( \delta_{k'} = 0 \) for all \( k' \in \{k, k+1, k+2, \cdots\} \).

Since \( \alpha_k = \sum_{j=0}^{k-1} \delta_j \), it is easy to look that \( \{\alpha_k\} \) is also monotone and that the sign of \( \alpha_k \) is “same” as that of \( \delta_k \) in the sense considering the sign of 0 to belong to both of plus and minus one. Hence, we get the following.

Corollary 3. For the sequences \( \{\delta_k\} \) and \( \{\alpha_k\} \) generated by Algorithm I, it holds that

\[ \alpha_k > 0 \Rightarrow \delta_k \geq 0 \quad \text{and} \quad \alpha_k < 0 \Rightarrow \delta_k \leq 0 \quad (k = 0, 1, 2, \cdots). \]

We use this property in the proof of Lemma 4.

Lemma 4 (boundedness). In Algorithm I, the sequence \( \{\alpha_k\} \) is bounded such as

\[ |\alpha_k| \leq \left( \frac{2}{b-a} \right)^{1/2} \|u\|_{H_0^1(a, b)} + 1 \quad (k = 0, 1, 2, \cdots). \]

(Proof) When \( u = 0 \) in \( H_0^1(a, b) \), it is clear that \( \alpha_k = 0 \) for any \( k \in \{0, 1, 2, \cdots\} \). Then, we take \( u \neq 0 \), namely, \( \|u\|_{H_0^1(a, b)} = ||u'||_{L^2(a,b)} > 0 \). And we will show only for the case \( \alpha_k > 0 \) here. Almost the same proof works for the case \( \alpha_k < 0 \).

For each fixed \( \varepsilon > 0 \), assume that

\[ \exists k \in \mathbb{N} \quad \text{s.t.} \quad \alpha_k \geq \left( \frac{2+\varepsilon}{b-a} \right)^{1/2} \|u\|_{H_0^1(a, b)} + 1. \quad (*) \]

Note that \( \delta_k \geq 0 \) since \( \alpha_k > 0 \). Putting

\[ \Omega_1 := \{x \in \Omega; u'(x) - \alpha_k \geq -1\}, \quad \Omega_2 := (a, b) \setminus \Omega_1, \]

we get the inequality

\[ (b-a)\delta_k = \int_{\Omega_1} C(u'(\xi) - \alpha_k) d\xi + \int_{\Omega_2} C(u'(\xi) - \alpha_k) d\xi \]

\[ \leq \int_{\Omega_1} |C(u'(\xi) - \alpha_k)| d\xi - \int_{\Omega_2} d\xi \leq |\Omega_1| - |\Omega_2|, \quad (\dagger) \]
where $|\Omega_i| := \int_{\Omega_i} dx$. Since $|\Omega_2| = (b - a) - |\Omega_1|$, $|\Omega_1| = 0$ implies that $\delta_k < 0$ which contradicts to the assumption (*). Then, we assume $|\Omega_1| > 0$ hereafter. By using (*) and the definition of $\Omega_1$, we can easily show that

$$\xi \in \Omega_1 \Rightarrow |u'(\xi)|^2 \geq (\alpha_k - 1)^2 \geq \frac{2 + \varepsilon}{b - a} \|u'\|^2_{L^2(a,b)}.$$ 

Hence, it follows that

$$\|u'\|^2_{L^2(a,b)} \geq \int_{\Omega_1} |u'(\xi)|^2 d\xi \geq \frac{2 + \varepsilon}{b - a} \|u'\|^2_{L^2(a,b)} |\Omega_1|,$$

and then,

$$|\Omega_2| - |\Omega_1| \geq \varepsilon |\Omega_1|.$$ 

This and (†) lead that $\delta_k < 0$ which contradicts to (*). \[\square\]

Lemma 1 (Corollary 2) and Lemma 4 show the convergence of $\{\alpha_k\}$ generated by Algorithm I. Then, we will show the convergence of $\{v_k\}$ in $H^1(a,b)$.

**Lemma 5** For $\{\alpha_k\}$ and $\{v_k\}$ generated by Algorithm I, denoting

$$\alpha := \lim_{k \to \infty} \alpha_k \quad \text{and} \quad v(x) := \int_a^x C(u'(\xi) - \alpha) \, d\xi \quad (a \leq x \leq b),$$

it holds that $v_k \to v \ (k \to \infty)$ in $H^1(a,b)$ and $v \in H^1_0(a,b)$.

**(Proof)** It is easy to see that

$$\forall z_1, z_2 \in \mathbb{R}, \quad |C(z_1) - C(z_2)| \leq |z_1 - z_2|.$$ 

By using this property and the definitions of $v$ and $v_k$, we get

$$|v'(x) - v_k'(x)| = |C(u'(x) - \alpha) - C(u'(x) - \alpha_k)| \leq |\alpha - \alpha_k| \quad (\text{a.e. in } \Omega).$$

Therefore, we obtain

$$\|v - v_k\|^2_{H^1(\Omega)} := \int_a^b |v(x) - v_k(x)|^2 \, dx + \int_a^b |v'(x) - v_k'(x)|^2 \, dx$$

\[= \int_a^b \int_a^x (v'(\xi) - v_k'(\xi)) \, d\xi \, dx + \int_a^b |v'(x) - v_k'(x)|^2 \, dx \]

\[\leq \int_a^b |\alpha - \alpha_k|^2 (x - a)^2 \, dx + \int_a^b |\alpha - \alpha_k|^2 \, dx \]

\[= |\alpha - \alpha_k|^2 \left( \frac{1}{3} (b - a)^3 + (b - a) \right),\]
and then the convergence $v_k \rightarrow v$ in $H^1(a, b)$. Furthermore, since

$$|v(b) - v_k(b)| = \left| \int_a^b (v'(x) - v'_k(x)) \, dx \right| \leq \int_a^b |v'(x) - v'_k(x)| \, dx \leq |\alpha - \alpha_k| (b - a),$$

it holds that $v_k(b) \rightarrow v(b) \ (k \rightarrow \infty)$. In the other hand,

$$v_k(b) = \delta_k(b - a) = (\alpha_{k+1} - \alpha_k)(b - a)$$

implies $v_k(b) \rightarrow 0$, hence we get $v(b) = 0$, namely, $v \in H^1_0(a, b)$. \hfill \Box

## 4 Radial symmetric case

For higher dimensional cases, the system of equations (4) may not have any solution, in general. But, when both of domain $\Omega$ and given function $u$ are radial symmetric, the problem is reducible to 1 dimensional one, and can be solved. In this section, we consider that both $\Omega$ and $u$ are radial symmetric.

At first, we mention about the most simple (trivial) case, namely, the domain $\Omega$ is spherical one:

$$\Omega = \{ x \in \mathbb{R}^N; |x| < a \} \quad \text{with} \quad 0 < a < \infty.$$ 

In this case, it is obvious that $v = P_K u$ can be obtained such as

$$v(x) := - \int_{|x|}^{a} C(\tilde{u}'(\rho)) \, d\rho \quad (x \in \Omega),$$

where $\tilde{u} : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\tilde{u}(|x|) := u(x)$.

For more interesting case, we consider a ring domain:

$$\Omega = \{ x \in \mathbb{R}^N; a < |x| < b \} \quad \text{with} \quad 0 < a < b < \infty. \quad (6)$$

In this case, the system of equations (4) can be written as

$$\begin{cases}
    v_r = C(u_r - w_r) \quad (\text{a.e. in } \Omega),
    \\
    w_{rr} + \frac{N - 1}{r} w_r = 0 \quad (\text{weak sense})
\end{cases}$$

with $r := |x|$. Since the 2nd equation of this system is solvable such as

$$w_r(x) = \alpha r^{1-N} \quad (\text{a.e. } x \in \Omega),$$
with arbitrary constant $\alpha$, it suffices to solve
\[
\tilde{v}'(r) = C \left( \tilde{u}'(r) - \alpha r^{1-N} \right) \quad \text{(a.e. } r \in [a, b])
\]
for $\tilde{v} \in H^1_0(a, b)$ and $\alpha \in \mathbb{R}$. The equation (7) is similar to (5) and we can expand Algorithm I to solve it as followings.

**Algorithm II**  
Put $\alpha_0 := 0$ and iterate the followings for $k = 0, 1, 2, \ldots$.

1. Define $v_k(x)$ by using $\alpha_k$ such as
   \[
v_k(x) := \int_a^{[x]} C \left( \tilde{u}'(\rho) - \frac{\alpha_k}{\rho^{N-1}} \right) d\rho \quad (x \in \Omega).
   \]

2. Put $\delta_k := \frac{a^{N-1}}{b-a} \lim_{|x| \to b} v_k(x)$ and $\alpha_{k+1} := \alpha_k + \delta_k$.

This algorithm is justified by the next theorem.

**Theorem 3**  
If $\Omega$ is a ring domain such as (6) and $u \in H^1_0(\Omega)$ is radial symmetric one, then each sequence of $\{\alpha_k\}$ and $\{v_k\}$ in Algorithm II converges.

The sequence $\{\alpha_k\}$ generated by Algorithm II also has the monotonicity and the boundedness, and the convergence of $\{\alpha_k\}$ is direct result of them. Once the convergence of $\{\alpha_k\}$ was shown, one can also show the convergence of $\{v_k\}$. These lemmas written below prove Theorem 3.

**Lemma 6 (monotonicity)**  
In Algorithm II, if
   \[
   \alpha_1 = \delta_0 := \frac{a^{N-1}}{b-a} \int_a^b C(\tilde{u}'(\rho)) d\rho > 0,
   \]
then the sequence $\{\delta_k\}$ satisfies that $0 \leq \delta_{k+1} \leq \delta_k$ ($k = 0, 1, 2, \ldots$).

**Lemma 7 (boundedness)**  
In Algorithm II, the sequence $\{\alpha_k\}$ is bounded such as
   \[
   |\alpha_k| \leq b^{N-1} \left( \frac{2}{b-a} \right)^{1/2} \|\tilde{u}'\|_{L^2(a,b)} + 1 \quad (k = 0, 1, 2, \ldots).
   \]

**Lemma 8**  
For $\{\alpha_k\}$ and $\{v_k\}$ generated by Algorithm II, denoting
   \[
   \alpha := \lim_{k \to \infty} \alpha_k \quad \text{and} \quad v(x) := \int_a^{[x]} C \left( \tilde{u}'(\rho) - \frac{\alpha}{\rho^{N-1}} \right) d\rho \quad (x \in \Omega),
   \]
then it holds that $v_k \to v$ ($k \to \infty$) in $H^1(\Omega)$ and $v \in H^1_0(\Omega)$. 

The proofs of Lemma 6, 7 and 8 are done by almost same arguments as Lemma 1, 4 and 5, respectively, and we omit them here.

Finally, we will show an example of numerical result of Algorithm II. In Fig. 6, \( u \) and \( P_Ku \) defined in 2 dimensional ring domain \( \Omega \) such as

\[
u(x) = 4(|x| + 1)^2(|x| + \frac{1}{2})(|x| - \frac{1}{5})(|x| - \frac{3}{5})(|x| - \frac{4}{5})(|x| - 1),
\]

\[
\Omega = \{x \in \mathbb{R}^2; 0.5 \leq |x| \leq 2.5\},
\]

are plotted as 3D graphs.

![Graph](image)

Figure 6: \( u \) and \( P_Ku \) for 2 dimensional ring domain case:

\[
u(r) = 4(r+1)^2(r+\frac{1}{2})(r-\frac{1}{5})(r-\frac{3}{5})(r-\frac{4}{5})(r-1).
\]

In Fig. 7, the same \( u \) and \( P_Ku \) expressed above but for 1, 2 and 3 dimensional domains are plotted as \( r-u \) and \( r-P_Ku \) graphs. One may notice that the difference between the values of \( u \) and those of \( P_Ku \) is rather uniform in 1 dimensional case. But in a higher dimensional case, the difference between the values of \( u \) and those of \( P_Ku \) near the origin is larger than that of them far from the origin.
Figure 7: $u$ and $P_K u$ for higher dimensional cases:

$$u(r) = 4(r+1)^2(r+\frac{1}{2})(r-\frac{1}{6})(r-\frac{3}{6})(r-\frac{4}{6})(r-1);$$

$v_n$ denotes $P_K u$ for $n$ dimensional case.

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References


