NOTE ON CLASSIFICATION OF SEXTICS

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ABSTRACT. We classify the local and global singularities of sextics which are tame torus curves of type (2,3) and we also show the degenerations among these classes. As an application, two Zariski pairs are found.

INTRODUCTION

The classification of complex algebraic plane curves is a classical problem. For a survey for the classification of curves of degree less than 6, see [N]. The classification of curves of degree 6 is still not completed, though there are some partial answers for instance in a series papers by Urabe on sextics which has only simple singularities (see [U] and its references), and then by Yang extended Urabe’s result in [Y].

According to [Li], singular plane curves of interest are given by

(a) a curve which appears as a branch of generic projections of surfaces,
(b) generic plane sections of discriminants of linear systems on projective space \( \mathbb{P}^2 \) (this includes dual curves),
(c) a curve defined by explicit equation.

Deformation theory is also useful to prove the existence of curves which can be degenerated into curves of type (a), (b), (c).

In this paper, we mainly use the method (c) for a special class of sextics, which is so called torus curves of type (2,3), and actually these curves have also non-simple singularities, by that reason we can not use the methods of Urabe or of Yang.

Let \( C = \{(X; Y; Z) \in \mathbb{P}^2; F(X; Y; Z) = 0\} \) be a complex projective plane curve and let \( (x, y) \) be the affine coordinates given by \( x = X/Z, y = Y/Z \) on \( \mathbb{C}^2 := \mathbb{P}^2 - \{Z = 0\} \). For simply we also keep the notation \( C \) in the affine chart \( \mathbb{C}^2 \); where its defining equation \( f(x, y) \) is given by \( f(x, y) := F(x, y, 1) \). Then \( C \) is called torus curve of type \( (p, q) \) (or \( (p, q) \)-torus curve for shortly) if we can write \( f = f_p^q + f_q^p \) for some polynomials \( f_p, f_q \) of degree \( p \) and \( q \) respectively in \( \mathbb{C}[x, y] \). In this paper, we consider reduced \( (2,3) \)-torus curve. A sextic \( C = \{f_2^3 + f_3^2 = 0\} \) of type \( (2,3) \) is called tame if its singularities are sitting only at the intersection of conic \( C_2 \) and cubic \( C_3 \), where \( C_2 \) and \( C_3 \) are respectively defined by \( f_2 = 0 \) and \( f_3 = 0 \).

In the first section, we assume that conic and cubic passing through the origin of the affine chart \( \mathbb{C}^2 \), and \( (C, O) \) is an isolated singularity. We will classify all the local singularities at the origin (in the sense of topological equivalence) by considering the geometrical relation of conic and cubic. For that purpose the intersection number \( \iota = I(C_2, C_3; O) \) plays an important role. The result is contained in Theorem 1.

In the second section, we study the possible configurations of singularities on tame torus curves of type \( (2,3) \) using the local result obtained in the first section. The main result is Theorem 2.

In the third section we study the spaces with fixed configurations and the possible degenerations among them. Actually these degenerations will be used in the investigation of topological
problem, namely, the computation of the fundamental group of the complement \( \mathbb{P}^2 \setminus C \), where \( C \) is a tame torus curve of type (2,3). The details will be given in [OP].

Finally in the last section, we give two new examples of weak Zariski pairs. In each pair, both curves are torus curves of type (2,3).

We omit several proofs in this paper, the full version is given in [P].

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1. LOCAL CLASSIFICATION PROBLEM

1.1. Some classes of singularities. We use the following standard notations for “simple” singularities, which have the normal form:

\[
\begin{align*}
A_n &: x^2 + y^{n+1} = 0 \quad (n \geq 1) \\
D_n &: x^2 y + y^{n-1} = 0 \quad (n \geq 4) \\
E_6 &: x^3 + y^4 = 0, E_7 &: x^3 + xy^3 = 0, E_8 &: x^3 + y^5 = 0
\end{align*}
\]

Furthermore we define notations of some other (topological equivalence classes of) singularities which we use later.

\[
\begin{align*}
B_{p,q} &: x^p + y^q = 0 \text{(Brieskorn-Pham type)} \\
C_{p,q} &: x^p + y^q + x^2y^2 = 0 \\
D_{4,7} &: y^4 + x^3y^2 + x^7 = 0 \\
S_{p1} &: (y^2 - x^3)^2 + (xy)^3 = 0 \\
S_{p2} &: (y^2 - x^3)^2 - y^6 = 0
\end{align*}
\]

(1.1)

Note the symmetry \( B_{p,q} = B_{q,p}, C_{p,q} = C_{q,p} \) and the identities \( B_{2,q} = A_{p-1}, B_{3,4} = E_6 \).

Suppose we have a germ \((C, O)\) of a plane curve. Recall three local invariants: \( \mu(C, O) \) is the Milnor number at \( O \), \( \delta(C, O) \) is the \( \delta \)-invariant which is the maximal number of nodes in a generic deformation of \( C \), and \( r(C, O) \) is the number of the analytic branches. It is well-known that \( \delta = (\mu + r - 1)/2 \) (see [M]). The invariant triple \((\mu, r, \delta)\) will be used for the later arguments.

**Definition 1.1.** Suppose that we have two germs of plane curve singularities \((C, p)\) and \((C', p')\). We say that \((C, p)\) and \((C', p')\) are **topologically equivalent** if there exists a local homeomorphism \( \phi \) of the respective ambient neighborhoods \( U, U' \) such that \( \phi(p) = p' \) and \( \phi(C \cap U) = C' \cap U' \). We denote this equivalence relation as \((C, p) \sim (C', p')\). When \( p = p' = O \), let \( f \) and \( f' \) are defining polynomials of \( C \) and \( C' \), then we also write \( f \sim f' \).

For a reduced plane curve germ \((C, p)\), the topological equivalence type (or shortly topological type) of the germ \((C, p)\) is determined by the following discrete characteristics (for instance [Z2, Lé1, BK]): the Puiseux pairs of its irreducible components and their linking numbers. Alternatively, it is known that the embedded resolution graph of \((C, p)\) and the multiplicities of the total transforms of \((C, p)\) at infinitely near points (including \( p \)) determine the topological type.

1.2. Setting. Hereafter, \((x, y)\) is the affine coordinates of \( C^2 \) and \( f_2(x, y), f_3(x, y) \) are the affine defining polynomial of \( C_2 \) and \( C_3 \). The sextic \( C \) is defined by \( f_2(x, y)^3 + f_3(x, y)^2 = 0 \).

We assume that the origin \( O \) is an intersection point of \( C_2 \) and \( C_3 \) and it is an isolated singularity of \( C \). This implies, in particular, that the conic \( C_2 \) and the cubic \( C_3 \) have no common component (such a line or the conic itself). We will classify the topological types of the germ \((C, O)\), using the geometry of the respective singularities of the conic \( C_2 \) and the cubic \( C_3 \) and their mutual position.
1.3. **Strategy of classification.** We frequently use the fact that the topological type of a non-degenerate germ depends only on its Newton boundary. See [K, O2] for more details. In our classification, most of the cases can be transformed into non-degenerate forms after several changes of coordinates, and then the Newton principal part (NPP for short) gives us a normal form. However there are two special cases, namely $Sp_1$ and $Sp_2$ which can not be transformed into non-degenerate form. For these cases, the topological type can be read off from the explicit resolution of the respective singularity.

From now on, we use $(u,v)$ for local coordinates (i.e. in the sense that we use an analytic change of coordinates), and $(x,y)$ for global coordinates (i.e. affine coordinates). To determine the local type of singularities, it is convenient to use suitable changes of local analytic coordinates. However to see the existence of a sextic of $(2,3)$-torus type, it is usually better to keep the affine coordinates.

We recall a following lemma [O3, lemma 4.3].

**Lemma 1.2.** Assume that $C_f = \{(u,v) \in \mathbb{C}^2; f(u,v) = 0\}$ a germ of a smooth curve at the origin. Let $C_g = \{(u,v) \in \mathbb{C}^2; g(u,v) = 0\}$ be another germ of a curve at the origin. Let $d$ be the multiplicity of $g$ at the origin and let $g_d(u,v)$ be the homogeneous polynomial of degree $d$, which defines the tangent cone of $C_g$. Let $p,q$ be positive integers such that $p < dq$. Consider the germ of a plane curve $C = \{(u,v) \in \mathbb{C}^2; f(u,v)^p - g(u,v)^q = 0\}$. Assume that each irreducible component of $g_d(u,v) = 0$ intersects transversely with $C_f$ at the origin. Then topological type of $(C, O)$ is $B_{p,dq}$ and the tangential direction at the origin coincides with that of $f = 0$, where $B_{p,dq}$ is the Brieskorn-Pham singularity introduced in (1.1).

For $(2,3)$-torus curve, we apply the above lemma with $(p,q) = (2,3)$ or $(p,q) = (3,2)$ and obtain the following result:

**Corollary 1.3.** With the transverse intersection condition as above, we have:

(i) If $(C_3, O)$ is smooth, then $(C, O)$ is of type $A_{3\iota-1}$ for $\iota = 1, 2, ..., 6$.

(ii) If $(C_3, O)$ is singular and $(C_2, O)$ is smooth, then $(C, O)$ is of type $B_{3,2\iota}$, for $\iota = 2, 3, ..., 6$.

1.4. **Classification steps.** We will divide the situation into 3 cases by the multiplicity $m_3$ of the cubic $C_3$ at the origin. The simplest case is the case when $(C_3, O)$ is smooth.

**Case A:** $m_3 = 1$. In the corollary 1.3 (i), we have to assume that the irreducible components of the tangent cone of $C_2$ intersects transversely with $C_3$. The claim is also true without this condition.

**Proposition 1.4.** If $(C_3, O)$ is smooth, and it intersects the conic $C_2$ with multiplicity $\iota$, then $(C, O)$ is of type $A_{3\iota-1}$.

**Proof.** Up to an analytic change of coordinates, we may assume that $f_3 = u$ and the conic has the following form $f_2 = cu^2 + uh$, where $h \in \mathbb{C}\{u,v\}, c \in \mathbb{C}\{v\}$ and $c(0) \neq 0$. Putting weights $w(u) = 3\iota$ and $w(v) = 2$, we can see $f = u^2 + (cv^\iota + uh)^3 = u^3 + c(0)^3v^{3\iota}$ + higher terms. This implies $(C, O) \sim A_{3\iota-1}$, for $\iota = 1, ..., 6$.

We recall here that the invariant triple $(\mu, r, \delta)$ is $(k, (k \mod 2) + 1, [k/2])$ for $A_k$ singularity, where $[\alpha]$ is the greatest integer less than or equal to $\alpha$.

**Proposition 1.5.** For any $1 \leq \iota \leq 6$ there exists a smooth cubic $C_3$ and a conic $C_2$, such that the $(C, O)$ is of type $A_{3\iota-1}$. Furthermore we have a degeneration family: $A_2 \to A_5 \to A_8 \to A_{11} \to A_{14} \to A_{17}$

**Proof.** See in [P].
Remark 1.6. One can compute the Newton principal part by hand, but usually it is a boring computation. We make a Maple package SCURVE\(^1\), it can be used for compute invariants of singular plane curves, such that intersection number, Milnor number, NPP, toric modification, etc.

The case \(m_3 = 1\) is done by Proposition 1.5.

**Case B:** \(m_3 = 2\). We divide Case B into two subcases by \(m_2\).

(B-I) \((C_2, O)\) is smooth \((m_2 = 1)\): there are 8 cases, indicated in the following figures, where the dotted lines denote the cubic \(C_3\), and the straight lines are affine lines. We remark here that \(C_2\) may not irreducible globally, however in the cases I-4 and I-8 it must be irreducible (because \(C_2\) has a tangent line which is a component of \(C_3\)).

In the pictures I-3 and I-4, the fat dotted line denotes a line with multiplicity 2.

![Figure 1. Case B-I \((m_3 = 2, m_2 = 1)\).](image)

In the above figure, the cases I-1, I-3, I-5, I-7 satisfy the conditions of the lemma 1.2, and \(d = 2\), so that we obtain \((C, O)\) is of \(B_{3,4}\) type. In these cases, \(\ell = 2\).

The next case I-2 is the most interesting case in this classification, and we consider it at the end of this part.

I-4. We assume \(f_2 = y - x^2\) and \(f_3 = y^2h\), where \(h\) is a linear term and \(c := h(0, 0) \neq 0\). We have \(\ell = 4\). Putting \(y_1 = y - x^2\), in the new coordinates \((x, y_1)\), we have \(\text{NPP}(f) = y_1^4 + c^2x^8\).

Thus \((C, O) \sim B_{3,8}\).

I-6. As \(C_2\) is smooth at \(O\), we can take a local system of coordinates \((u, v)\) so that \(f_2 = v\) and \(\text{NPP}(f_3) = \alpha v^3 - \beta u^3\), where \(\alpha, \beta \neq 0\). Putting weights \(w(u) = 1\) and \(w(v) = 2\), we obtain \(\ell = 3\) and \(\text{NPP}(f) = v^3 + \beta^2 u^6\). We can easily see that this polynomial is non-degenerate for any \(\beta \neq 0\). Thus \((C, O) \sim B_{3,6}\). This computation applies also for the case of \(C_2\) being two lines.

I-8. The cubic \(C_3\) consists of a line \(\ell\) and a conic \(C_2'\). We may assume that \(f_2 = y - x^2\) and \(f_3 = y(x + ax^2 + bxy + cy^2)\), where \(a\) or \(b\) is non-zero. Putting \(t_1 = a + 1, t_2 = b, t_3 = c,\) and substitute \(y = x^2\) we get \(f_3(x, x^2) = t_3x^6 + t_2x^5 + t_1x^4\). Hence \(\ell\) is equal to 4, 5 or 6 (respectively \(1(C_2', C_2; O) = 2, 3\) or 4). By this setting we have:

**Proposition 1.7.** Under the above situation the germ \((C, O)\) can be of type \(B_{3,4}\), for \(\ell = 4, 5, 6\). Furthermore there is a degeneration family: \(B_{3,8} \rightarrow B_{3,10} \rightarrow B_{3,12}\).

**Proof.** See in [P].

\(^1\)You can get SCURVE by a request e-mail to the author or download from the web-page:
http://www.comp.metro-u.ac.jp/~pdtai/scurve/
Now we come back to the case I-2.

We may assume the conic $C_2$ is defined by $f_2 = b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2$ and the cubic $C_3$ is defined by $f_3(x, y) = y(a_{11}x + a_{02}y) + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3$, where $b_{01} \neq 0$ and $a_{11} \neq 0$. Because $C_2 \cap \{y = 0\} = C_3 \cap \{y = 0\} = \{O\}$, therefore $\iota = \text{val}_{x=0} \text{Res}_y(f)$.

We have $\text{Res}_y(f) = x^3 \varphi(x)$ and
\begin{equation}
(1.2) \quad \varphi(0) = b_{01}(b_{01}a_{30} - a_{11}b_{20})(b_{02}a_{02} - a_{03}b_{01})
\end{equation}

Thus $\iota = 3$ iff none of these factors is zero.

First we consider the case $\iota = 3$. Putting weights $w(x) = 1, w(y) = 2$, by computation we have $\text{NPP}(f; (x, y))$ is a weighted homogeneous polynomial of degree 6. So that $f(x, y)$ is non-degenerate if and only if
\begin{align*}
(27a_{30}b_{01}^3 - 27a_{11}b_{01}^2b_{20} - 4a_{11}^3)(a_{30}b_{01} - a_{11}b_{20}) \neq 0
\end{align*}

Thus $f \sim B_{3,6}$ for a generic case. Note that the term $(a_{30}b_{01} - a_{11}b_{20})$ is non-zero by (1.2).

Thus to see further degeneration, we solve
\begin{align*}
(27a_{30}b_{01}^3 - 27a_{11}b_{01}^2b_{20} - 4a_{11}^3) = 0
\end{align*}
in $a_{30}$ and then $\text{NPP}(f; (x, y))$ gets a multiply factor. Namely, $(9b_{01}^3y + 9b_{01}b_{20}x^2 + 4a_{11}^2x^2)^2$.

This factor is a multiply factor. Namely, $(9b_{01}^3y + 9b_{01}b_{20}x^2 + 4a_{11}^2x^2)^2$. Thus we need to take the new coordinates $(x, y_1)$ with $y_1 = y - (9b_{01}^3b_{20} + 4a_{11}^2)x^2/(9b_{01}^3)$. In this case we get $C_{3,7}$ as long as the coefficient of $x^7$ is non-zero. In fact, this coefficient is given by
\begin{align*}
c(4a_{11}^2 + 9b_{01}^2b_{20}^2)(9a_{02}b_{20}b_{01}^2 - 9a_{21}b_{01}^3 + 9a_{11}b_{11}b_{01}^2 + 4a_{02}a_{11}^2)
\end{align*}

where $c$ is a non-zero constant.

Thus there are two exceptional cases:

(i-3-1) $4a_{11}^2 + 9b_{01}^2b_{01} = 0$ or

(i-3-2) $9a_{02}b_{20}b_{01}^2 - 9a_{21}b_{01}^3 + 9a_{11}b_{11}b_{01}^2 + 4a_{02}a_{11}^2 = 0$.

It turns out that the two cases give completely different geometries.

In the first case (i-3-1), putting $b_{20} = -c_{20}^2$, thus the equivalence condition is $a_{11} = \pm 3b_{01}c_{20}/2$. Let assume $a_{11} = 3b_{01}c_{20}/2$, we get $\text{NPP}(f; (x, y_1)) = c_1x^2y_1 + c_2x^2y_1^2 + c_3y_1^3$ with $c_1, c_2, c_3$ are non-zero constants. One can check that this form is topological equivalent to $x^3 + x^2y_1 + y_1^3$ (see Case B-II-5 for a detailed explanation). Thus $(C, O) \sim C_{3,8}$. We also obtain same result for the case $a_{11} = -3b_{01}c_{20}/2$. An important observation is that the generic member of this family is not irreducible but it is a union of a line and a quintic.

Next we consider the case (i-3-2), i.e. $a_{21} = (9b_{01}^3b_{20}a_{02} + 9b_{01}^2a_{11}b_{11} + 4a_{02}a_{11}^2)/(9b_{01}^3)$. We look at the Newton principal part of $f$ with respect to the coordinates $(x, y_1)$. It has two faces, $AB$ and $BC$ where $A = (0, 3), B = (2, 2)$ and $C = (8, 0)$. The face function with respect to $AB$ is non-degenerate, while the discriminant in $y_1$ of the face function with respect to $BC$ is
\begin{align*}
d = cx^{10}(4a_{11}^2 + 9b_{01}^2b_{20})^2(-9b_{11}b_{01}^2 - 16a_{11}a_{12}b_{01} + 16b_{02}a_{11}^2), \quad (c \neq 0)
\end{align*}

Since the factor $4a_{11}^2 + 9b_{01}^2b_{20}$ is already considered in the case (i-3-1), so the generic case take place if the last factor of $d$, say $d_2$, is non-zero. In that case we have $(C, O) \sim C_{3,8}$. In comparison with the previous class of $C_{3,8}$ in case (i-3-1) a generic member of this class is irreducible.

Finally if $d_2 = 0$, it turns out that $(C, O) \sim C_{3,9}$. And it does not degenerate any further (as long as $(C, O)$ is smooth and $(C, O)$ is $A_1$).

**Summary:** In the case $\iota = 3$ there are 4 possibilities $B_{3,6}, C_{3,7}, C_{3,8}$ and $C_{3,9}$. We remark that in this case, the intersection multiplity is not enough to determine the topology.
Next we consider the case \( \iota \geq 4 \). By Bezout theorem, \( C_2 \) is necessarily a smooth conic. Thus we can assume that \( f_2(x, y) = y - x^2 \). We have \( a_{30} = -a_{11} \) and

\[
\begin{align*}
\iota &= 4 & \text{iff} & \quad a_{02} + a_{21} \neq 0 \\
\iota &= 5 & \text{iff} & \quad a_{02} + a_{21} = 0, a_{12} \neq 0 \\
\iota &= 6 & \text{iff} & \quad a_{02} + a_{21} = a_{12} = 0, a_{03} \neq 0
\end{align*}
\]

Let consider the case \( \iota = 4 \). We take the coordinate change \( (x, y_1) \) with \( y_1 = y - x^2 \). The Newton principal part is given by

\[
\text{NPP}(f; (x, y_1)) = (a_{02} + a_{21})^2 x^8 + 2a_{11}(a_{02} + a_{21}) x^5 y_1 + a_{11}^2 x^2 y_1^2 + y_1^3
\]

It is degenerate for the weight \( P = \iota(1, 3) \), where \( f_P = x^2(x^3a_{21} + x^3a_{02} + a_{11}y_1)^2 \). Taking the coordinate change \( (x, y_2) \) with \( y_2 = x^3a_{21} + x^3a_{02} + a_{11}y_1 \). We have

\[
\text{NPP}(f; (x, y_2)) = -(a_{02} + a_{21})^3 x^9 / a_{11}^3 + x^2 y_2^2 + y_2^3 / a_{11}^2
\]

Hence for generic \( a_{ij} \)'s (i.e. \( a_{02} + a_{21} \neq 0 \)) we have \( (C, O) \sim C_{3,9} \). Continuing by the same method, we get \( C_{3,12} \) and \( C_{3,15} \) for \( \iota = 5 \) and \( 6 \) respectively.

**Proposition 1.8.** There exists a cubic \( C_3 \) and a conic \( C_2 \), such that the \( (C, O) \) is of type \( B_{3,6}, C_{3,k} \) (for \( k = 7, 8, 9 \)) and \( C_{3,3k} \) (for \( k = 3, 4, 5 \)). Furthermore there are the following degeneration families:

(i) \( B_{3,6} \rightarrow C_{3,7} \rightarrow C_{3,8} \rightarrow C_{3,9} \), with the same \( \iota = 3 \).

(ii) \( B_{3,6} \rightarrow C_{3,9} \rightarrow C_{3,12} \rightarrow C_{3,15} \), with \( \iota : 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \).

**Proof.** See in [P].

**Remark 1.9.** (a) There are two moduli components for the singularity classes \( C_{3,8} \) with \( \iota = 3 \). The generic curve in the component corresponding to the case (II-1) decomposes into a line and a quintic, later we will use the notation \( C_{3,8}^2 \) for the singularity in this moduli.

(b) There are two components for the moduli space of the class \( C_{3,9} \), with different intersection numbers. Later we will distinguish by the new notation \( C_{3,9}^2 \) for the case \( \iota = 4 \).

(c) There are more degenerations for the singularities if we admit \( \iota \) to increase or if we admit the degenerations for the singularities of \( C_2 \) and \( C_3 \). For example, we can degenerate \( C_{3,7} \rightarrow B_{3,8} \rightarrow B_{3,10} \rightarrow B_{3,12} \), \( C_{3,8} \rightarrow B_{3,10} \) and \( C_{3,9} \rightarrow B_{3,12} \) by increasing the intersection number \( \iota \). \( B_{3,8} \) degenerates also into \( B_{4,6} \). For the explicit construction, see §3.

(B-II) \( (C_2, O) \) is singular \( (m_2 = 2) \). There are 9 cases, indicated in Figure 2, where the straight lines are again affine lines.

**Attention.** In the cases II-1,5,8,9 though in the pictures of \( C_2 \) are 2 distinct lines, but 2 lines may also coincide.

II-1. Up to an analytic change of coordinates we may assume \( f_3 = uv \) and \( f_2 = (a_1u + b_1v + h_1)(a_2u + b_2v + h_2) \), where \( a_i, b_i \) are non-zero constants, and \( h_i \in m^2 \) for \( i = 1, 2 \). Here \( m \) is the maximal ideal in \( \mathbb{C}\{u, v\} \). We have \( \iota = 4 \). Putting weights \( w(u) = w(v) = 1 \), we get \( f \sim u^6 + v^6 + u^2v^2 \). Thus \( (C, O) \sim C_{6,6} \).

II-2. Since conic consists of 2 distinct lines, say \( \ell_1 \) and \( \ell_2 \), where \( \ell_1 \) intersects \( C_3 \) transversely and \( \ell_2 \) is a tangent cone direction of \( C_3 \), hence \( I(\ell_1, C_3; O) = 2 \) and \( I(\ell_2, C_3; O) = 3 \). Therefore we may assume \( f_3 = uv \), and \( f_2 = (a_1u + b_1v + h_1)(a_2u^2 + b_2v + h_2) \) so that \( \iota = 5 \). And we obtain \( (C, O) \sim C_{6,9} \).

II-3,4. In these case, the intersection number \( \iota = 6 \). By the same way, we get respectively \( C_{9,9}, C_{6,12} \).
Proposition 1.10. Under the situations II-1~II-4, the type of \((C, O)\) can be \(C_{6,6}, C_{6,9}, C_{9,9}\) and \(C_{6,12}\). Furthermore, there are the following degeneration families:

\[
C_{6,6} \rightarrow C_{6,9} \rightarrow C_{9,9} \quad C_{6,9} \rightarrow C_{6,12}
\]

Proof. See in [P].

In the next three cases II-5~II-7, the cubic \(C_{3}\) has a cusp, and by a linear change of coordinates we may assume that \(\text{NPP}(f_3) = y^2 - x^3\), i.e. \(f_3 = y^2 - x^3 + h\) where \(h\) is the higher term, and \(f_2 = (a_1 x + b_1 y)(a_2 x + b_2 y)\).

II-5. This case \(a_1, a_2 \neq 0\), the intersection number \(\iota = 4\), we may assume \(a_1 = 1\). We have \(\text{NPP}(f) = (1 + a_3^2)x^6 - 2x^3y^2 + y^4\), and its discriminant in \(x\) is \(-46656(1 + a_3^2)^2a_2^6y^20\). Hence if \(1 + a_3^2 \neq 0\), we have \((C, O) \sim B_{4,6}\).

II-6. One branch of the conic is tangent to \((C_3, O)\), we may assume that

\[
f_2(x, y) = axy, \quad f_3(x, y) = y^2 - x^3 + b_1 y x^2 + b_2 y y^2 + b_3 y^3
\]

In this case \(\iota = 5\). This is a degenerate singularity, since \(\text{NPP}(f) = (y^2 - x^3)^2\). Taking a canonical toric modification \(\pi_1 : X_1 \rightarrow C^2\), we find that \(\pi_1^* f = 0\) has again a \((2,3)\)-cusp and it has a simultaneous resolution by one more toric modification for any \(b_i \in C_i\), \(i = 1, 2, 3\). The tower of the weight vectors in the sense of [AO] is given by \((2, 3), (2, 3)\). Thus the topology of \((C, O)\) does not depend on the parameters \(a \neq 0\) and \(b_1, b_2, b_3\).

II-7. The conic is a double line and it coincides with the tangent direction of \((C_3, O)\). We assume that \(f_2(x, y) = -a_3^2 y^2, a \neq 0\) and \(f_3(x, y)\) is as above. The intersection number \(\iota = 6\). By the same method as \(S_{P_1}\) case, we obtain this singularity is equivalent to \(S_{P_2}\) and the resolution graph is given in Figure 4. We see that \(S_{P_2}\) has 2 irreducible components \(f_3 \pm a_3^2 y^3 = 0\), they have same Puiseux pair \((2, 3)\) and their linking number is 9. Thus \(\mu(S_{P_2}) = 21\) and \(\delta(S_{P_2}) = 11\).

Proposition 1.11. There are following degeneration families:

(i) \(B_{4,6} \rightarrow D_{4,7}\) in Case II-5.
(ii) $B_{4,6} \to S_{P_{1}} \to S_{P_{2}}$ corresponding to II-5 $\to$ II-6 $\to$ II-7.

Proof. The proof is obvious from the explanations in Cases II-5,6,7.

II-8. Assuming $f_{3} = y(y - x^{2})$ and $f_{2} = a(x + by)(x + cy)$, where $a$ is non-zero. We have $\text{NPP}(f) = y^{4} + a^{3}x^{6}$. Thus $(C, O) \sim B_{4,6}$.

II-9. Assuming $f_{3} = x^{2}(ax + by + k)$ and $f_{2} = (cx + y)(dx + y)$, where $k$ is non-zero. We have $\text{NPP}(f) = y^{6} + k^{2}x^{4}$. Thus $(C, O) \sim B_{4,6}$.

Finally we consider singular cubic with multiplicity 3 ($m_{3} = 3$).

Case C: $m_{3} = 3$. Similarly, we also divide 2 cases by the multiplicity of the conic.

(C-I) Conic $C_{2}$ is smooth at $O$. Obviously we have $\iota \geq 3$.

C-I-I. $\iota = 3$, applying the corollary 1.3, we get $(C, O) \sim B_{3,6}$.

Now we consider the case $\iota \geq 4$. Then $C_{2}$ is irreducible, as $(C_{2}, O)$ is assumed to be smooth.

Therefore we may assume that $f_{2} = y - x^{2}$ and $f_{3} = (a_{1}x + b_{1}y)(a_{2}x + b_{2}y)(a_{3}x + b_{3}y)$. We have $\iota$ is equal to the lowest degree in $x$ of $f_{2}(x, x^{2}) = x^{3}(a_{1}x + b_{1}y)(a_{2}x + b_{2}y)(a_{3}x + b_{3}y)$.

C-I-II. $\iota = 4$, by symmetry of $a_{i}x$ we may assume $a_{1} = 0$ and $a_{2}, a_{3}$ are none-zero. Putting $y_{1} = y - x^{2}$, then $f(x, y_{1}) = y_{1}^{3} + b_{1}^{2}a_{3}^{2}x^{8} + \text{higher terms} \sim y_{1}^{3} + b_{1}^{2}a_{3}^{2}x^{8}$. Since $a_{1} = 0$ then $b_{1}$ must be non-zero. Thus $(C, O) \sim B_{3,8}$.

Similarly, we obtain

C-I-III. $\iota = 5$ if $a_{1} = a_{2} = 0$ and $a_{3} \neq 0$: result is $(C, O) \sim B_{3,10}$.

C-I-IV. $\iota = 6$ if $a_{1} = a_{2} = a_{3} = 0$: result is $(C, O) \sim B_{3,12}$.

(C-II) Conic $C_{2}$ is singular. We have $\iota = 6$. Since both conic and cubic are products of linear terms, then $f$ is a homogeneous polynomial of degree 6. Since $O$ is an isolated singularity, then $f$ should be non-degenerate and it is a product of linear terms. Thus $(C, O) \sim B_{6,6}$.

1.5. Statement of the result on the local classification. Now we can state the result in the local classification:

Theorem 1 (Local Classification). Let $C = \{f = f_{3}^{2} + f_{2}^{3} = 0\}$ is a tame $(2,3)$-torus curve. Put $C_{i} = \{f_{i} = 0\}$ for $i = 2, 3$. The topological type of the germ $(C, O)$ can be read off from $C_{2}$ and $C_{3}$ as follows:

1. Cubic $(C_{3}, O)$ is smooth: $A_{3\iota-1}$ ($\iota = 1, \ldots, 6$).
2. Cubic is not smooth at $O$ (i.e., $m_{3} \geq 2$).
(a) Conic $(C_2, O)$ is smooth.
   (i) $(C_3, O)$ is $A_1$ and $t = 2 : E_6 = B_{3,4}.$
   (ii) $(C_3, O)$ is $A_1$ and $(C_2, O)$ is tangent to one of the branch:
      $B_{3,6}, C_{3,7}, C_{3,8}, C_{3,9}$ for $t = 3,$ and
      $C_{3,3t-3}$ for $t = 4, 5, 6.$
   (iii) $(C_3, O)$ is not $A_1 : B_{3,2t}, t = 2, \ldots, 6.$

(b) Conic $(C_2, O)$ is $A_1 :$
   (i) $(C_3, O)$ is $A_1 :$
      $C_{6,6}$ for $t = 4,$ no common tangential cone,
      $C_{6,9}$ for $t = 5,$ one common tangential cone,
      $C_{9,9}$ for $t = 6,$ two common tangential cones.
   (ii) $(C_3, O)$ is either $A_2$ or $A_3 :$
      $B_{4,6}$ or $D_{4,7},$ for $t = 4,$ no common tangential cone,
      $Sp_1$ if $t = 5$ and $(C_3, O) = A_2$ and the tangential cones coincide.

(c) $C_2$ is a line with multiplicity 2:
   (i) $(C_3, O)$ is a $A_1 :$
      $B_{4,6}$ for $t = 4,$ no common tangential cone,
      $C_{6,12}$ if $t = 6,$ a common tangential cone.
   (ii) $(C_3, O)$ is either $A_2$ or $A_3 :$
      $B_{4,6}, D_{4,7}$ for $t = 4,$ no common tangential cone,
      $Sp_2$ for $t = 6,$ $(C_3, O)$ is $A_2,$ the same tangential cone.

(d) Cubic $C_3$ consists of three lines passing through $O,$ and $C_2$ consists of two lines passing through $O : B_{6,6} (t = 6).$

Remark 1.12. When $(C_3, O)$ is a node (i.e., $A_1),$ $C_3$ can be reduced (either a line and a conic meeting transversely at $O$ or three lines where two of them are passing through $O$).
$(C_3, O)$ is $A_3$ if and only if $C_3$ consists of a line and a conic which are tangent at $O.$ $(C_2, O)$ has a node if it consists of two lines.

Theorem 1-D (Local Degenerations). Under the notation in Theorem 1, we have following degenerations:

1. $(C_3, O)$ is smooth: $A_2 \rightarrow A_3 \rightarrow A_8 \rightarrow A_{11} \rightarrow A_{14} \rightarrow A_{18}$
2. $(C_3, O)$ is $A_1$ and $(C_2, O)$ is smooth:
   (a) $B_{3,4} \rightarrow B_{3,6} \rightarrow C_{3,7} \rightarrow C_{3,8} \rightarrow C_{3,9}$
   (b) $B_{3,4} \rightarrow B_{3,6} \rightarrow C_{3,9} \rightarrow C_{3,12} \rightarrow C_{3,15}$
3. $(C_3, O)$ is $A_2$ or $A_3,$ $(C_2, O)$ is smooth: $B_{3,4} \rightarrow B_{3,8} \rightarrow B_{3,10} \rightarrow B_{3,12}$
4. $(C_3, O)$ is $A_1,$ $(C_2, O)$ is $A_1 :$ $B_{4,6} \rightarrow D_{4,7}, B_{4,6} \rightarrow Sp_1 \rightarrow Sp_2.$
5. $(C_3, O)$ is $A_2$ or $A_3,$ $(C_2, O)$ is $A_1 :$ $C_{6,6} \rightarrow C_{6,9} \rightarrow C_{9,9}, C_{6,9} \rightarrow C_{6,12}.$

The following is well known.

Proposition 1.13. Let $C$ be a reduced curve of degree $d$ in $\mathbb{P}^2$ defined by $F(X, Y, Z).$ Then there is a family $C_t$ for $0 \leq t \leq 1$ of curves of degree $d$ such that $C_t \cong C$ for $t \neq 0$ and $C_0 \cong B_{d,d}$ where $B_{d,d}$ is the class of $d$ lines meeting at $O.$

Proof. We follow the method in [OS]. We may assume that the line at infinity $Z = 0$ meets $C$ transversely. Take $F(X, Y, Z, t) := F(X/t, Y/t, Z) t^d$ and define the family $C_t = \{ F(X, Y, Z, t) = 0 \}.$

Remark 1.14. For the sake of the global study of sextics, we distinguish $C_{3,8}$ and $C^d_{3,8},$ and also $C_{3,9}$ and $C^d_{3,9}$ though they are locally topologically equivalent.
Table A: Local classification

<table>
<thead>
<tr>
<th>$t$</th>
<th>Singularity type $T$ and the invariant triple $(\mu, r, \delta)$ of (C,O)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A_2(2,1,1)$</td>
</tr>
<tr>
<td>2</td>
<td>$A_5(5,2,3), E_6(6,1,3)$</td>
</tr>
<tr>
<td>3</td>
<td>$A_8(8,1,4), B_{3,6}(10,3,6), C_{3,7}(11,2,6), C_{3,8}(12,3,7), C_{3,8}^2(12,3,7), C_{3,9}(13,2,7)$</td>
</tr>
<tr>
<td>4</td>
<td>$A_{11}(11,2,6), C_{3,9}^2(13,2,7), B_{3,8}(14,1,7), C_{6,6}(13,4,8), B_{4,6}(15,2,8), D_{4,7}(16,3,9)$</td>
</tr>
<tr>
<td>5</td>
<td>$A_{14}(14,1,7), C_{3,12}(16,3,9), B_{3,10}(18,1,9), C_{6,9}(16,3,9), Sp_1(18,1,9)$</td>
</tr>
<tr>
<td>6</td>
<td>$A_{17}(17,2,9), C_{3,15}(19,2,10), C_{9,9}(19,2,10), B_{3,12}(22,3,12), C_{6,12}(19,3,11), Sp_2(21,2,11), B_{6,6}(25,6,15)$</td>
</tr>
</tbody>
</table>

Table A': The local degeneration series

\[ \begin{array}{ccc}
 t & m_3 = 1 & m_2 = 1, m_3 \geq 2 & m_2 = m_3 = 2 \\
 1 & A_2 & E_6 = B_{3,4} & "\text{node}" \\
 2 & A_5 & C_{3,6} & C_{3,8} & C_{3,8}^2 & "\text{cusp}" \\
 3 & A_8 & B_{3,6} & C_{3,7} & C_{3,8} & C_{3,9} \\
 4 & A_{11} & C_{3,9} & B_{3,8} & C_{6,6} & B_{4,6} & D_{4,7} \\
 5 & A_{14} & C_{3,12} & B_{3,10} & C_{6,9} & Sp_1 & "Sp_2" \\
 6 & A_{17} & C_{3,15} & B_{3,12} & C_{9,9}, C_{6,12} & Sp_2 & B_{6,6} \\
\end{array} \]
2. Global classification

In this section, we consider the possible combination of the local singularities of reduced tame (2,3)-torus curves, using the local classification obtained in Section 1. Assume that $C$ is an irreducible curve of degree $d$ and denote the set of singularities of $C$ by $\Sigma(C)$. We recall the genus formula

\begin{align}
\tag{2.1}
g &= \frac{(d-1)(d-2)}{2} - \sum_{P \in \Sigma(C)} \delta(C, P) \geq 0 \quad \text{or}
\tag{2.2}
\chi(C') &= 3d - d^2 + \sum_{P \in \Sigma(C)} \mu'(C, P) \leq 2
\end{align}

where $C'$ is the normalization of $C$, $\chi(C')$ is the topological Euler characteristic, $\mu'(C, P) := \mu(C, P) + r(C, P) - 1$ and $r(C, P)$ is the number of analytic branches at $P$ (see [BK]). We call $\mu'(C, P)$ the normalized Milnor defect of $C$ at $P$. When $C$ is a tame (2,3)-torus curve, then its singularities set $\Sigma(C)$ is given by $C_2 \cap C_3 =: \{P_1, P_2, \ldots, P_n\}$. Therefore (2.1) is equivalent to the following

\begin{align}
\tag{2.3}
\sum_{i=1}^{n} \delta(C, P_i) &\leq (6-1)(6-2)/2 = 10.
\end{align}

Denote $i_k := \text{I}(f_2, f_3; P_k)$, we call $(i_1, \ldots, i_n)$ an i-vector and $n$ the length of i-vector. By Bezout theorem, we have

\begin{align}
\tag{2.4}
\sum_{i=1}^{n} i_k &= 6.
\end{align}

When $C$ is reducible curve with $r(C)$ irreducible components, the inequality (2.1) does not hold and (2.2) has to be replaced by the following.

\begin{align}
\tag{2.5}
\chi(C') &= 3d - d^2 + \sum_{P \in \Sigma(C)} \mu'(C, P) \leq 2r(C)
\end{align}

Our strategy for the global classification is the following steps:

\textit{Step 1}. Consider every possible i-vector, which satisfies (2.4).

\textit{Step 2}. List up every possible combination of the local singularities having prescribed i-vector.

\textit{Step 3}. Prove or disprove the existence of a reduced tame (2,3)-torus curve with the configurations. In this stage, if the inequality (2.2) is not satisfied, we have to look for the reduced curves with the given configuration.

For the later discussion, we recall

\textbf{Lemma 2.1.} The possible topological types of a singularity on an irreducible quartic curve are $A_1, \ldots, A_6, D_4, D_5$ and $E_6$.

\textit{Proof}. See for instance in [W] \hfill \Box

From the above strategy we can enumerate singularities as follows:

\textbf{Theorem 2 (Global Classification).} The configuration of singularities of a tame torus curves $C$ of type (2,3) is given by Table B, where the notation $\{\Sigma(C)\}^*$ is for the cases of reducible $C$. 


Table B: Global classification

<table>
<thead>
<tr>
<th>n</th>
<th>i-vector ((i_1,i_2,...,i_n))</th>
<th>Configuration of singularities (\Sigma(C))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(6)</td>
<td>{A_{17}, {C_{3,15}}, {C_{9,9}}} {B_{3,12}}^<em>, {C_{6,12}}^</em>, {Sp_2}^<em>, {B_{6,6}}^</em></td>
</tr>
<tr>
<td>2</td>
<td>(1,5)</td>
<td>{A_2, \sigma}, \ \sigma = A_{14}, C_{3,12}, B_{3,10}, C_{6,9}, Sp_1</td>
</tr>
<tr>
<td>3</td>
<td>(2,4)</td>
<td>{A_5, \sigma}, \ \sigma = A_{11}, C_{3,9}^, B_{3,8} {A_5, \sigma'}^*, \ \sigma' = C_{6,6}, B_{4,6}, D_{4,7}</td>
</tr>
<tr>
<td>4</td>
<td>(3,3)</td>
<td>{2A_8}, \ {A_8, B_{3,6}}, \ {A_8, C_{3,7}}, \ {A_8, C_{3,8}}^<em>, \ {2B_{3,6}}^</em></td>
</tr>
<tr>
<td>5</td>
<td>(1,1,1,2)</td>
<td>{4A_2, A_5}, \ {4A_2, E_6}</td>
</tr>
<tr>
<td>6</td>
<td>(1,1,1,1,1)</td>
<td>{6A_2}</td>
</tr>
</tbody>
</table>

**Proof.** We will prove the assertion by three steps (corresponds to 3 steps in the strategy) as follows.

**Step 1 + Step 2:** The i-vector of length 6 \((1,1,1,1,1,1)\) is obviously given under the generic situation where \(C_2\) and \(C_3\) intersect transversely.

The i-vector with \(n = 5\) is given by \((1,1,1,1,2)\) and the possible configurations are \\(\{A_5, 4A_2\}\) and \\(\{E_6, 4A_2\}\).

The i-vectors with \(n = 4\) is either \((1,1,1,3)\) or \((1,1,2,2)\). The possible configurations are \\((1,1,1,3)\) : \\(\{3A_2, \sigma\}\), where \(\sigma = A_{8}, B_{3,6}, C_{3,7}, C_{3,8}, C_{3,9}\)
\(\{1,1,2,2\}\) : \\(\{2A_2, 2A_5\}, \ \{2A_2, A_5, E_6\}, \ \{2A_2, 2E_6\}\)

The i-vectors with \(n = 3\) are \((1,1,4), (1,2,3), (2,2,2)\) and the possible configurations are
\((1,1,4)\) : \\(\{2A_2, \sigma\}\), where \(\sigma = A_{11}, C_{3,9}^, B_{3,8}, C_{6,6}, B_{4,6}, D_{4,7}\)
\((1,2,3)\) : \\(\{A_2, A_5, \sigma\}, \ \{A_2, E_6, \sigma\}\), where \(\sigma = A_{8}, B_{3,6}, C_{3,7}, C_{3,8}, C_{3,9}\)
\((2,2,2)\) : \\(\{3A_5\}, \ \{3E_6\}, \ \{2A_5, E_6\}, \ \{A_5, 2E_6\}\)
The i-vectors with $n = 2$ are $(1,5),(2,4),(3,3)$ and the possible configurations are

$$(1,5): \{A_2, \sigma\}, \text{ where } \sigma = A_{14}, C_{3,12}, B_{3,10}, C_{6,9}, Sp_1$$

$$(2,4): \{\tau, \sigma\} \text{ where } \tau = A_5, E_6, \sigma = A_{11}, C_{3,9}, B_{3,8}, C_{6,6}, B_{4,6}, D_{4,7}$$

$$(3,3): \{\tau, \xi\} \text{ where } \tau, \xi = A_8, B_{3,6}, C_{3,7}, C_{3,8}, C_{3,9}, C_{3,9}$$

For the case $n = 1$, we have the following obvious possibility.

$$(6): \{A_{17}, \{C_{3,15}\}, \{C_{9,9}\}, \{B_{3,12}\}, \{C_{6,12}\}, \{Sp_2\}, \{B_{6,6}\}$$

**Step 3. Existence and Non-existence.**

$n = 6$: The configuration $\{6A_2\}$ is given by generic conic and cubic which intersect transversely and this case was first studied by Zariski, [Z1].

$n = 5$: The configuration $\{4A_2, \sigma\}$, $\sigma = A_5$ or $E_6$ is given by the generic member of the moduli of the singularity $A_5$ or $E_6$ discussed in §1.

$n = 4$: Any configurations $\{3A_2, \sigma\}$ where $\sigma = A_8, B_{3,6}, C_{3,7}, C_{3,8}$ are obtained by a generic curve in the moduli of singularity $\sigma$ at the origin. This holds also for $\{3A_2, C_{3,8}\}$, but we notice that a generic member of this class consists of a line and a quintic, though it satisfies the inequality $2(2, 2)$.

The configuration $\{2A_2, 2A_3\}$ is given by a smooth conic $C_2$ and a smooth cubic $C_3$ which are tangent at two points and have two other transverse intersections. The configurations $\{2A_2, A_5, E_6\}$ and $\{2A_2, 2E_6\}$ are obtained by the similar device. The existence of these configurations also shown by explicit equations, in Proposition 3.1 of Section 3.

$n = 3$: For $(1,1,4)$, the existence of the configurations $\{2A_2, \sigma\}$ where $\sigma = A_{11}, C_{3,9}, B_{3,8}, C_{6,6}, B_{4,6}$ is proved as the above. Similarly the configuration $\{2A_2, D_{4,7}\}$ exists and an example is given by $f_2 = (x + y)(-x + 2y), f_3 = y^2 - x^3$. In this case $C$ has two irreducible components, a line $\Gamma_1$ and a quintic $\Gamma_5$. Note that $\Gamma_1 \cap \Gamma_5 = \{O\}$, $\Gamma_5$ has a singularity $E_7$ at $O$ and $2A_2$ singularities.

Now we consider the configuration with i-vector $(2,2,2)$. The configuration $\{3A_5\}$ is given for instance by $f_2 = (x - 1)^2 + y^2 - 1$ and $f_3 = x(y^2 - 1)$. Note that $C_2 \cap C_3$ consists of three simple tangent points. The configuration $\{3E_6\}$ is given by $f_2 = (x - 1)^2 + y^2 - 1$ and $f_3 = (x - 1)(x^2 - y^2)$. This case was studied by Oka [O4]. The existence of the configurations $\{2A_2, E_6\}$, $\{A_5, 2E_6\}$ is shown in Proposition 3.1.

Now we consider the configurations with i-vector $(1,2,3)$. Examples of the configurations $\{A_2, A_5, \sigma\}$ and $\{A_2, E_6, \sigma\}$ with $\sigma = A_8, B_{3,6}, C_{3,7}$ are given in Proposition 3.1.

The configuration $\{A_2, A_5, C_{3,8}\}$ exists and an example is given by $f_2 = y + x^2, f_3 = (-23x^3 - 15xy^2 + 27xy^2 - 16y^2 + 27y)^2/27$. In this case, $C$ has two irreducible components, a conic $\Gamma_2$ which is defined by $5x^2 - 16xy - 16y^2 - 9y = 0$ and a quartic $\Gamma_4$ which defined by $40x^4 - 10y^3 - 117yx^3 + 51y^2x^2 + 18y^2 - 16xy^3 + 81y^2 + 16y^4 - 63y^3 = 0$. Note that $\Gamma_2 \cap \Gamma_4 = \{O, A\}$ where $(C, O)$ is $C_{3,8}$, $(C, A)$ is $A_5$ and $\Gamma_4$ has two singularities, $A_3$ at $O$ and $A_2$ at $B = (1, 1)$.

The configuration $\{A_2, A_5, C_{3,8}\}$ exists, by taking $f_2 = y + x^2, f_3 = 2y^3 + (3/2)yx - (7/2)xy^2 + x^2 - x^3$. In this case, $C$ is also has two irreducible components: they are a line $\Gamma_1 = \{y = 0\}$ and a quintic $\Gamma_5$. The line $\Gamma_1$ is a tangent direction of $\Gamma_5$ with multiplicity 5 at $O$, and they have no other intersection point except $O$. Moreover, $\Gamma_5$ has 3 singularities, an $A_3$ at $O$, an $A_5$ at $(1, 1)$ and an $A_2$ at $(-1/4, 1/16)$.

The configuration $\{A_2, E_6, C_{3,8}\}$ exists, by taking $f_2 = y^2 + y(x - 1/4) - x^2, f_3 = y(-3/8)x + (9/16)y - x^3 + (19/8)x^2y - (63/32)y^3$. Again $C$ is a union of a line and a quintic.

$n = 2$: The configurations with respect to i-vector $(1,5)$ do exist by similar reason as the cases $(1,1,4), (1,1,1,3)$ or $(1,1,1,1,2)$. 
Now we consider the configuration with i-vector (2,4). Examples of the configurations \{A_5, \sigma\} and \{E_6, \sigma\} with \(\sigma = A_{11}, C_{3,9}, B_{3,8}\) are given in Proposition 3.1.

There are 3 more configurations for i-vector is (2,4), in these cases the conic is a line with multiplicity 2. Hence at least f has two factors. Explicit examples are given as follows.

\{A_5, B_{4,6}\}^* : \(f_2(x, y) = x^2\) and \(f_3(x, y) = y^2 - x^3\). C has two cubic components, defined by \(y^2 - (1 \pm \sqrt{-1})x^3 = 0\). The singularities of C are a \(B_{4,6}\) at O and an \(A_5\) at (0,1).

\{A_5, D_{4,7}\}^* : \(f_2 = -(x-y)^2\) and \(f_3 = y^2 - x^3\). In this case, C has 3 irreducible components, a line \(\Gamma_1\), a conic \(\Gamma_2\) and a cuspidal cubic \(\Gamma_3\). The singularities of C are a \(D_{4,7}\) at O and an \(A_5\) at (1,1). \(\Gamma_2\) intersects \(\Gamma_3\) at O and (1,1) with respective intersection multiplicity 3 and 1, while \(\Gamma_1\) is tangent to both \(\Gamma_2\) and \(\Gamma_3\) at O.

\{A_5, C_{6,6}\}^* : \(f_2 = -x^2\) and \(f_3 = (y - 1)(x^2 - y^2)\). In this case, C has 2 irreducible components, both of them are nodal cubics. The singularities of C are a \(C_{6,6}\) at O and an \(A_5\) at (0,1).

i-vector is (3,3): Examples of the configurations \{2A_8\}, \{A_8, B_{3,6}\} and \{A_8, C_{3,7}\} are given in Proposition 3.1.

An example of \(A_8, C_{3,8}\)^* is given by \(f_2 = y - x^2, f_3 = -x^3 + x^2y + (9y + 16y^2)x/6 + y^2 + 32y^3/27\). In this case, C is a union of a line \(\Gamma_1\) and a quintic \(\Gamma_5\), where \(\Gamma_1\) is a tangent direction of \(\Gamma_5\) at O, and \(\Sigma(\Gamma_5) = \{A_8, A_3\}\). The singularities of C are a \(C_{3,8}\) at O and an \(A_8\) at (1,1).

An example of \(2B_{3,6}\)^* is given by \(f_3 = x(x-2)y, f_2 = y^2 + x^2 - 2x\). In this case, C is a union of three smooth conics \(\Gamma^{(i)}_2\), and they intersect each other at two points O and A = (2,0) with multiplicity 2.

\(n = 1\). The existence in these cases is obvious from the local classification. The explicit equations for irreducible curves are given in Proposition 3.1. Explicit examples for the reducible cases are given as following.

\(B_{3,12}\)^* : Let \(f_2 = y - x^2, f_3 = y^2\). In this case, C consists of three smooth conics \(\Gamma^{(i)}_2\), which are tangent to each other at O with multiplicity 4.

\(C_{6,12}\)^* : Let \(f_2 = -y^2, f_3 = x(y - x^2)\). In this case, \(f = (x^3 - xy - y^2)(x^3 - xy + y^3)\), thus C is a union of two nodal cubics.

\(S_{P_2}\)^* : Let \(f_2 = -y^2, f_3 = y^2 - x^3\). In this case, \(f = (x^3 - y^2 - y^3)(x^3 - y^2 + y^3)\), thus C is a union of two cuspidal cubics.

\(B_{6,6}\)^* : Let \(f_2 = y^2, f_3 = x^3\). Thus C consists of 6 concurrent lines.

To complete the proof of Theorem 2, it suffices to show the non-existence of the remaining configurations.

\[\square\]

**Lemma 2.2.** There are no tame (2,3)-torus curves having a configuration of singularities in the following list.

1. \(\{A_2, E_6, C_{3,8}\}^*, \{A_2, E_6, C_{3,9}\}^*\) and \(\{A_2, A_5, C_{3,9}\}^*\),
2. \(\{C_{3,k}, C_{3,l}\}^*\) for any \(6 \leq k, l \leq 9, (k, l) \neq (6, 6)\). Here we use the notation C_{3,8} for B_{3,6} for notation’s consistency, and C_{3,8} is also included in these pairs.
3. \(\{E_6, \sigma\}^*\) where \(\sigma = C_{6,6}, B_{4,6}, D_{4,7}\).
4. \(\{A_8, C_{3,8}\}^*, \{A_8, C_{3,9}\}^*\).

**Proof.** (1) For a singular point \(P \in C\), recall that \(\mu'(P) := \mu(P) + r(C, P) - 1\). We first consider the case \(\{A_2, E_6, C_{3,8}\}^*, \{A_2, E_6, C_{3,9}\}^*\) and \(\{A_2, A_5, C_{3,9}\}^*\). Note that \(\chi(C') = 4\) for each of them. Thus \(r(C) \geq 2\). In fact, we have observed before any generic member of the moduli of \(\{A_2, A_5, C_{3,8}\}^*\) consists of a conic and a quartic. Each of the above three configurations should be a degeneration of a family of \(\{A_2, A_5, C_{3,8}\}^*\). In the case of \(\{A_2, E_6, C_{3,8}\}^*\)
and \(\{A_2, E_6, C_{3,9}\}^*\) (respectively in the case of \(\{A_2, A_5, C_{3,9}\}^*\)), a quartic can not contain singularities \(A_2\) and \(E_6\) simultaneously (resp. \(A_2\) and \(A_6\)) by the inequality (2.2), which proves the non-existence of the case (1).

(2) Now we consider the configurations \(\{C_{3,k}, C_{3,l}\}^*\) for any \(6 \leq k, l \leq 9\). Note that \(\mu'(C_{3,l}) = 12, 12, 14, 14\) respectively and the tangent cones are irreducible. Thus if \(C\) has any two of them simultaneously, we need to have \(r(C) \geq 3\). Furthermore if \(C_{3,8}\) or \(C_{3,9}\) is included, \(r(C) \geq 4\), in which case, \(C\) has at least two line components. If \(C\) has two line components, we have an obvious contradiction by the number of the tangential cone argument. Thus the remaining cases are \(\{C_{3,k}, C_{3,l}\}^*, k, l = 6, 7\) and \(C\) has either 3 conics or one line and two other components of degree 2 and 3 respectively. Observe that \(C_{3,7}\) has irreducible singularity \(A_4\), which cannot exist on a cubic or a conic. Thus the non-existence is proved except \(\{C_{3,6}, C_{3,6}\}^* = \{B_{3,6}, B_{3,6}\}^*\). This exceptional case exists as we have seen before.

(3) Assume that the singularity at \(O\) is \(C_{6,6}, B_{4,6}\) or \(D_{4,7}\). Then the conic \(C_2\) consists of two different lines \(\ell_1, \ell_2\) or a line with multiplicity two by the local classification argument in §2. Note that in any case, the intersection multiplicity of a line (or the reduced line) in \(C_2\) and \(C_3\) at \(O\) is 2. Thus there exists a simple intersection outside of \(O\). In the first case, the other partner singularities are \(A_2\), which is not the case (3). If \(C_2\) is a double line, the other partner singularity is \(A_5\) as we have seen in Corollary 1.3, which is also not in case (3).

(4) Assume that \(C\) has a configuration \(\{A_8, C_{3,8}\}^*\) or \(\{A_8, C_{3,9}\}^*\). Then they must be a degeneration of the configuration of type \(\{A_2, A_5, C_{3,8}\}^*\) whose generic member consists of a conic and a quartic. Thus by the same argument as above, this is impossible. 

I am grateful to Professor Oka for pointing out some missing cases in Table B and showing the above non-computational proof of Lemma 2.2.

3. THE GLOBAL DEGENERATION

3.1. A certain degenerations. In this section we consider the degenerations of irreducible tame torus curves of type (2,3), our aim is finding the list of all maximal curves. Here an irreducible sextic \(C\) of a torus type is called maximal if \(C\) does not have any degeneration in the space of irreducible sextics of torus type.

The degenerations between the curves in the table B may come from the configurations in the same level of i-vectors in the table B, and also from different levels of i-vectors. We first show the following.

**Proposition 3.1.** In the same i-vectors level, the degeneration families are given in Table B'.

**Proof.** See in [P].

A configuration at the end of each degeneration family in the above proposition is called semi-maximal. Next we consider degenerations between the semi-maximal curves. For later purpose, we first give the following degenerations, which will be used to prove Theorem 3.4.

**Proposition 3.2.** Among the semi-maximal curves and together \(\{6A_2\}\), there are following degeneration families:

1. \(\{6A_2\} \rightarrow \{4A_2, E_6\} \rightarrow \{2A_2, 2E_6\} \rightarrow \{3E_6\} \rightarrow \{E_6, B_{3,8}\}\)
2. \(\{A_2, E_6, C_{3,7}\} \rightarrow \{E_6, B_{3,8}\}\)
3. \(\{2A_2, B_{4,6}\} \rightarrow \{A_2, Sp_1\}\)

**Proof.** See in [P].
Table B': Global degenerations (with same i-vector)

| {A_{17}} | → | {C_{3,15}} | {A_{17}} | → | {C_{9,9}} |
| {A_{2}, A_{14}} | → | {A_{2}, C_{3,12}} | {A_{2}, B_{3,10}} |
| {A_{2}, A_{14}} | → | {A_{2}, C_{6,9}} | {A_{2}, Sp_{1}} |
| {A_{5}, A_{11}} | → | {A_{5}, C_{3,9}^{\#}} | {A_{5}, B_{3,8}} |
| {E_{6}, A_{11}} | → | {E_{6}, C_{3,9}^{\#}} | {E_{6}, B_{3,8}} |
| {2A_{8}} | → | {A_{8}, B_{3,6}} | {A_{8}, C_{3,7}} |
| {2A_{2}, A_{11}} | → | {2A_{2}, C_{3,9}^{\#}} | {2A_{2}, B_{3,8}} |
| {A_{2}, E_{6}, A_{8}} | → | {A_{2}, E_{6}, B_{3,6}} | {A_{2}, E_{6}, C_{3,7}} |
| {3A_{5}} | → | {2A_{5}, E_{6}} | {A_{5}, 2E_{6}} | → | {3E_{6}} |
| {3A_{2}, A_{8}} | → | {3A_{2}, B_{3,6}} | {3A_{2}, C_{3,7}} | → | {3A_{2}, C_{3,8}} | → | {3A_{2}, C_{3,9}} |
| {2A_{2}, 2A_{5}} | → | {2A_{2}, A_{5}, E_{6}} | → | {2A_{2}, 2E_{6}} |
| {4A_{2}, A_{5}} | → | {4A_{2}, E_{6}} |

3.2. Maximal sextics. First we remark the following.

Lemma 3.3. Assume that we have an analytic family of plane curves \(|t| \leq \varepsilon\) such that \(C_t\) has only isolated singularities in a fixed open neighborhood \(U\) of the origin for any \(t\) and \(O\) is the unique singularity of \(C_0\). We assume that \(P_{i,1}, \ldots, P_{i,\nu}\) are the singular points of \(C_t \cap U\) which converges to \(O\) for \(t = 0\). Then we have

1. If \(\nu \geq 2\), \(\sum_{i=1}^{\nu} \mu(C_t, P_{i,t}) > \mu(C_0, O)\).
2. If \(\nu = 1\), \(\mu(C_0, O) \geq \mu(C_t, P_{1,t})\) for \(t \neq 0\) and the equality holds if and only if \((C_0, O)\) is equivalent to \((C_t, P_{1,t})\) for \(t \neq 0\).

Proof. The first assertions follows from the vanishing theorem of Lefschetz number of the monodromy ([A1], [Lê2]) and the second assertion is due to ([Lê2],[LR]).

Using Propositions 3.1 and 3.2 we obtain the following theorem.

Theorem 3.4. The maximal sextics of torus type has the following configurations.

1. \(n = 1: \{C_{3,15}\}, \{C_{9,9}\}\).
2. \(n = 2: \{A_{2}, B_{3,10}\}, \{A_{2}, Sp_{1}\}, \{E_{6}, B_{3,8}\}, \{A_{8}, C_{3,7}\}\).
3. \(n = 4: \{3A_{2}, C_{3,9}\}\).

Proof. Assume that there is a family of degeneration \(C_t \rightarrow C_0\) for \(t \rightarrow 0\). By Lemma 3.3, the sum of Milnor numbers is strictly increasing for \(t \rightarrow 0\). For \(\{C_{3,15}\}, \{C_{9,9}\}\), the assertion is obvious as they have \(\mu = 19\) and no other place to degenerate. Now we consider the
configurations \{A_2, B_{3,10}\}, \{A_2, S_{P1}\}, \{E_6, B_{3,8}\}. The total Milnor numbers are 20 for each of them and they are obviously maximal. For \{A_8, C_{3,7}\}, the total Milnor number is 19 and the possibility of the degeneration is to one of \{A_2, B_{3,10}\}, \{A_2, S_{P1}\}, \{E_6, B_{3,8}\}. If this is the case, there must be degenerations of each of the two singularities to the corresponding singularities in the above configurations. Then \(C_{3,7}\) has to degenerate into either \(S_{P1}\) or \(B_{3,10}\) or \(B_{3,8}\), and then \(A_8\) has no partner to degenerate.

It remains to show that the configuration \(\{3A_2, C_{3,9}\}\) can not degenerate to any configuration in (1) or (2). By the total Milnor number argument, the possibility is to \(\{A_2, B_{3,10}\}, \{A_2, S_{P1}\}\) or \(\{E_6, B_{3,8}\}\). Assume that \(\{3A_2, C_{3,9}\} \rightarrow \{E_6, B_{3,8}\}\). As \(2A_2 \rightarrow E_6\) but \(3A_2 \not\rightarrow E_6\), we need to have a degeneration \(A_2 + C_{3,9} \rightarrow B_{3,8}\) which is ridiculous as the Milnor number is decreasing. The imposibility of the degeneration \(\{3A_2, C_{3,9}\} \rightarrow \{A_2, B_{3,10}\}\) or \(\{A_2, S_{P1}\}\) can be proved by computation, see in [P].

**Remark 3.5.** The imposibility of the above degeneration can be shown by looking the dual curves. Suppose that we have a degenerating family \(C_t\) with \(C_t\) has \(\{3A_2, C_{3,9}\}\) for \(t \neq 0\) and \(C_0\) has the configuration \(\{A_2, S_{P1}\}\). We assume also that the singularities \(C_{3,9}\) and \(S_{P1}\) are at the origin, with \(y = 0\) as the tangent cone. Then we first notice that the dual curve of \(C_t\) and \(C_0\) has degree 6 ([O4]). This implies that the corresponding dual curve can be considered as an analytic family of sextics. Secondly we can see that the dual singularity of \(C_{3,9}\) is again \(C_{3,9}\) at \((0,1,0)\) and \(C_t^*\) has further three \(A_2\) singularities also. On the other hand, the dual singularity of \(S_{P1}\) is \(A_8\) by Theorem 14, [O4]. Let \(L\) be the line supporting the tangent cone of \(C_0\) at the singularity \(S_{P1}\). Then \(L \cap C_0 = \{O\}\). This implies that in the dual curve \(C_0^*\), it has \(A_8\) singularity at \((0,1,0)\). Then in the family \(C_t^* \rightarrow C_0^*\), we have the degeneration \(C_{3,9} \rightarrow A_8\). This is impossible as the Milnor number is decreasing. The detail will be studied in our next paper [OP].

4. **New Zariski pairs**

We first modify the definition of a Zariski pair given in [Ar]. Our definition is weaker than the original one.

**Definition 4.1.** A pair of plane curves \((C, C')\) is called a **weak Zariski pair** if they have same degree and \(\Sigma(C) \sim \Sigma(C')\), but \(\mathbb{P}^2 - C\) is not homeomorphic to \(\mathbb{P}^2 - C'\).

We do not ask the respective tubular neighborhood of \(C\) and \(C'\) are homeomorphic. The notation \(\Sigma(C) \sim \Sigma(C')\) means that there is a bijection \(\phi : \Sigma(C) \rightarrow \Sigma(C')\) such that \((C, \xi)\) is topological equivalent to \((C', \phi(\xi))\) for any \(\xi \in \Sigma(C)\).

As a corollary of Theorem 2 we have two new Zariski pairs with respect to the following configurations

a) \(\{A_2, A_5, C_{3,8}\}^*\) and \(\{A_2, A_5, C_{3,8}^d\}^*\).

b) \(\{3A_2, C_{3,8}\}\) and \(\{3A_2, C_{3,8}^d\}^*\).

In the first pair, \(\{A_2, A_5, C_{3,8}\}^*\) is a union of a conic and a quartic, while \(\{A_2, A_5, C_{3,8}^d\}^*\) is a union of a line and a quintic.

In the second pair, \(\{3A_2, C_{3,8}\}\) is irreducible, while \(\{3A_2, C_{3,8}^d\}^*\) is reducible, a union of a line and a quintic.

The explicit equations of the above pairs and the discussions about geometry of these curves are given in the previous section.
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