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Moduli of Rational Functions and Rational Plane Curves

Makoto Namba

1. Introduction

Holomorphic mappings \( f : X \to M \) and \( f' : X' \to M' \) of complex spaces are said to be **equivalent** if there are biholomorphic mapping \( \varphi : X \to X' \) and \( \varphi : M \to M' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X' \\
\downarrow f & & \downarrow f' \\
M & \xleftarrow{\varphi} & M'
\end{array}
\]

If it is possible to introduce a natural complex space structure on the set of equivalence classes (for a given type of mappings), then we may call it the **moduli space of holomorphic mappings** (for the given type).

It is a difficult problem in general to show the existence of moduli space. In this lecture, we mainly consider the case in which \( X \) and \( X' \) are compact Riemann surfaces and \( M \) and \( M' \) are the \( m \)-dimensional complex projective space \( \mathbb{P}^m = \mathbb{P}^m(\mathbb{C}) \).

2. Linearly non-degenerate holomorphic mappings

Some of our results may be rewritten in terms of stability (see Mumford [3]). But it is difficult to analyze the stability. Our discussion is topological and complex analytic. In particular we make use of the following 2 theorems in order to prove our theorems.
Theorem 1 (Holmán [2]) Let $X$ be a (resp. normal) complex space and $G$ be a complex Lie group acting properly on $X$. Then the quotient space $X/G$ is a (resp. normal) complex space and the projection $\pi : X \rightarrow X/G$ is holomorphic. If moreover $G$ acts on $X$ without fixed point, then $\pi : X \rightarrow X/G$ is a principal $G$-bundle.

Theorem 2 (Popp [5]) Let $X$ be a quasi-projective $\mathbb{C}$-scheme and $G$ be an algebraic group acting properly on $X$. Assume that every stabilizer is a finite group. Then the quotient space $X/G$ is an algebraic space.

In the above 2 theorems, "$G$ acts properly on $X"$ means that the following mapping is proper, that is, the inverse image of every compact set is compact:

$$(\varphi, p) \in G \times X \mapsto (\varphi(p), p) \in X \times X.$$ 

Now, for a compact complex space $X$, we put

$$H(X, \mathbb{P}^n) = \{ f : X \rightarrow \mathbb{P}^n \mid f(X) \text{ is not contained in any hyperplane, that is, } f \text{ is linearly non-degenerate} \}$$

Then $H(X, \mathbb{P}^n)$ is a complex space (so called the Douady space, Douady [1]), whose underlying topology is the compact-open topology. $\text{Aut}(\mathbb{P}^n)$ acts on $H(X, \mathbb{P}^n)$ as the composition of mappings:

$$(\varphi, f) \mapsto \varphi \circ f$$

Theorem 3 $\text{Aut}(\mathbb{P}^n)$ acts on $H(X, \mathbb{P}^n)$ properly without fixed point. Hence $H(X, \mathbb{P}^n)/\text{Aut}(\mathbb{P}^n)$ is a complex space and $H(X, \mathbb{P}^n) \rightarrow H(X, \mathbb{P}^n)/\text{Aut}(\mathbb{P}^n)$ is a principal $\text{Aut}(\mathbb{P}^n)$-bundle.
Theorem 4 Let $\{X_t\}_{t \in \Gamma}$ be a family of compact complex spaces with the parameter space a connected complex space $T$. Then $\text{Aut}(\mathbb{P}^m)$ acts properly without fixed point on $H = \bigcup_t H(X_t, \mathbb{P}^m)$. Hence $H/\text{Aut}(\mathbb{P}^m)$ is a complex space and $H \longrightarrow H/\text{Aut}(\mathbb{P}^m)$ is a principal $\text{Aut}(\mathbb{P}^m)$-bundle.

Here $H$ is the relative Douady space (see Pourcin [6]). The proof of Theorems 3 and 4 can be done by taking a sequence of points and using the property of $\text{Aut}(\mathbb{P}^m)$ that every element $\varphi$ of $\text{Aut}(\mathbb{P}^m)$ is uniquely determined by $m+1$ points $p_1, \cdots, p_{m+1}$ in general position and $m+1$ points $q_1, \cdots, q_{m+1}$ in general position such that $\varphi(p_j) = q_j$ for $1 \leq j \leq m+1$.

Remark The quotient space $H(X, \mathbb{P}^m)/\text{Aut}(\mathbb{P}^m)$ can be regarded as the set of linear systems of dimension $m$ without base point on $X$.

3. Moduli of holomorphic mappings from compact Riemann surfaces

We solved the moduli problem of holomorphic mappings of compact Riemann surfaces of genus greater than 0 into $\mathbb{P}^n$ in Namba [4].

We constructed the moduli space as follows: Let $T$ be the Teichmüller space of compact Riemann surfaces of genus $g \geq 2$ and $X = \{X_t\}_{t \in \Gamma}$ be the Teichmüller family. Let $\Gamma$ be the Teichmüller modular group. Then $\Gamma$ acts properly discontinuously on both $T X$. Let $H_d^m = \bigcup_t H_d(X_t, \mathbb{P}^m)$ be the relative Douady space of linearly non-degenerate holomorphic mappings of $X_t$ for some $t$ into $\mathbb{P}^m$ of degree $d$. Here the degree of a non-degenerate holomorphic mapping of a compact Riemann surface $X_t$ into $\mathbb{P}^m$ is by definition $\deg[f : X_t \longrightarrow f(X_t)] = \deg[f(X_t)]$.

Theorem 5 (Namba [4]) $\text{Aut}(\mathbb{P}^m) \times \Gamma$ acts properly on $H_d^m$. Hence $M_d^m = H_d^m/(\text{Aut}(\mathbb{P}^m) \times \Gamma)$ is a complex space. If $m = 1$, then $M_d^m$ is a normal complex space of dimension $2d + 2g - 5$. 
The complex space $\mathcal{M}_d^m$ is nothing but the moduli space of non-degenerate holomorphic mappings of degree $d$ of compact Riemann surface of genus $g$ into $\mathbb{P}^m$.

The case $g = 1$ can be treated in a similar way and the moduli space of maps can be constructed using the theory of elliptic functions. In particular $\mathcal{M}_d^1$ for $m = 1$ is a normal complex space of dimension $2d - 3$.

4. Moduli of rational functions

In this lecture, we give some recent results on the case $g = 0$, that is, some results on the moduli problems of linearly non-degenerate holomorphic mappings from the complex projective line $\mathbb{P}^1$ into $\mathbb{P}^m$.

A linearly non-degenerate holomorphic mapping of $\mathbb{P}^1$ to $\mathbb{P}^1$ is nothing but a non-constant rational function. A rational function $f$ of degree $d$ can be expressed as follows:

$$f(z) = \frac{a_0 z^d + \cdots + a_d}{b_0 z^d + \cdots + b_d} \quad (a_0 \neq 0 \text{ or } b_0 \neq 0),$$

where the denominator and the numerator do not have a common root. Hence the set of all rational functions of degree $d$ can be identified with the Zariski open set

$$H_d(\mathbb{P}^1, \mathbb{P}^1) = \{(a_0 : \cdots : a_d : b_0 : \cdots : b_d) \} = \mathbb{P}^{2d+1} \setminus R$$

of $\mathbb{P}^{2d+1}$, where $R$ is the zero locus of the resultant of the denominator and the numerator. The moduli problem in this case asks when there is a natural complex space structure (or an algebraic structure) on $H_d(\mathbb{P}^1, \mathbb{P}^1)/G$, where $G = \text{Aut}(\mathbb{P}^1) \times \text{Aut}(\mathbb{P}^1)$ acting on $H_d(\mathbb{P}^1, \mathbb{P}^1)$ by the composition of mappings as follows:

$$(\varphi, \psi, f) \in G \times H_d(\mathbb{P}^1, \mathbb{P}^1) \mapsto \varphi \circ f \circ \psi^{-1} \in H_d(\mathbb{P}^1, \mathbb{P}^1).$$
But this action is not proper:

**Example 1**  Put $f(z) = z^3 - 3tz \ (t \in \mathbb{C}).$ Then $f_t \ (t \neq 0)$ is equivalent to $f_1,$ for $f_t(z) = a(u^3 - 3u)$ where $u = z/\sqrt{t}$ and $a = (\sqrt{t})^3,$ while $f_0$ is not equivalent to $f_1.$

**Example 2**  Let $P(z) = a_0 z^d + \cdots + a_d$ be any polynomial of degree $d$ such that $P(-n) \neq 0$ for $n = 1, 2, \cdots.$ Put

$$f_n(z) = \frac{P(z)}{(1/n)z + 1}$$

$$g_n(u) = \frac{u^d + (a_i/a_0 n^i)u^{d-i} + \cdots + (a_d/a_0 n^d)}{u + 1}$$

Then $f_n$ converges to $P(z)$ and $g_n(u)$ converges to

$$g(u) = \frac{u^d}{u + 1}$$

as $n \to \infty.$ Note that $f_n$ and $g_n$ are equivalent, for

$$f_n = \phi_n \circ q_n \circ \psi_n^{-1}$$

where $\phi_n(w) = a_0 n^d w$ and $\psi_n(u) = n u.$ But $g$ is not equivalent to $P$ for a general $P.$

Now by the Riemann–Hurwitz formula for the rational function $f$ as a branched covering from $\mathbb{P}^1$ onto $\mathbb{P}^1$ of degree $d,$

$$\sum_{P \in R_f} (e_P - 1) = 2d - 2,$$

where the summation runs over the set $R_f$ of all ramification points
and $e_p$ is the ramification index at the ramification point $p$. Put

$$H_{d,k} = H_d^k(P^1, P^1) = \{ f \in H_d(P^1, P^1) \mid \text{there is a ramification point } p \text{ such that } e_p \geq k \}.$$  

Then $H_{d,k}$ is a closed algebraic set of $H_d = H_d(P^1, P^1)$.

**Theorem 6.** Let $d \geq 3$. Then $G = \text{Aut}(P^1) \times \text{Aut}(P^1)$ acts properly on $H_d - H_{d,d}$ such that every stabilizer is finite. Hence the quotient space $(H_d - H_{d,d})/G$ is an algebraic space of finite type.

The quotient space $(H_d - H_{d,d})/G$ can be regarded as the moduli space of the rational functions of degree $d$. For the proof of Theorem 6, we use the following lemma in combinatorics:

**Lemma** Let $m$ be an integer greater than or equal to 3 and let $A$ and $B$ be finite sets. Suppose that $F$ and $G$ be surjective mappings of the set $\{1, \ldots, n\}$ onto the sets $A$ and $B$ respectively such that (i) for every point $\alpha$ in $A$, the number of the points $F^{-1}(\alpha)$ is less than $n/2$ and (ii) for every point $\beta$ in $B$, the number of the points $G^{-1}(\beta)$ is less than $n/2$. Then there are distinct 3 numbers $a, b, c$ in $\{1, \ldots, n\}$ such that (1) $F(a), F(b)$ and $F(c)$ are distinct and (2) $G(a), G(b)$ and $G(c)$ are distinct.

5. **Moduli of plane rational curves**

Put

$$B_d = B_d(P^1, P^2) = \{ f : P^1 \rightarrow P^2 \mid f \text{ is a birational holomorphic mappings of } P^1 \text{ onto the image curve } C = f(P^1) \text{ of degree } d \}$$

Then $B_d$ is an Zariski open set of $H_d(P^1, P^2)$ and $G = \text{Aut}(P^2) \times \text{Aut}(P^1)$ acts on $B_d$ as the composition of mappings.
By the genus formula for the rational curve $C = f(\mathbb{P}^4)$,

$$\sum_{p \in \text{Sing} C} \delta_p = (d-1)(d-2)/2,$$

where the summation runs over the singular locus $\text{Sing} (C)$ of the curve $C$ and

$$\delta_p = \dim_{\mathbb{C}} (\hat{O}_p / \Theta_p) = \frac{\mu + r - 1}{2}$$

($\hat{O}_p$ is the integral closure of the ring $\Theta_p$ of germs of holomorphic functions on $C$, $r$ is the number of branches of $C$ at $p$, $\mu$ is the Milnor number.)

Put

$$B_{d, k} = \{ f \in B_d \mid \text{there is a point } p \in \text{Sing} f(\mathbb{P}^1) \text{ such that } \delta_p \geq k \}.$$

Then $B_{d, k}$ is a closed algebraic set of $B_d$.

**Theorem 7** Let $d \geq 4$. Put $l = (d-1)(d-2)/4$. Then $G = \text{Aut} (\mathbb{P}^2) \times \text{Aut} (\mathbb{P}^1)$ acts properly on

$$B = B_d - B_{d, l}$$

such that every stabilizer is finite. Hence the quotient space $B/G$ is an algebraic space of finite type.

Since $B/G$ can be written as $B/G = (B/\text{Aut} (\mathbb{P}^1))/\text{Aut} (\mathbb{P}^2)$, this can be regarded as the moduli space of rational plane curves of degree $d$.

**Remark** Theorem 6 and Theorem 7 can be generalized. But we do not discuss it here.
References


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