Problem of Fenchel on the complex projective plane and representations of the 3rd braid group (Fundamental Groups and Algebraic Functions)

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Problem of Fenchel on the complex projective plane and representations of the 3rd braid group

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1 Abstract

We denote by $\mathbb{P}^2$ the complex projective plane. Let $C = \{(X_0 : X_1 : X_2) \in \mathbb{P}^2 | X_2X_0^2 - X_1^3 = 0\}$ be a curve of $\mathbb{P}^2$. Let $L_\infty = \{(X_0 : X_1 : X_2) \in \mathbb{P}^2 | X_2 = 0\}$ be the line of $\mathbb{P}^2$, which we call line at infinity. $C$ is a rational curve of degree 3 with a cusp at $(0 : 0 : 1)$. $C$ and $L_\infty$ are tangent at $(1 : 0 : 0)$. Let $e_1$, $e_2$ be positive integers greater than 1. Put $D = e_1C + e_2L_\infty$. We consider the following problem and give here a partial answer by constructing representations of the 3rd braid group.

Fenchel's Problem Give a condition on the pair $(e_1, e_2)$ for the existence of a finite Galois covering $\pi : X \rightarrow \mathbb{P}^2$ which branches at $D$.

2 Elementary facts

We choose a point $p_0 \in \mathbb{P}^2 - \{C \cup L_\infty\}$ and fix it. The fundamental group $\pi_1(\mathbb{P}^2 - \{C \cup L_\infty\}, p_0)$ is isomorphic to $< \alpha, \beta, \delta | \alpha\beta\alpha = \beta\alpha\beta = \delta^{-1} >$ the 3rd braid group. This group is isomorphic to $< \gamma, \delta | \gamma^3 = \delta^2 >$. This isomorphism is given by $\gamma \mapsto (\alpha\beta)^{-1}, \delta \mapsto (\alpha\beta\alpha)^{-1}$. We identify $\alpha$ (resp. $\beta$, resp. $\delta$) with a closed path in $\mathbb{P}^2 - \{C \cup L_\infty\}$ which rounds counterclockwise direction once around non-singular points $P_\alpha$ of $C$ (resp. $P_\beta$ of $C$, resp. $P_\delta$ of $L_\infty$). Let $J$ be the smallest normal subgroup of $\pi_1(\mathbb{P}^2 - \{C \cup L_\infty\}, p_0)$ which contains $\alpha^{e_1}$ and $\delta^{e_2}$. There is a finite Galois covering which branches at $D$ if and only if there is a normal subgroup $K$ of $\pi_1(\mathbb{P}^2 - \{C \cup L_\infty\}, p_0)$ of finite index with $J \subset K$, which satisfies the following conditions: (1) If $\alpha^k \in K$ then $k \equiv 0 \pmod{e_1}$ and (2) If $\delta^l \in K$ then $l \equiv 0 \pmod{e_2}$.

Let $G$ be a finite group generated by two elements $A$, $B$, which satisfy the relation $ABA = BAB, A^{e_1} = B^{e_1} = 1, (ABA)^{e_2} = 1$. Obviously $A$ and $B$ are conjugate to each other. If there is a finite group $G$ as above, we have a surjective homomorphism $\Phi : \pi_1(\mathbb{P}^2 - \{C \cup L_\infty\}, p_0) \rightarrow G$. Then the kernel of $\Phi$ corresponds to a finite Galois covering $\pi : X \rightarrow \mathbb{P}^2$ which branches at $D$.

Put $Q = ABA$. It is easy to see:

Lemma 2.1 If $G$ is abelian, then $G$ is a cyclic group.

Since $Q^2$ is an element of the center of $G$,

Lemma 2.2 If the order of $Q$ is odd, then $G$ is abelian (G is a cyclic group).

Hence we have:

Theorem 2.1 If $e_2$ is odd, then any covering $\pi : X \rightarrow \mathbb{P}^2$ which branches at $D$ is cyclic.

Trivially we have:
Proposition 2.1 For given odd number \(e_2\), if \(e_2 \equiv 0 \pmod{3}\) put \(e_1 = e_2/3\), otherwise put \(e_1 = e_2\). Then there exists \(\pi : X \to \mathbb{P}^2\) which branches at \(D\).

It is well-known (see for example [1]):

Lemma 2.3 For given positive integer \(n\) there is a finite group \(G\) generated by two elements \(Q\) of order 2 and \(R\) of order 3 with \(QR\) of order \(n\).

By putting \(Q = \hat{A}\hat{B}\hat{A}\) and \(R = \hat{A}\hat{B}\), we have:

Theorem 2.2 If \(e_2\) is 2, then for any positive integer \(e_1\) greater than 1 there is a covering \(\pi : X \to \mathbb{P}^2\) which branches at \(D\).

Let \(D\) be as before and let \(D' = e_1'C + e_2'L_\infty\). Let \(e_j''\) be the LCM \(<e_j, e_j'>\) \((j = 1, 2)\) and put \(D'' = e_1''C + e_2''L_\infty\).

By constructing the fiber product, we have:

Proposition 2.2 If there is a covering \(\pi : X \to \mathbb{P}^2\) which branches at \(D\) and there is a covering \(\pi' : X' \to \mathbb{P}^2\) which branches at \(D'\), then there is a covering \(\pi'' : X'' \to \mathbb{P}^2\) which branches at \(D''\).

3 Cyclic extension

We denote by \(S_n\) the symmetric group of \(n\) letters. Let \(\hat{G} \subset S_r\) be a finite group generated by two permutations \(Q, R\), which satisfy the relation \(Q^2 = R^3 = 1\). Then \(\hat{Q}\) is a product of cycles of length 2 with no common letters and \(\hat{R}\) is a product of cycles of length 3 with no common letters.

We may assume \(\hat{G}\) has the following properties. (1) transitivity: For each letters \(x, y\) there is a permutation of \(\hat{G}\) which maps \(x\) to \(y\). (2) simplicity: If a permutation of \(\hat{G}\) fixes a letter, then it is the unit element of \(\hat{G}\).

Now by showing examples, we give a method to construct a cyclic extension \(G \subset S_{er}\) of \(\hat{G} \subset S_r\) by an element of its center.

The case \(r = 3\). Put \(\hat{Q} = (a b)\) and \(\hat{R} = (a b c)\). In this case \(\hat{G} = S_3\) and non-abelian. We need to assume \(q\) is odd. Put

\[
Q = \begin{pmatrix} a_1 & a_2 & \ldots & a_q & b_1 & b_2 & \ldots & b_{q-1} & b_q \\ b_1 & b_2 & \ldots & b_q & a_2 & a_3 & \ldots & a_q & a_1 \end{pmatrix} \quad R = \begin{pmatrix} c_1 & c_2 & \ldots & c_q & c_{p+1} & c_{p+2} & \ldots & c_q \\ c_{p+2} & c_{p+3} & \ldots & c_q & c_1 & c_2 & \ldots & c_{p+1} \end{pmatrix}
\]

Then

\[
F = Q^2 = R^3 = (a_1 \ldots a_q)(b_1 \ldots b_q)(c_1 \ldots c_q)
\]

and

\[
A = R^{-1}Q = (a_1 c_{p+1} a_{p+2} c_1 \ldots)
\]

where \(q = 2p + 1\). The order of \(A\) is 2q.

Let \(G\) be a finite group generated by two permutations \(Q, R. F\) is a center of \(G\). In a natural way we have the following exact sequence:

\[
1 \to F \to G \to \hat{G} \to 1
\]

where \(< F >^G\) is a subgroup of \(G\) generated by \(F. < F >^G\) is a cyclic group of order \(q\). Then we can have a surjective homomorphism \(\Phi : \pi_1(\mathbb{P}^2 - \{C \cup L_\infty\}, p_0) \to G\). Hence we have:
Theorem 3.1 If $q$ is odd, then there is a finite Galois covering $\pi : X \rightarrow \mathbb{P}^2$ which branches at $2qC + 2qL_\infty$.

The case $r = 4$. Put $Q = (a \ b)$ and $R = (b \ c \ d)$. In this case $G \subset S_4$ and non-abelian. For the extension we need to assume the LCM $< 6, q \geq 1$. In a similar way, we have:

Theorem 3.2 If $q$ is as above, then there is a finite Galois covering $\pi : X \rightarrow \mathbb{P}^2$ which branches at $4qC + 2qL_\infty$.

The case $r = 12$. Put $Q = (a \ j)(b \ d)(c \ h)(e \ i)(f \ l)(g \ k)$ and $R = (a \ b \ c)(d \ e \ f)(g \ h \ i)(j \ k \ l)$. In this case $G \subset S_{12}$ and non-abelian. In a similar way, we have:

Theorem 3.3 There is a finite Galois covering $\pi : X \rightarrow \mathbb{P}^2$ which branches at $3q(q-1)C + 2qL_\infty$.

References
