Moduli of Rational Functions and Rational Plane Curves

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1. Introduction

Holomorphic mappings \( f : X \rightarrow M \) and \( f' : X' \rightarrow M' \) of complex spaces are said to be equivalent if there are biholomorphic mapping \( \psi : X \rightarrow X' \) and \( \phi : M \rightarrow M' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\psi} & X' \\
\downarrow f & & \downarrow f' \\
M & \xrightarrow{\phi} & M'
\end{array}
\]

If it is possible to introduce a natural complex space structure on the set of equivalence classes (for a given type of mappings), then we may call it the moduli space of holomorphic mappings (for the given type).

It is a difficult problem in general to show the existence of moduli space. In this lecture, we mainly consider the case in which \( X \) and \( X' \) are compact Riemann surfaces and \( M \) and \( M' \) are the \( m \)-dimensional complex projective space \( \mathbb{P}^n = \mathbb{P}^n(\mathbb{C}) \).

2. Linearly non-degenerate holomorphic mappings

Some of our results may be rewritten in terms of stability (see Mumford [3]). But it is difficult to analyze the stability. Our discussion is topological and complex analytic. In particular we make use of the following 2 theorems in order to prove our theorems.
Theorem 1 (Holmann [2]) Let $X$ be a (resp. normal) complex space and $G$ be a complex Lie group acting properly on $X$. Then the quotient space $X/G$ is a (resp. normal) complex space and the projection $\pi : X \rightarrow X/G$ is holomorphic. If moreover $G$ acts on $X$ without fixed point, then $\pi : X \rightarrow X/G$ is a principal $G$-bundle.

Theorem 2 (Popp [5]) Let $X$ be a quasi-projective $\mathbb{C}$-scheme and $G$ be an algebraic group acting properly on $X$. Assume that every stabilizer is a finite group. Then the quotient space $X/G$ is an algebraic space.

In the above 2 theorems, "$G$ acts properly on $X$" means that the following mapping is proper, that is, the inverse image of every compact set is compact:

$$(\varphi, p) \in G \times X \mapsto (\varphi(p), p) \in X \times X.$$ 

Now, for a compact complex space $X$, we put

$$H(X, \mathbb{P}^m) = \{ f : X \rightarrow \mathbb{P}^m \mid f(X) \text{ is not contained in any hyperplane, that is, } f \text{ is linearly non-degenerate} \}$$

Then $H(X, \mathbb{P}^m)$ is a complex space (so called the Douady space, Douady [1]), whose underlying topology is the compact-open topology. $\text{Aut}(\mathbb{P}^m)$ acts on $H(X, \mathbb{P}^m)$ as the composition of mappings:

$$(\varphi, f) \mapsto \varphi \circ f$$

Theorem 3 $\text{Aut}(\mathbb{P}^m)$ acts on $H(X, \mathbb{P}^m)$ properly without fixed point. Hence $H(X, \mathbb{P}^m)/\text{Aut}(\mathbb{P}^m)$ is a complex space and $H(X, \mathbb{P}^m) \rightarrow H(X, \mathbb{P}^m)/\text{Aut}(\mathbb{P}^m)$ is a principal $\text{Aut}(\mathbb{P}^m)$-bundle.
Theorem 4. Let \( \{ X_t \}_{t \in \mathbb{T}} \) be a family of compact complex spaces with the parameter space a connected complex space \( T \). Then \( \text{Aut}(\mathbb{P}^m) \) acts properly without fixed point on \( H = \bigcup_t H(X_t, \mathbb{P}^m) \). Hence \( H/\text{Aut}(\mathbb{P}^m) \) is a complex space and \( H \longrightarrow H/\text{Aut}(\mathbb{P}^m) \) is a principal \( \text{Aut}(\mathbb{P}^m) \)-bundle.

Here \( H \) is the relative Douady space (see Pourcin [6]). The proof of Theorems 3 and 4 can be done by taking a sequence of points and using the property of \( \text{Aut}(\mathbb{P}^m) \) that every element \( \varphi \) of \( \text{Aut}(\mathbb{P}^m) \) is uniquely determined by \( m+1 \) points \( p_1, \ldots, p_{m+1} \) in general position and \( m+1 \) points \( q_1, \ldots, q_{m+1} \) in general position such that \( \varphi(p_j) = q_j \) for \( 1 \leq j \leq m+1 \).

Remark. The quotient space \( H(\mathbb{X}, \mathbb{P}^m)/\text{Aut}(\mathbb{P}^m) \) can be regarded as the set of linear systems of dimension \( m \) without base point on \( \mathbb{X} \).

3. Moduli of holomorphic mappings from compact Riemann surfaces

We solved the moduli problem of holomorphic mappings of compact Riemann surfaces of genus greater than 0 into \( \mathbb{P}^m \) in Namba [4].

We constructed the moduli space as follows: Let \( T \) be the Teichmüller space of compact Riemann surfaces of genus \( g \) \( (g \geq 2) \) and \( X = \{ X_t \}_{t \in \mathbb{T}} \) be the Teichmüller family. Let \( \Gamma \) be the Teichmüller modular group. Then \( \Gamma \) acts properly discontinuously on both \( T \) \( X \). Let \( H_d^m = \bigcup_t H_d(X_t, \mathbb{P}^m) \) be the relative Douady space of linearly non-degenerate holomorphic mappings of \( X_t \) for some \( t \) into \( \mathbb{P}^m \) of degree \( d \). Here the degree of a non-degenerate holomorphic mapping of a compact Riemann surface \( X_t \) into \( \mathbb{P}^m \) is by definition \( \text{deg} \left[ f : X_t \longrightarrow f(X_t) \right] \).

Theorem 5 (Namba [4]) \( \text{Aut}(\mathbb{P}^m) \times \Gamma \) acts properly on \( H_d^m \). Hence \( M_d^m = H_d^m/(\text{Aut}(\mathbb{P}^m) \times \Gamma) \) is a complex space. If \( m = 1 \), then \( M_d^1 \) is a normal complex space of dimension \( 2d + 2g - 5 \).
The complex space $M_d^m$ is nothing but the moduli space of non-degenerate holomorphic mappings of degree $d$ of compact Riemann surface of genus $g$ into $P^m$.

The case $g = 1$ can be treated in a similar way and the moduli space space can be constructed using the theory of elliptic functions. In particular $M_d^1$ for $m = 1$ is a normal complex space of dimension $2d - 3$.

4. Moduli of rational functions

In this lecture, we give some recent results on the case $g = 0$, that is some results on the moduli problems of linearly non-degenerate holomorphic mappings from the complex projective line $P^1$ into $P^m$.

A linearly non-degenerate holomorphic mapping of $P^1$ to $P^1$ is nothing but a non-constant rational function. A rational function $f$ of degree $d$ can be expressed as follows:

$$f(z) = \frac{a_0 z^d + \cdots + a_d}{b_0 z^d + \cdots + b_d} \quad (a_0 \neq 0 \text{ or } b_0 \neq 0),$$

where the denominator and the numerator do not have a common root. Hence the set of all rational functions of degree $d$ can be identified with the Zariski open set

$$H_d(P^1, P^1) = \left \{ (a_0: \cdots : a_d: b_0: \cdots : b_d) \right \} = P^{2d+1} - R$$

of $P^{2d+1}$, where $R$ is the zero locus of the resultant of the denominator and the numerator. The moduli problem in this case asks when there is a natural complex space structure (or an algebraic structure) on $H_d(P^1, P^1)/G$, where $G = \text{Aut}(P^1) \times \text{Aut}(P^1)$ acting on $H_d(P^1, P^1)$ by the composition of mappings as follows:

$$(\varphi, \psi, f) \in G \times H_d(P^1, P^1) \quad \longrightarrow \quad \varphi \circ f \circ \psi^{-1} \in H_d(P^1, P^1).$$
But this action is not proper:

**Example 1.** Put $f(z) = z^3 - 3tz$ ($t \in \mathbb{C}$). Then $f_t$ ($t \neq 0$) is equivalent to $f_1$, for $f_t(z) = a(u^3 - 3u)$ where $u = z/\sqrt{t}$ and $a = (\sqrt{t})^3$, while $f_0$ is not equivalent to $f_1$.

**Example 2.** Let $P(z) = a_0z^d + \cdots + a_d$ be any polynomial of degree $d$ such that $P(-n) \neq 0$ for $n = 1, 2, \cdots$. Put

$$f_n(z) = \frac{P(z)}{(1/n)z + 1}$$

$$g_n(u) = \frac{u^d + (a_1/a_0n)u^{d-1} + \cdots + (a_d/a_0n^d)}{u + 1}$$

Then $f_n$ converges to $P(z)$ and $g_n(u)$ converges to

$$g(u) = \frac{u^d}{u + 1}$$

as $n \to \infty$. Note that $f_n$ and $g_n$ are equivalent, for

$$f_n = \varphi_n \circ \delta_n \circ \psi_n^{-1}$$

where $\varphi_n(w) = a_0n^d w$ and $\psi_n(u) = nu$. But $g$ is not equivalent to $P$ for a general $P$.

Now by the Riemann–Hurwitz formula for the rational function $f$ as a branched covering from $\mathbb{P}^1$ onto $\mathbb{P}^1$ of degree $d$,

$$\sum_{P \in R_f} (e_P - 1) = 2d - 2,$$

where the summation runs over the set $R_f$ of all ramification points.
and \( e_p \) is the ramification index at the ramification point \( p \). Put

\[
H_{d,k} = H_{d,k}(\mathbb{P}^1, \mathbb{P}^1) = \{ f \in H_d(\mathbb{P}^1, \mathbb{P}^1) \mid \text{there is a ramification point } p \text{ such that } e_p \geq k \}.
\]

Then \( H_{d,k} \) is a closed algebraic set of \( H_d = H_d(\mathbb{P}^1, \mathbb{P}^1) \).

**Theorem 6.** Let \( d \geq 3 \). Then \( G = \text{Aut}(\mathbb{P}^1) \times \text{Aut}(\mathbb{P}^1) \) acts properly on \( H_d - H_{d,d} \) such that every stabilizer is finite. Hence the quotient space \( (H_d - H_{d,d})/G \) is an algebraic space of finite type.

The quotient space \( (H_d - H_{d,d})/G \) can be regarded as the moduli space of the rational functions of degree \( d \). For the proof of Theorem 6, we use the following lemma in combinatorics:

**Lemma** Let \( m \) be an integer greater than or equal to 3 and let \( A \) and \( B \) be finite sets. Suppose that \( F \) and \( G \) be surjective mappings of the set \( \{1, \ldots, n\} \) onto the sets \( A \) and \( B \) respectively such that (i) for every point \( \alpha \) in \( A \), the number of the points \( F^{-1}(\alpha) \) is less than \( n/2 \) and (ii) for every point \( \beta \) in \( B \), the number of the points \( G^{-1}(\beta) \) is less than \( n/2 \). Then there are distinct 3 numbers \( a, b, c \) in \( \{1, \ldots, n\} \) such that (1) \( F(a), F(b) \) and \( F(c) \) are distinct and (2) \( G(a), G(b) \) and \( G(c) \) are distinct.

5. **Moduli of plane rational curves**

Put

\[
B_d = B_d(\mathbb{P}^1, \mathbb{P}^2) = \{ f: \mathbb{P}^1 \to \mathbb{P}^2 \mid f \text{ is a birational holomorphic mappings of } \mathbb{P}^1 \text{ onto the image curve } C = f(\mathbb{P}^1) \text{ of degree } d \}
\]

Then \( B_d \) is an Zariski open set of \( H_d(\mathbb{P}^1, \mathbb{P}^2) \) and \( G = \text{Aut}(\mathbb{P}^2) \times \text{Aut}(\mathbb{P}^1) \) acts on \( B_d \) as the composition of mappings.
By the genus formula for the rational curve $C = f(\mathbb{P}^4)$,

$$\sum_{p \in \text{Sing} C} \delta_p = (d-1)(d-2)/2,$$

where the summation runs over the singular locus $\text{Sing} (C)$ of the curve $C$ and

$$\delta_p = \dim \mathbb{C}(\mathcal{O}_p / \mathcal{O}_C) = \frac{\mu + r - 1}{2}$$

($\mathcal{O}_p$ is the integral closure of the ring $\mathcal{O}_p$ of germs of holomorphic functions on $C$. $r$ is the number of branches of $C$ at $p$. $\mu$ is the Milnor number.)

Put

$$B_{d,k} = \{ f \in B_d \mid \text{there is a point } p \in \text{Sing} f(\mathbb{P}^1) \text{ such that } \delta_p \geq k \}.$$

Then $B_{d,k}$ is a closed algebraic set of $B_d$.

**Theorem 7** Let $d \geq 4$. Put $l = (d-1)(d-2)/4$. Then $G = \text{Aut} (\mathbb{P}^2) \times \text{Aut} (\mathbb{P}^1)$ acts properly on

$$B = B_d - B_{d,l}$$

such that every stabilizer is finite. Hence the quotient space $B/G$ is an algebraic space of finite type.

Since $B/G$ can be written as $B/G = (B/\text{Aut} (\mathbb{P}^1))/\text{Aut} (\mathbb{P}^2)$, this can be regarded as the moduli space of rational plane curves of degree $d$.

**Remark** Theorem 6 and Theorem 7 can be generalized. But we do not discuss it here.
References


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