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Moduli of Rational Functions and Rational Plane Curves

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1. Introduction

Holomorphic mappings $f : X \longrightarrow M$ and $f' : X' \longrightarrow M'$ of complex spaces are said to be equivalent if there are biholomorphic mapping $\psi : X \longrightarrow X'$ and $\varphi : M \longrightarrow M'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ M \arrow[r, "\varphi"] & M' \end{tikzcd}$$

If it is possible to introduce a natural complex space structure on the set of equivalence classes (for a given type of mappings), then we may call it the moduli space of holomorphic mappings (for the given type).

It is a difficult problem in general to show the existence of moduli space.

In this lecture, we mainly consider the case in which X and X' are compcat Riemann surfaces and M and M' are the m-dimensional complex projective space $\mathbb{P}^m = \mathbb{P}^m(\mathbb{C})$.

2. Linearly non-degenerate holomorphic mappings

Some of our results may be rewritten in terms of stability (see Mumford [3]). But it is difficult to analyze the stability. Our discussion is topological and complex analytic. In particular we make use of the following 2 theorems in order to prove our theorems.

<u>Theorem 1</u> (Holmann [2]) Let X be a (resp. normal) complex space and G be a complex Lie group acting properly on X. Then the the quotient space X/G is a (resp. normal) complex space and the projection $\pi : X \longrightarrow X/G$ is holomorphic. If moreover G acts on X without fixed point, then $\pi : X \longrightarrow X/G$ is a principal G-bundle.

<u>Theorem 2</u> (Popp [5]) Let X be a quasi-projective $\mathbb{C}$-scheme and G be an algebraic group acting properly on X. Assume that every stabilizer is a finite group. Then the quotient space X/G is an algebraic space.

In the above 2 theorems, "G acts properly on X" means that the following mapping is proper, that is, the inverse image of every compact set is compact:

$$(\psi, p) \in G \times X \longmapsto (\psi(p), p) \in X \times X.$$

Now, for a compact complex space X, we put

$$H(X, \mathbb{P}^m) = \{ f : X \longrightarrow \mathbb{P}^m \mid f(X) \text{ is not contained in any hyperplane, that is, } f \text{ is linearly non-degenerate} \}$$

Then $H(X, \mathbb{P}^m)$ is a complex space (so called the Douady space , Douady [1]), whose underlying topology is the compact-open topology. $\mathrm{Aut}(\mathbb{P}^m)$ acts on $H(X, \mathbb{P}^m)$ as the composition of mappings:

$$(\varphi, f) \longmapsto \varphi \circ f$$

<u>Theorem 3</u> $\mathrm{Aut}(\mathbb{P}^m)$ acts on $H(X, \mathbb{P}^m)$ properly without fixed point. Hence $H(X, \mathbb{P}^m)/\mathrm{Aut}(\mathbb{P}^m)$ is a complex space and $H(X, \mathbb{P}^m) \longrightarrow H(X, \mathbb{P}^m)/\mathrm{Aut}(\mathbb{P}^m)$ is a principal $\mathrm{Aut}(\mathbb{P}^m)$-bundle.

Theorem 4 Let $\{X_t\}_{t\in T}$ be a family of compact complex spaces with the parameter space a connected complex space T. Then $\mathrm{Aut}(\mathbb{P}^m)$ acts properly without fixed point on $H = \bigcup_t H(X_t, \mathbb{P}^m)$. Hence $H/\mathrm{Aut}(\mathbb{P}^m)$ is a complex space and $H \longrightarrow H/\mathrm{Aut}(\mathbb{P}^m)$ is a principal $\mathrm{Aut}(\mathbb{P}^m)$-bundle.

Here H is the relative Douady space (see Pourcin [6]). The proof of Theorems 3 and 4 can be done by taking a sequence of points and using the property of $\mathrm{Aut}(\mathbb{P}^m)$ that every element φ of $\mathrm{Aut}(\mathbb{P}^m)$ is uniquely determined by $m+1$ points $p_1, \cdots, p_{m+1}$ in general position and $m+1$ points $q_1, \cdots, q_{m+1}$ in general position such that $\varphi(p_j) = q_j$ for $1 \leq j \leq m+1$.

Remark The quotient space $H(X, \mathbb{P}^m)/\mathrm{Aut}(\mathbb{P}^m)$ can be regarded as the set of linear systems of dimension m without base point on X.

3. Moduli of holomorphic mappings from compact Riemann surfaces

We solved the moduli problem of holomorphic mappings of compact Riemann surfaces of genus greater than 0 into $\mathbb{P}^m$ in Namba [4].

We constructed the moduli space as follows: Let T be the Teichmüller space of compact Riemann surfaces of genus g $(g \geq 2)$ and $X = \{X_t\}_{t\in T}$ be the Teichmüller family. Let Γ be the Teichmüller modular group. Then Γ acts properly discontinuously on both T X. Let $H_d^m = \bigcup_t H_d(X_t, \mathbb{P}^m)$ be the relative Douady space of linearly non-degenerate holomorphic mappings of X_t for some t into $\mathbb{P}^m$ of degree d. Here the degree of a non-degenerate holomorphic mapping of a compact Riemann surface X_t into $\mathbb{P}^m$ is by definition $\deg[f : X_t \to f(X_t)]\deg[f(X_t)]$.

Theorem 5 (Namba [4]) $\mathrm{Aut}(\mathbb{P}^m)\times\Gamma$ acts properly on H_d^m. Hence $M_d^m = H_d^m/(\mathrm{Aut}(\mathbb{P}^m)\times\Gamma)$ is a complex space. If $m = 1$, then M_d^1 is a normal complex space of dimension $2d + 2g - 5$.

The complex space M_d^m is nothing but the moduli space of non-degenerate holomorphic mappings of degree d of compact Riemann surface of genus g into $\mathbb{P}^m$.

The case $g = 1$ can be treated in a similar way and the moduli space space can be constructed using the theory of elliptic functions. In particular M_d^1 for $m = 1$ is a normal complex space of dimension $2d - 3$.

4. Moduli of rational functions

In this lecture, we give some recent results on the case $g = 0$, that is some results on the moduli problems of linearly non-degenerate holomorphic mappings from the complex projective line $\mathbb{P}^1$ into $\mathbb{P}^m$.

A linearly non-degenerate holomorphic mapping of $\mathbb{P}^1$ to $\mathbb{P}^1$ is nothing but a non-constant rational function. A rational function f of degree d can be expressed as follows:

$$f(z) = \frac{a_0 z^d + \cdots + a_d}{b_0 z^d + \cdots + b_d} \qquad (a_0 \neq 0 \ \text{ or } \ b_0 \neq 0),$$

where the denominator and the numerator do not have a common root. Hence the set of all rational functions of degree d can be identified with the Zariski open set

$$H_d(\mathbb{P}^1, \mathbb{P}^1) = \{(a_0 : \cdots : a_d : b_0 : \cdots : b_d)\} = \mathbb{P}^{2d+1} - R$$

of $\mathbb{P}^{2d+1}$, where R is the zero locus of the resultant of the donominator and the numerator.The moduli problem in this case asks when there is a natural complex space structure (or an algebraic structure) on $H_d(\mathbb{P}^1, \mathbb{P}^1)/G$, where $G = \mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$ acting on $H_d(\mathbb{P}^1, \mathbb{P}^1)$ by the composition of mappings as follows:

$$(\varphi, \psi, f) \in G \times H_d(\mathbb{P}^1, \mathbb{P}^1) \longmapsto \varphi \circ f \circ \psi^{-1} \in H_d(\mathbb{P}^1, \mathbb{P}^1).$$

But this action is not proper:

Example 1 Put $f(z) = z^3 - 3tz$ $(t \in \mathbb{C})$. Then f_t $(t \neq 0)$ is equivalent to f_1, for $f_t(z) = a(u^3 - 3u)$ where $u = z/\sqrt{t}$ and $a = (\sqrt{t}\,)^3$, while f_0 is not equivalent to f_1.

Example 2 Let $P(z) = a_0 z^d + \cdots + a_d$ be any polynomial of degree d such that $P(-n) \neq 0$ for $n = 1, 2, \cdots$. Put

$$f_n(z) = \frac{P(z)}{(1/n)z + 1}$$

$$g_n(u) = \frac{u^d + (a_1/a_0 n)u^{d-1} + \cdots + (a_d/a_0 n^d)}{u + 1}$$

Then f_n converges to $P(z)$ and $g_n(u)$ converges to

$$g(u) = \frac{u^d}{u+1}$$

as $n \longrightarrow \infty$. Note that f_n and g_n are equivalent, for

$$f_n = \varphi_n \circ g_n \circ \psi_n^{-1}$$

where $\varphi_n(w) = a_0 n^d w$ and $\psi_n(u) = n\,u$. But g is not equivalent to P for a general P.

Now by the Riemann-Hurwitz formula for the rational function f as a branched covering from $\mathbb{P}^1$ onto $\mathbb{P}^1$ of degree d,

$$\sum_{p \in R_f} (e_p - 1) = 2d - 2,$$

where the summation runs over the set R_f of all ramification points and e_p is the ramification index at the ramification point p. Put

$$H_{d,k} = H_{d,k}(\mathbb{P}^1, \mathbb{P}^1) = \{ f \in H_d(\mathbb{P}^1, \mathbb{P}^1) \mid \text{there is a ramification point } p \text{ such that } e_p \geq k \}.$$

Then $H_{d,k}$ is a closed algebraic set of $H_d = H_d(\mathbb{P}^1, \mathbb{P}^1)$.

Theorem 6 Let $d \geq 3$. Then $G = \mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$ acts properly on $H_d - H_{d,d}$ such that every stabilizer is finite. Hence the quotient space $(H_d - H_{d,d})/G$ is an algebraic space of finite type.

The quotient space $(H_d - H_{d,d})/G$ can be regarded as the moduli space of the rational functions of degree d. For the proof of Theorem 6, we use the following lemma in combinatorics:

Lemma Let m be an integer greater than or equal 3 and let A and B be finite sets. Suppose that F and G be surjective mappings of the set $\{1, \cdots, n\}$ onto the sets A and B respectively such that (i) for every point α in A, the number of the points $F^{-1}(\alpha)$ is less than $n/2$ and (ii) for every point β in B, the number of the points $G^{-1}(\beta)$ is less than $n/2$. Then there are distinct 3 numbers a, b, c in $\{1, \cdots, n\}$ such that (1) $F(a)$, $F(b)$ and $F(c)$ are distinct and (2) $G(a)$, $G(b)$ and $G(c)$ are distinct.

5. Moduli of plane rational curves

Put

$$B_d = B_d(\mathbb{P}^1, \mathbb{P}^2) = \{ f : \mathbb{P} \longrightarrow \mathbb{P}^2 \mid f \text{ is a birational holomorphic mappings of } \mathbb{P}^1 \text{ onto the image curve } C = f(\mathbb{P}^1) \text{ of degree } d \}$$

Then B_d is an Zariski open set of $H_d(\mathbb{P}^1, \mathbb{P}^2)$ and $G = \mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^1)$ acts on B_d as the composition of mappings.

By the genus formula for the rational curve $C = f(\mathbb{P}^1)$,

$$\sum_{p \in \operatorname{Sing} C} \delta_p = (d-1)(d-2)/2,$$

where the summation runs over the singular locus Sing (C) of the curve C and

$$\delta_p = \dim_{\mathbb{C}} (\hat{\mathcal{O}}_p / \mathcal{O}_p) = \frac{\mu + r - 1}{2}$$

($\hat{\mathcal{O}}_p$ is the integral closure of the ring $\mathcal{O}_p$ of germs of holomorphic functions on C. r is the number of branches of C at p. μ is the Milnor number.)

Put

$$B_{d,k} = \{ f \in B_d \mid \text{there is a point } p \text{ in } \operatorname{Sing} f(\mathbb{P}^1) \text{ such that } \delta_p \geq k \}.$$

Then $B_{d,k}$ is a closed algebraic set of B_d.

<u>Theorem 7</u> Let $d \geq 4$. Put $\ell = (d-1)(d-2)/4$. Then $G = \operatorname{Aut}(\mathbb{P}^2) \times \operatorname{Aut}(\mathbb{P}^1)$ acts properly on

$$B = B_d - B_{d,\ell}$$

such that every stabilizer is finite. Hence the quotient space B/G is an algebraic space of finite type.

Since B/G can be written as $B/G = (B/\operatorname{Aut}(\mathbb{P}^1))/\operatorname{Aut}(\mathbb{P}^2)$, this can be regarded as the moduli space of rational plane curves of degree d.

<u>Remark</u> Theorem 6 and Theorem 7 can be generalized. But we do not discuss it here.

References

[1] A. Douady : Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytiques donne. Ann. Inst. Fourier, Grenoble 16 (1966), 1-98.

[2] H. Holmann : Quotienten komplexer Räume. Math. Ann. 142 (1961), 407-440.

[3] D. Mumford et al. : Geometric Invariant Theory. Springer-Verlag, 3rd Ed. 1994.

[4] M. Namba : Families of Meromorphic Functions on Compact Riemann surfaces. Springer Lec. Notes 767 (1979).

[5] H. Popp : On moduli of algebraic varieties. Compos. Math. 28 (1974), 51-81.

[6] G. Pourcin : Théorème de Douady au-dessus de S. Ann. Scuola Norm.Sup. Pisa 23 (1969), 451-459.

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