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Kyoto University
A Quantization of Conjugacy Classes of Matrices

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概要: 一般の放物型部分群のスカラー表現から誘導された $\mathfrak{gl}(n, \mathbb{C})$ の
Generalized Verma 加群の零化イデアルの具体的な生成元を、行列における小行列式や単因子の「量子化」として構成する。「量子化」のパラメータ $\epsilon$ を 0 とした「古典極限」では、構成した生成元は正方行列の相
似類の作る集合の定義イデアルとなる。

1. Introduction

Let $A$ be an element of the space $M(n, \mathbb{C})$ of square matrices of size $n$ with
components in $\mathbb{C}$. Then the conjugacy class containing $A$ is the algebraic variety
$V_A = \bigcup_{g \in G} \text{Ad}(g)A$ by denoting $G = GL(n, \mathbb{C})$ and $\text{Ad}(g)A = gAg^{-1}$. Under the
$G$-action on $M(n, \mathbb{C})$, we will study a quantization of $V_A$ interpreted as follows:

For the defining equations of $V_A$ or the $G$-invariant defining ideal of $V_A$ in the
ring of polynomial functions of $M(n, \mathbb{C})$, we will associate left invariant differential
operators on $G$ or an ideal $J_A$ of the ring of the left invariant differential operators
on $G$. The Lie algebra $\mathfrak{g}$ of $GL(n, \mathbb{C})$ is identified with $M(n, \mathbb{C})$ and we identify the
left invariant differential operators on $G$ with the universal enveloping algebra $U(\mathfrak{g})$
of $\mathfrak{g}$. Then our quantization of $V_A$ is a $U(\mathfrak{g})$-homomorphism of $U(\mathfrak{g})/J_A$ to a suitable
$U(\mathfrak{g})$ module $M$. Note that the quantization of $V_A$ becomes a representation space
of a real form $G_{\mathbb{R}}$ of $G$ if $M$ is a function space on a homogeneous space of $G_{\mathbb{R}}$ or
a space of sections of a $G_{\mathbb{R}}$-homogeneous vector bundle.

$$V_A = \bigcup_{g \in G} \text{Ad}(g)A \quad \longrightarrow \quad \text{G-invariant defining ideal of } V_A$$

Representations of $U(\mathfrak{g})$ or $G_{\mathbb{R}} \quad \longleftarrow \quad \text{Ideal of } U(\mathfrak{g})$

In §2 we introduce a homogenized universal enveloping algebra $U^\epsilon(\mathfrak{g})$ to study
our quantization together with "the classical limit" ($\epsilon = 0$). We construct gener-
ators of $J_A$ from the generalized Capelli operators introduced by [O2] which can be
considered as quantizations of minors and we show in Theorem 2.8 that they
generate the annihilator of a generalized Verma module induced from a character
of a parabolic subalgebra of $\mathfrak{g}$. When $\epsilon = 0$ and $A$ is a nilpotent matrix, the corre-
spanding result is Tanisaki's conjecture [Ta], which is solved by Weyman [We]. In
particular, if $A$ is a regular nilpotent matrix, the result is due to Kostant [Ko].
In §3 we examine how the annihilator determines the difference between the
generalized Verma module and the Verma module, which is important for applica-
tions. For example, the theorem on boundary value problems for symmetric spaces
studied in [O2, Theorem 5.1] is improved by the generator system defined in this
note.

We can also quantize the minimal polynomial of $V_A$ from which we can construct
another generator system of the annihilator. This is valid for other general reductive
Lie algebras and is studied in another paper [O3].

2. Elementary divisors

The Lie algebra $\mathfrak{g}$ of $G = GL(n, \mathbb{C})$ is identified with $M(n, \mathbb{C})$ and also with
the space of left $G$-invariant holomorphic vector fields on $G$. Then $\mathfrak{g}$ is spanned by
$E_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$ where $E_{ij}$ is the fundamental matrix unit whose
$(p, q)$-component equals $\delta_{ip}\delta_{jq}$ and

\begin{equation}
E_{ij} = \sum_{\nu=1}^{n} x_{\nu i} \frac{\partial}{\partial x_{\nu j}}
\end{equation}

with the coordinate $(x_{ij}) \in G$. Then $\mathfrak{g}$ is naturally a $(\mathfrak{g}, G)$-module.

Using the non-degenerate symmetric bilinear form $\langle X, Y \rangle = \text{Trace}(XY)$ on
$M(n, \mathbb{C}) \times M(n, \mathbb{C})$ we identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. The dual basis $\{E_{ij}^*\}$ of $\{E_{ij}\}$ is
given by $E_{ij}^* = E_{ji}$. For simplicity, we will denote $E_i = E_{ii}$ and $e_i = E_{ii}^*$.

**DEFINITION 2.1.** The *homogenized universal enveloping algebra* $U^\epsilon(\mathfrak{g})$ of $\mathfrak{g}$ is
defined by

\begin{equation}
U^\epsilon(\mathfrak{g}) = \left( \sum_{k=0}^{\infty} \otimes^k \mathfrak{g} \right) / (X \otimes Y - Y \otimes X - \epsilon[X, Y]; X, Y \in \mathfrak{g})
\end{equation}

and the subalgebra of $G$-invariants in $U^\epsilon(\mathfrak{g})$ is denoted by $U^\epsilon(\mathfrak{g})^G$. Here $\epsilon$ is a
complex number (or an element commuting with $\mathfrak{g}$) and the denominator is the
span as a two-sided ideal of the numerator, the tensor algebra of $\mathfrak{g}$.

Note that $U^\epsilon(\mathfrak{g})$ is naturally a $(\mathfrak{g}, G)$-module induced from the tensor algebra.
$U^1(\mathfrak{g})$ and $U^0(\mathfrak{g})$ are the universal enveloping algebra $U(\mathfrak{g})$ and the symmetric
algebra $S(\mathfrak{g})$ of $\mathfrak{g}$, respectively. If $\epsilon \neq 0$, the map defined by $E_{ij} \mapsto \epsilon E_{ij}$ gives an
algebra isomorphism of $U^\epsilon(\mathfrak{g})$ onto $U(\mathfrak{g})$.

The residue class of the element $X_1 \otimes X_2 \otimes \cdots \otimes X_m$ $(X_j \in \mathfrak{g})$ in $U^\epsilon(\mathfrak{g})$ will
be denoted by $X_1X_2\cdots X_m$ and the image of $\sum_{k=0}^{m} \otimes^k \mathfrak{g}$ in $U^\epsilon(\mathfrak{g})$ is denoted by
$U^\epsilon(\mathfrak{g})^{(m)}$. 
For an ordered partition \( \{n'_1, \ldots, n'_L\} \) of a positive integer \( n \) into \( L \) positive integers put

\[
\begin{align*}
\{n_j\} &= n'_1 + \cdots + n'_j \quad (1 \leq j \leq L), \quad n_0 = 0, \\
\Theta &= \{n_1, n_2, \ldots, n_L\}, \\
\iota_\Theta(\nu) &= j \quad \text{if } n_{j-1} < \nu \leq n_j \quad (1 \leq \nu \leq n).
\end{align*}
\]

The ordered partition of \( n \) is expressed by the set \( \Theta \) of strictly increasing positive integers ending at \( n \). Define Lie subalgebras \( n_\Theta, \bar{n}_\Theta \) and \( m_\Theta \) by the span of \( E_{ij} \) with \( \iota_\Theta(i) > \iota_\Theta(j), \iota_\Theta(i) < \iota_\Theta(j) \) and \( \iota_\Theta(i) = \iota_\Theta(j) \), respectively, and put \( p_\Theta = m_\Theta + n_\Theta \).

We denote \( m_\Theta^k = \sum_{\iota_\Theta(i) = \iota_\Theta(j) = k} \mathbb{C}E_{ij}, \quad n = \sum_{1 \leq j < i \leq n} \mathbb{C}E_{ij}, \quad \bar{n} = \sum_{1 \leq i < j \leq n} \mathbb{C}E_{ij}, \quad a = \sum_{j=1}^{n} \mathbb{C}E_{ij} \) and \( p = a + n \). Then \( m_\Theta = m_\Theta^1 \oplus \cdots \oplus m_\Theta^L \) and \( p_\Theta \) is a parabolic subalgebra containing the minimal parabolic subalgebra \( p \). We remark that \( p_\Theta = \{X \in g; \langle X, Y \rangle = 0 \quad (\forall Y \in n_\Theta)\} \).

Fix \( \lambda = (\lambda_1, \ldots, \lambda_L) \in \mathbb{C} \) and define a closed subset of \( p \):

\[
A_{\Theta, \lambda} = \sum_{j=1}^{n} \lambda_{\iota_\Theta(j)} E_j + n_\Theta
\]

(2.4)

Here \( I_m \) denotes the identity matrix of size \( m \) and \( M(k, \ell; \mathbb{C}) \) denotes the space of matrices of size \( k \times \ell \) with components in \( \mathbb{C} \). The generic element of \( A_{\Theta, \lambda} \) corresponds to a unique Jordan's canonical form and any Jordan's canonical form is obtained by this correspondence with a suitable choice of \( \Theta \) and \( \lambda \).

The set \( \bigcup_{g \in G} \text{Ad}(g)A_{\Theta, \lambda} \) is a closed algebraic variety of \( M(n, \mathbb{C}) \) because any element of \( M(n, \mathbb{C}) \) can be transformed into an element in \( p \) under the \( \text{Ad} \)-action of the unitary group \( U(n) \). Then if \( \epsilon = 0 \), for \( f \in U^0(g) = S(g) \) we have

\[
f(\bigcup_{g \in G} \text{Ad}(g)A_{\Theta, \lambda}) = 0 \iff (\text{Ad}(g)f)(A_{\Theta, \lambda}) = 0 \quad (\forall g \in G)
\]

\[
\iff \text{Ad}(g)f \in J_{\Theta}^\epsilon(\lambda) \quad (\forall g \in G)
\]

\[
\iff f \in \text{Ann}_G(M_{\Theta}^\epsilon(\lambda)).
\]
where

\[ J_{\ominus}^\epsilon(\lambda) = \sum_{X \in \mathfrak{p}_{\ominus}} U^\epsilon(\mathfrak{g}) (X - \lambda \ominus (X)) \],

(2.5)

\[ M_{\ominus}^\epsilon(\lambda) = U^\epsilon(\mathfrak{g}) / J_{\ominus}^\epsilon(\lambda) \],

\[ \text{Ann} \left( M_{\ominus}^\epsilon(\lambda) \right) = \{ D \in U^\epsilon(\mathfrak{g}); DM_{\ominus}^\epsilon(\lambda) = 0 \}, \]

\[ \text{Ann}_G \left( M_{\ominus}^\epsilon(\lambda) \right) = \{ D \in U^\epsilon(\mathfrak{g}); \text{Ad}(g)D \in \text{Ann} \left( M_{\ominus}^\epsilon(\lambda) \right) \ (\forall g \in G) \} \]

and the character \( \lambda_{\ominus} \) of \( \mathfrak{p}_{\ominus} \) is defined by

\[ \lambda_{\ominus}(Y + \sum_{k=1}^{L} x_k) = \sum_{1k=}^{L} \lambda_k \text{Trace}(x_k) \]

for \( X_k \in \mathfrak{m}_{\ominus}^k \) and \( Y \in \mathfrak{n}_{\ominus} \).

When \( \epsilon = 1 \), \( M_{\ominus}(\lambda) = M_{\ominus}^1(\lambda) \) is a generalized Verma module induced from the character \( \lambda_{\ominus} \) of \( \mathfrak{m}_{\ominus} \), which is a quotient of the Verma module

(2.7)

\[ M(\lambda_{\ominus}) = U(\mathfrak{g}) / J(\lambda_{\ominus}) \]

with

(2.8)

\[ J^\epsilon(\lambda_{\ominus}) = \sum_{X \in \mathfrak{p}} U^\epsilon(\mathfrak{g}) (x - \lambda_{\ominus}(X)) \]

and

\[ J(\lambda_{\ominus}) = J^1(\lambda_{\ominus}) \].

In general we will omit the superfix \( \epsilon \) if \( \epsilon = 1 \).

**PROPOSITION 2.2.**

(2.9)

\[ \text{Ann}_G \left( M_{\ominus}^\epsilon(\lambda) \right) = \text{Ann} \left( M_{\ominus}^\epsilon(\lambda) \right) \text{ if } \epsilon \neq 0, \]

(2.10)

\[ \text{Ann}_G \left( M_{\ominus}^\epsilon(\lambda) \right) = \bigcap_{g \in G} \text{Ad}(g)J_{\ominus}^\epsilon(\lambda). \]

**Proof.** We may assume \( \epsilon \neq 0 \) to prove the proposition.

Let \( D \in \text{Ann} \left( M_{\ominus}^\epsilon(\lambda) \right) \). Then for \( X \in \mathfrak{g} \) and \( v \in M_{\ominus}^\epsilon(\lambda) \), \( (XD - DX)v = X(Dv) - D(Xv) = 0 \) and therefore \( XD - DX \in \text{Ann} \left( M_{\ominus}^\epsilon(\lambda) \right) \). Since \( XD - DX = \epsilon \text{ad}(X)D \) in \( U^\epsilon(\mathfrak{g}) \), \( \text{ad}(X)D \in \text{Ann} \left( M_{\ominus}^\epsilon(\lambda) \right) \) and therefore \( \text{Ad}(g)D \in \text{Ann} \left( M_{\ominus}^\epsilon(\lambda) \right) \) for \( g \in G \).

Put \( I = \bigcap_{g \in G} \text{Ad}(g)J_{\ominus}^\epsilon(\lambda) \). Since \( \text{Ann}(M_{\ominus}^\epsilon(\lambda)) \subset J_{\ominus}^\epsilon(\lambda) \), \( \text{Ann}_G \left( M_{\ominus}^\epsilon(\lambda) \right) \subset I \).

For \( P \in U^\epsilon(\mathfrak{g}) \), \( IP = PI \equiv 0 \mod J_{\ominus}^\epsilon(\lambda) \) because \( I \) is a two-sided ideal of \( U^\epsilon(\mathfrak{g}) \), which means \( I \subset \text{Ann} \left( M_{\ominus}^\epsilon(\lambda) \right) \).

**DEFINITION 2.3.** Define the polynomials and an integer

\[ d_x^\epsilon(x) = d_x^\epsilon(x; \Theta, \lambda) = \prod_{j=1}^{L} (x - \lambda_j - n_{j-1} \epsilon)^{n_j + m - n}, \]

\( d_m = d_m(\Theta) = \deg_x d_x^\epsilon(x; \Theta, \lambda) = \sum_{j=1}^{L} \max\{n_j + m - n, 0\}, \)

\( e_x^\epsilon(x) = e_x^\epsilon(x; \Theta, \lambda) = d_x^\epsilon(x)/d_{x-1}^\epsilon(x), \)

\( q(x) = q(x; \Theta, \lambda) = \prod_{j=1}^{L} (x - \lambda_j - n_{j-1} \epsilon) \)
by putting

\[(2.12)\]

\[
z^{(\ell)} = \begin{cases} 
  z(z - \epsilon) \cdots (z - (\ell - 1)\epsilon) & \text{if } \ell > 0, \\
  1 & \text{if } \ell \leq 0 
\end{cases}
\]

and call \(d_m^n(x), q^\epsilon(x)\) and \(\{e_m^\epsilon(x); 1 \leq m \leq n\}\) the characteristic polynomial, the minimal polynomial and the elementary divisors of \(M_{\ominus}^\epsilon(\lambda)\), respectively.

**Remark 2.4.**

i) The set \(\{e_m^\epsilon(x)\}\) recovers \(\{d_m^n(x)\}\) because \(e_m^\epsilon(x) \in \mathbb{C}[x]e_{m-1}^\epsilon(x - \epsilon)\).

ii) For the generic element \(A\) of \(J_{\ominus}^0(\lambda)\), the greatest common divisor of \(m\)-minors of the matrix \(xI_n - A\) equals \(d_m^0(x)\) and therefore when \(\epsilon = 0\), the above definition coincides with that in the linear algebra.

iii) The meaning of the minimal polynomial for \(\epsilon \neq 0\) will be clear in [O3].

Now we introduce quantized minors.

**Definition 2.5.** For set of indices \(I = \{i_1, \ldots, i_m\}\) and \(J = \{j_1, \ldots, j_n\}\) with \(i_\mu, j_\nu \in \{1, \ldots, n\}\), define a generalized Capelli operator (cf. [O2])

\[(2.13)\]

\[
\det^\epsilon(x; E_{IJ}) = \det \left((x + (\nu - m)\epsilon)\delta_{i_\mu j_\nu} - E_{i_\mu j_\nu}\right)_{1 \leq \mu \leq m, 1 \leq \nu \leq m}
\]

in \(U^\epsilon(\mathfrak{g})[x]\) by the column determinant:

\[(2.14)\]

\[
\det \left(A_{\mu \nu}\right)_{1 \leq \mu \leq m} = \sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma)A_{\sigma(1)1}A_{\sigma(2)2} \cdots A_{\sigma(m)m}.
\]

**Proposition 2.6.** The Capelli operators satisfy

\[(2.15)\]

\[
\det^\epsilon(x; E_{\sigma(I)\sigma'(J)}) = \text{sgn}(\sigma)\text{sgn}(\sigma') \det^\epsilon(x; E_{IJ}) \quad \text{for } \sigma, \sigma' \in \mathfrak{S}_m,
\]

\[(2.16)\]

\[
\text{ad}(E_{ij}) \det^\epsilon(x; E_{IJ}) = D_1 - D_2
\]

where

\[
\sigma(I) = \{i_{\sigma(1)}, \ldots, i_{\sigma(m)}\}, \quad \sigma'(J) = \{j_{\sigma'(1)}, \ldots, j_{\sigma'(m)}\},
\]

\[
D_1 = \begin{cases} 
  \det^\epsilon(x; E_{i_1, \ldots, i_{\mu-1}, i_{\mu+1}, \ldots, i_m}J) & \text{if there exists only one } i_\mu \text{ with } i_\mu = j, \\
  0 & \text{otherwise},
\end{cases}
\]

\[
D_2 = \begin{cases} 
  \det^\epsilon(x; E_{J_1, \ldots, J_{\nu-1}, i_{\nu+1}, \ldots, J_m}) & \text{if there exists only one } j_\nu \text{ with } j_\nu = i, \\
  0 & \text{otherwise}.
\end{cases}
\]

**Proof.** When \(\epsilon = 1\), (2.15) and (2.16) are proved by [O2, Lemma 2.2 and Proposition 2.4]. Combining this with the definition of \(U^\epsilon(\mathfrak{g})\), we have the proposition.

**Definition 2.7.** Under Definition 2.3 and Definition 2.5, put

\[(2.17)\]

\[
\det^\epsilon(x; E_{IJ}) = h_{IJ}(x)d_m^\epsilon(x) + r_{IJ}^{d_{m-1}}x^{d_{m-1}} + \cdots + r_{IJ}^1x + r_{IJ}^0
\]
in $U^\epsilon(\mathfrak{g})[x]$ with $h_{IJ}[x] \in U^\epsilon(\mathfrak{g})[x]$ and $r^j_{IJ} \in U^\epsilon(\mathfrak{g})^{(m-j)}$ for $j = 0, \ldots, d_m - 1$ and define the two-sided ideal of $U^\epsilon(\mathfrak{g})$:

\begin{equation}
I^\epsilon_\otimes(\lambda) = \sum_{m=1}^{n} \sum_{\#I = \#J = m} \sum_{j=0}^{d_m-1} U^\epsilon(\mathfrak{g}) r^j_{IJ}.
\end{equation}

Note that if $m \leq n - \max\{n_1', \ldots, n_L'\}$ the summand equals 0 because $d_m = 0$. Moreover note that $\{r^j_{IJ}\}$ with $\#I = n$ are in $U^\epsilon(\mathfrak{g})^G$. In particular, if $\Theta = \{1, 2, \ldots, n\}$, then $p_{\Theta} = p$ and $I^\epsilon_\otimes(\lambda)$ is generated by suitable $n$ elements in $U^\epsilon(\mathfrak{g})^G$.

Now we can state the main result in this section and we call $r^j_{IJ}$ quantized Tanisaki generators of $\text{Ann}_G(M^\epsilon_\otimes(\lambda))$. In the case when $\epsilon = \lambda = 0$, $d^0_m(x; \Theta, 0) = x^{d_m}$ and the generators are introduced by [Ta].

\textbf{Theorem 2.8.} Under the notation (2.5) and (2.18) \[ \text{Ann}_G(M^\epsilon_\otimes(\lambda)) = I^\epsilon_\otimes(\lambda). \]

If all the roots of $d^\epsilon_n(x) = 0$ are simple, which is equivalent to say that the infinitesimal character of $M^\epsilon_\otimes(\lambda)$ is regular (cf. Remark 2.14), then

\begin{equation}
\text{Ann}_G(M^\epsilon_\otimes(\lambda)) = \sum_{k=1}^{L} \sum_{\#I = \#J = n+1-n_k'} U^\epsilon(\mathfrak{g}) D^\epsilon_{IJ}(\lambda_k + n_k-1\epsilon).
\end{equation}

Here for $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_m\}$ we put

\begin{equation}
D^\epsilon_{IJ}(x) = (-1)^m \det \epsilon(x; E_{IJ}) = \det \left( E_{i_\mu j_\nu} - (x + (\nu - m)\epsilon) \delta_{i_\mu j_\nu} \right)_{1 \leq \mu \leq m, 1 \leq \nu \leq m}.
\end{equation}

If all the roots of $d^\epsilon_{n-1}(x) = 0$ are simple, (2.19) holds modulo the ideal generated by $\text{Ann}_G(M^\epsilon_\otimes(\lambda)) \cap U^\epsilon(\mathfrak{g})^G$.

When $\epsilon = 0$, (2.19) holds if $\lambda_i \neq \lambda_j$ for $1 \leq i < j \leq L$ and the last statement above holds if $\lambda_i \neq \lambda_j$ for $1 \leq i < j \leq L$ satisfying $n_i' > 1$ and $n_j' > 1$.

\textbf{Remark 2.9.} Let $\{\lambda'_1, \ldots, \lambda'_k\}$ be the set of the roots of $d^\epsilon_m(x) = 0$ and let $m_k$ be the multiplicity of the root $\lambda'_k$. Here $d_m = m_1 + \cdots + m_k$ and $\lambda'_\mu \neq \lambda'_\nu$ if $1 \leq \mu < \nu \leq k$. Then

\begin{equation}
\sum_{j=0}^{d_m-1} \mathbb{C}r^j_{IJ} = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \mathbb{C} \left( \frac{d^{j-1}}{dx^{j-1}} D^\epsilon_{IJ}(x) \right)_{x=\lambda'_i}.
\end{equation}

for $\#I = \#J = m$.

The rest of this section will be devoted to the proof of this theorem. First we will examine the image of our minors under the Harish-Chandra homomorphism.

Define the map $\omega$ of $U^\epsilon(\mathfrak{g})$ to $S(\mathfrak{a}) = U^\epsilon(\mathfrak{a})$ by

\begin{equation}
D - \omega(D) \in U^\epsilon(\mathfrak{g})\mathfrak{n} + \mathfrak{n}U^\epsilon(\mathfrak{n} + \mathfrak{a}).
\end{equation}
Fix $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_m\}$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq n$. Then [O2, Corollary 2.11] in the case $\epsilon = 1$ shows

\[ \omega(D^\epsilon_{I,J}(x)) = \begin{cases} 0 & \text{if } I \neq J, \\ \prod_{\nu=1}^m \left( E_i - x + (\nu - 1)\epsilon \right) & \text{if } I = J \end{cases} \]

under the notation in Theorem 2.8. Introducing the algebra isomorphism

\[ \tilde{\omega} : S(\mathfrak{a}) \rightarrow S(\mathfrak{a}) \]

(2.24) with $E_j = E_j - \left( -\frac{n-1}{2} + (j-1) \right) \epsilon$ for $j = 1, \ldots, n$

(cf. Remark 2.14), put

\[ \tilde{\omega}(P) = \omega(P). \]

Then $\tilde{\omega}$ defines the Harish-Chandra isomorphism of $U^\epsilon(\mathfrak{g})^G$ onto the algebra $S(\mathfrak{a})^W$ of $\mathfrak{S}_n$-invariants in $S(\mathfrak{a})$. Here we note that if $I = \{i_1 < i_2 < \cdots < i_m\}$,

\[ \tilde{\omega}(D^\epsilon_{I,I}(x)) = \prod_{\nu=1}^m \left( E_i - x + \left( \frac{n-1}{2} + \nu - i_\nu \right) \epsilon \right). \]

(2.26) Since $D^\epsilon_{\{1, \ldots, n\}\{1, \ldots, n\}}(x) \in U^\epsilon(\mathfrak{g})^G[x]$ (cf. Proposition 2.6), it is clear that the coefficients of $D^\epsilon_{\{1, \ldots, n\}\{1, \ldots, n\}}(x)$ as a polynomial of $x$ generate the algebra $U^\epsilon(\mathfrak{g})^G$.

**Lemma 2.10.** Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be a triangular decomposition of a reductive Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Here $\mathfrak{n}$ and $\mathfrak{n}$ are nilpotent subalgebras of $\mathfrak{g}$ and $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$. For an element $D$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, we define $\omega(D) \in S(\mathfrak{a})$ so that

\[ D - \omega(D) \in U(\mathfrak{g})\mathfrak{n} + \mathfrak{n}U(\mathfrak{n} + \mathfrak{a}). \]

For a subspace $V$ of $U(\mathfrak{g})$

\[ \langle \omega(V) \rangle_{S(\mathfrak{a})} = \sum_{p \in \omega(V)} S(\mathfrak{a})p. \]

Then if $\text{ad}(\mathfrak{g})V \subset V$, we have

\[ \omega(PDQ) \in \langle \omega(V) \rangle_{S(\mathfrak{a})} \quad \text{for any } P, Q \in U(\mathfrak{g}) \text{ and any } D \in V. \]

**Proof.** Let $\{X_1, \ldots, X_N\}$, $\{Y_1, \ldots, Y_N\}$ and $\{H_1, \ldots, H_M\}$ be the basis of $\mathfrak{n}$, $\bar{\mathfrak{n}}$ and $\mathfrak{a}$, respectively. Then $\{Y^\alpha H^\beta X^\gamma = Y_{1}^{\alpha_1} \cdots Y_{N}^{\alpha_N} H_{1}^{\beta_1} \cdots H_{M}^{\beta_M} X_{1}^{\gamma_1} \cdots X_{N}^{\gamma_N} ; \ \alpha \in \mathbb{N}^N, \ \beta \in \mathbb{N}^M, \ \gamma \in \mathbb{N}^N \}$ with $N = \{0, 1, 2, \ldots\}$ is a Poincare-Birkhoff-Witt’s basis of $U(\mathfrak{g})$.

Let $D \in V$. The assumption implies $PDQ \in U(\mathfrak{g})V$ and therefore we may assume $Q = 1$ in (2.29). Since $XD = \text{ad}(X)D + DX \in V + U(\mathfrak{g})\mathfrak{n}$ for $X \in \mathfrak{n}$, we have $X^\gamma D \in V + U(\mathfrak{g})\mathfrak{n}$. On the other hand, $Y^\alpha H^\beta D - Y^\alpha H^\beta \omega(D) = Y^\alpha H^\beta (\mathfrak{n}U(\mathfrak{n} + \mathfrak{a}) + U(\mathfrak{g})\mathfrak{n}) \subset \mathfrak{n}U(\mathfrak{n} + \mathfrak{a}) + U(\mathfrak{g})\mathfrak{n}$ and therefore $\omega(Y^\alpha H^\beta D) = H^\beta \omega(D)$ if $\alpha = 0$. 

and 0 otherwise. Hence $\omega(Y^{\alpha}H^{\beta}X^{\gamma}D) \in \langle \omega(V) \rangle_{S(a)}$ and $\omega(PD) \in \langle \omega(V) \rangle_{S(a)}$ for $P \in U(\mathfrak{g})$.

**Lemma 2.11.** Under the notation in Lemma 2.10, fix $H_{\Theta} \in \mathfrak{a}$ so that the condition $\text{ad}(H_{\Theta})Y = c_{Y}Y$ with $c_{Y} \in \mathbb{C}$ and $Y \in \mathfrak{n} \setminus \{0\}$ means $c_{Y} \geq 0$. Suppose $\text{ad}(H_{\Theta})\mathfrak{n} \neq \{0\}$. Let $m_{\Theta}$ be the centralizer of $H_{\Theta}$ in $\mathfrak{g}$ and let $\mathfrak{n}_{\Theta}$ and $\bar{\mathfrak{n}}_{\Theta}$ be subspaces spanned by the elements $Y$ in $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively, satisfying $\text{ad}(H_{\Theta})Y = c_{Y}Y$ with $c_{Y} \neq 0$. Then $p_{\Theta} = m_{\Theta} \oplus \mathfrak{n}_{\Theta}$ be a Levi decomposition of a parabolic subalgebra $p_{\Theta}$ containing $\mathfrak{p}$. Let $a_{\Theta}$ denote the center of $m_{\Theta}$. For an element $\lambda$ of the dual $a_{\Theta}^{\flat}$ of $a_{\Theta}$ we define a character $\lambda_{\Theta}$ of $p_{\Theta}$ so that $\lambda_{\Theta}(\mathfrak{n}_{\Theta} + [m_{\Theta}, m_{\Theta}]) = 0$ and $\lambda_{\Theta}(H) = \lambda(H)$ for $H \in a_{\Theta}$. Suppose there exist $D_{1}(\lambda), \ldots, D_{m}(\lambda)$ in $U(\mathfrak{g})[\lambda]$ so that

\begin{align*}
(2.30) & \quad \text{ad}(X)D_{k}(\lambda) \in \sum_{j=1}^{m} U(\mathfrak{g})[\lambda]D_{j}(\lambda) \quad \text{for } X \in \mathfrak{g} \text{ and } k = 1, \ldots, m, \\
(2.31) & \quad D_{k}(\lambda) \in \sum_{X \in \mathfrak{p}} U(\mathfrak{g})[\lambda] (X - \lambda_{\Theta}(X)) + \bar{\mathfrak{n}}U(\mathfrak{g})[\lambda] \quad \text{for } k = 1, \ldots, m.
\end{align*}

Then $D_{k}(\lambda) \in \sum_{X \in \mathfrak{p}_{\Theta}} U(\mathfrak{g})[\lambda](X - \lambda_{\Theta}(X))$ and therefore $D_{k}(\lambda) \in \text{Ann}(M_{\Theta}(\lambda))$ for $k = 1, \ldots, m$ under the same notation as in the case $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$.

**Proof.** Retain the notation in the proof of Lemma 2.10. We may assume $\{Y_{1}, \ldots, Y_{N'}\}$ is a basis of $\bar{\mathfrak{n}}_{\Theta}$ for a suitable $N'$. We note that for $D \in U(\mathfrak{g})[\lambda]$

\begin{equation}
(2.32) \quad D \equiv \sum_{\alpha \in \mathbb{N}^{N'}} c_{\alpha}(D; \lambda)Y^{\alpha} \mod \sum_{X \in \mathfrak{p}_{\Theta}} U(\mathfrak{g})[\lambda](X - \lambda_{\Theta}(X)).
\end{equation}

Here $c_{\alpha}(D; \lambda) \in \mathbb{C}[\lambda]$ are uniquely determined by $D$ because of the decomposition $U(\mathfrak{g}) = U(\bar{\mathfrak{n}}_{\Theta}) \oplus U(\mathfrak{p})p_{\Theta}$.

Put $I = \sum_{k=1}^{m} U(\mathfrak{g})D_{k}(\lambda)U(\mathfrak{g})$ and $I_{\lambda} = \sum_{H \in a} S(a)[\lambda](H - \lambda(H))$ and suppose $D \in I$. Then (2.31) implies $\omega(D_{k}(\lambda)) \in I_{\lambda}$ for $k = 1, \ldots, m$ and therefore $\omega(PD_{k}(\lambda)Q) \in I_{\lambda}$ for $P, Q \in U(\mathfrak{g})$ by Lemma 2.10 which implies $c_{0}(D; \lambda) = \omega(D)(\lambda) = 0$. Hence $\text{IM}_{\Theta}(\lambda)$ is a proper $\mathfrak{g}$-submodule of $M_{\Theta}(\lambda)$ for any fixed $\lambda \in a_{\Theta}^{\flat}$.

Since $M_{\Theta}(\lambda)$ is an irreducible $\mathfrak{g}$-module for a generic $\lambda$ (if the infinitesimal character of the Verma module with the highest weight which equals to the weight $Y^{\alpha}$ with $\alpha \neq 0$ plus $\lambda$ is different from that of $M_{\Theta}(\lambda)$, then $M_{\Theta}(\lambda)$ is irreducible), $\text{IM}_{\Theta}(\lambda) = 0$ for a generic $\lambda$. Hence $c_{\alpha}(D; \lambda) = 0$ for $\alpha \in \mathbb{N}^{N'}$ and $\text{IM}_{\Theta}(\lambda) = 0$ for any $\lambda$.

The following remark is clear from the argument in the proof of Lemma 2.11.

**Remark 2.12.** i) Let $\ell$ be a positive integer and let $r(\lambda, \epsilon)$ be a polynomial function of $(\lambda, \epsilon) \in \mathbb{C}^{\ell+1}$ valued in $U^*(\mathfrak{g})$. If $r(\lambda, \epsilon) \in \text{Ann}_{G}(M_{\Theta}(\lambda))$ for generic $(\lambda, \epsilon)$, then $r(\lambda, \epsilon) \in \text{Ann}_{G}(M_{\Theta}(\lambda))$ for any $(\lambda, \epsilon)$. 

ii) Let $p$ be a suitable polynomial function of $C^k$ to $a^*$. Replacing $D_k(\lambda)$, $U(\mathfrak{g})|\lambda$ and $\lambda$ by $D_k(\mu)$, $U(\mathfrak{g})|\mu$ and $\mu$, respectively, in Lemma 2.11, we have the same conclusion if $M_\phi(p(\mu))$ is irreducible for generic $\mu \in C^k$.

**Remark 2.13.** Use the notation in Lemma 2.10. Let $\lambda \in a^*$ and consider the Verma module $M(\lambda) = U(\mathfrak{g})/(U(\mathfrak{g})\mathfrak{n} + \sum_{H \in a} U(\mathfrak{g})(H - \lambda(\mathfrak{h})))$. Then
\begin{equation}
P_\lambda = \{ D \in U(\mathfrak{g}); \omega(D)(\lambda) = \omega\left(\text{ad}(X)D\right)(\lambda) = 0 \ (\forall X \in \mathfrak{g}) \}
\end{equation}
is the annihilator $\text{Ann} \left(L(\lambda)\right)$ of the unique irreducible quotient $L(\lambda)$ of $M(\lambda)$. Here we identify $S(a)$ with the space of polynomial functions of $a^*$. This may be also considered to be a *quantization* of the conjugacy class of semisimple matrices.

**Proof.** Lemma 2.10 proves that $P_\lambda$ is a two-sided ideal of $U(\mathfrak{g})$. Since the assumption means that the projection of $P_\lambda L(\lambda)$ into the highest weight space of $L(\lambda)$ vanishes, $P_\lambda L(\lambda) = 0$ because of the irreducibility of $L(\lambda)$. On the other hand, if $DL(\lambda) = 0$, $D \in U(\mathfrak{g})\mathfrak{n} + \sum_{H \in a} U(\mathfrak{g})(H - \lambda(\mathfrak{h}))$ and therefore $\omega(D)(\lambda) = 0$. Since $\text{Ann} \left(L(\lambda)\right)$ is a two-sided ideal of $U(\mathfrak{g})$, we have $\text{Ann} \left(L(\lambda)\right) \subseteq P_\lambda$. \hfill \Box

**Remark 2.14.** Define $\rho \in a^*$ by $\rho(X) = \frac{1}{2} \text{Trace } \text{ad}(H)|_a$ and $w.\lambda = w(\lambda + \rho) - \rho$ for the element $w$ of the Weyl group $W$ of the pair $(\mathfrak{g}, a)$. Then the infinitesimal character of the highest weight module $M(\lambda)$ is parametrized by $W.\lambda$. We say that the infinitesimal character is *regular* if $w.\lambda \neq \lambda$ for any $w \in W$ with $w \neq e$.

If $\mathfrak{g} = \mathfrak{g}(n, \mathbb{C})$, then
\begin{equation}
\rho = \left(-\frac{n-1}{2} + (1 - 1)\right)e_1 + \cdots + \left(-\frac{n-1}{2} + (n - 1)\right)e_n,
\end{equation}
$W \simeq S_n$ and
\begin{equation}
w\left(\sum_{j=1}^{n} \mu_j e_j\right) = \sum_{j=1}^{n} \mu_j e_{w^{-1}(j)} = \sum_{j=1}^{n} \mu_{w(j)} e_j \quad \text{for } (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n \text{ and } w \in W.
\end{equation}
In $U^e(\mathfrak{g})$, $\rho$ changes into $\rho^e = e\rho$ and the infinitesimal character of $M_\phi^e(\lambda)$ equals that of $M^e(\lambda_\phi)$. Hence the infinitesimal character is regular if and only if all the roots of $d_n^e(x) = 0$ are simple because the set of roots is $\{\bar{\lambda}_\nu + \frac{n-1}{2}; \nu = 1, \ldots, n\}$ by putting
\begin{equation}
\lambda_\phi + \rho^e = \bar{\lambda}_1 e_1 + \cdots + \bar{\lambda}_n e_n.
\end{equation}

**Lemma 2.15.** Let $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_{m-1}\}$ be sets of positive numbers with $m > 0$, $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_{m-1}$. Then there exists a positive number $\mu \leq m$ such that $\# \{j \in J; j < i_\mu\} = \mu - 1$ and $i_\mu \notin J$.

**Proof.** Suppose $m > 1$ since the lemma is clear when $m = 1$. If $j_{m-1} < i_m$, we can put $\mu = m$. If $j_{m-1} \geq i_m$, we can reduce to the case when $I = \{i_1, \ldots, i_{m-1}\}$ and $J = \{j_1, \ldots, j_{m-2}\}$. \hfill \Box

Retain the notation in Theorem 2.8. Fix $k$ with $1 \leq k \leq L$ and put $m = n+1-n'_k$ and $J = \{1, 2, \ldots, n\}\backslash\{n_{k-1}+1, n_{k-1}+2, \ldots, n_k\}$. Note that $\# J = m-1$. 


For $I = \{i_1, \ldots, i_m\}$ with $1 \leq i_1 < \cdots < i_m \leq n$, choose an integer $\mu$ as in Lemma 2.15. Then $n_{k-1} < i_\mu \leq n_k$ and $\#\{1, 2, \ldots, n_{k-1}\} = \mu - 1$, from which we have $\mu = n_{k-1} + 1$ and $\lambda(E_{i_\mu}) - (\lambda_k + n_{k-1}\epsilon) + (\mu - 1)\epsilon = 0$ and therefore (2.23) and Proposition 2.6 show

$$\omega(D^\epsilon_{IJ}(\lambda_k + n_{k-1}\epsilon)) \in \sum_{H\in\alpha} S(a)(H - \lambda(H))$$

if $\# I = \neq J = n + 1 - n_k$.

Denoting (2.37)

$$J(m, x) = \sum_{\# I = \# J = m} CD^\epsilon_{IJ}(x),$$

the basis of $J(n + 1 - n_k', \lambda_k + n_{k-1}\epsilon)$ satisfies the assumption in Lemma 2.11 for $\epsilon = 1$ and therefore

(2.38)  
$$J(n + 1 - n_k', \lambda_k + n_{k-1}\epsilon) \subset \text{Ann}_G(M^\epsilon_\theta(\lambda)) \quad \text{for } k = 1, \ldots, L.$$  

for $\epsilon = 1$. But this holds for any $\epsilon$ because of Remark 2.12 i) with the isomorphism between $U(g)$ and $U^\epsilon(g)$.

Now the Laplace expansions of $D^\epsilon_{IJ}(x)$ with respect to the first and the last column show (cf. [O2, Proposition 2.6 i])

(2.39)  
$$J(m + 1, \lambda) + J(m + 1, \lambda + \epsilon) \subset U^\epsilon(g)J(m, \lambda) \quad \text{if } m < n$$

and therefore

(2.40)  
$$J(n + 1 - n_k' + j, \lambda_k + (n_{k-1} + i)\epsilon) \in \text{Ann}_G(M^\epsilon_\theta(\lambda)) \quad \text{for } 0 \leq i \leq j \leq n_k' - 1.$$  

When $\epsilon = 0$, it is obvious by the Laplace expansion of $D^0_{IJ}(x)$ that

$$\left(\frac{d^i}{dx^i} D^0_{IJ}(x)\right)|_{x = \lambda_k} = 0 \quad \text{for } \# I = \# J = n + 1 - n_k' + j \text{ with } 0 \leq i \leq j \leq n_k' - 1.$$  

Hence if $c \in \mathbb{C}$ satisfies $d^\epsilon_m(c; \lambda) = 0$, then $\text{det}^\epsilon_m(c; E_{IJ}) \in I^\epsilon_\theta(\lambda)'$ for $\# I = \# J = m$ by denoting

(2.41)  
$$I^\epsilon_\theta(\lambda)' = \sum_{k=1}^{L} U^\epsilon(g)J(n + 1 - n_k', \lambda_k + n_{k-1}\epsilon).$$  

We have proved

(2.42)  
$$I^\epsilon_\theta(\lambda)' \subset I^\epsilon_\theta(\lambda) \quad \text{and } I^\epsilon_\theta(\lambda)' \subset \text{Ann}_G(M^\epsilon_\theta(\lambda))$$

and $I^\epsilon_\theta(\lambda)' = I^\epsilon_\theta(\lambda)$ if all the root of $d^\epsilon_m(x; \lambda) = 0$ are simple for $m = 1, \ldots, n$ (cf. Remark 2.9). Hence it follows from Remark 2.12 i) that

(2.43)  
$$I^\epsilon_\theta(\lambda) \subset \text{Ann}_G(M^\epsilon_\theta(\lambda)).$$  

Note that the element $r^1_{IJ}$ for $\# I = n$ in (2.17) are contained in $J^\epsilon(\lambda_\theta)$ because they are in the center $U^\epsilon(g)^G$ of $U^\epsilon(g)$ and $U^\epsilon(g)^G \equiv \mathbb{C}$ mod $J^\epsilon(\lambda_\theta)$.  

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Thus we have only to show $I^e_\emptyset(\lambda) \supset \text{Ann}_G(M^e_\emptyset(\lambda))$ to complete the proof of Theorem 2.8. We can prove this for generic $\lambda$ with $\epsilon \neq 0$ using the result in the next section (cf. [O3]) or Theorem 2.21 but we reduce it to the claim

$$I^0_\emptyset(0) = \text{Ann}_G(M^0_\emptyset(0)).$$

For $\epsilon = \lambda = 0$, this is conjectured by [Ta] and is proved by [We]. In this case $r^I_{J,J} \in S(g)$ are of homogeneous polynomials of $g^*$ with degree $# I - j$. Here we note that $\text{det}^\epsilon(x; E_{IJ})$ is homogeneous of degree $# I$ with respect to $(g, \epsilon, \lambda)$, which is well-defined under any choice of Poincare-Birkhoff-Witt basis because of the homogenized universal enveloping algebra.

Let $S(g)_m$ be the space of homogeneous elements of $S(g)$ with degree $m$. Then $U^\epsilon(g)^{(m)} / U^\epsilon(g)^{(m)} \simeq S(g)_m$ and for $D \in U^\epsilon(g)^{(m)}$, we denote by $\sigma_m(D)$ the corresponding element in $S(g)_m$. Note that $\sigma_{# I-j}(r^I_{J,J})$ in (2.17) does not depend on $\lambda$ and $\epsilon$. Hence

$$I^e_\emptyset(0) = \sum_{m=n+1}^{\infty} \sum_{\sum n_i = m} \sum_{j=0}^{d_m-1} S(g) \sigma_{# I-j} r^I_{J,J}$$

Put $R^\epsilon(\lambda)^{(m)} = \text{Ann}_G(M^e_\emptyset(\lambda)) \cap U^\epsilon(g)^{(m)}$ and $D \in R^\epsilon(\lambda)^{(m)} \setminus R^\epsilon(\lambda)^{(m-1)}$. We will prove $D \in I^e_\emptyset(\lambda)$ by the induction on $m$. Since (2.10) implies $\text{Ad}(g)D \equiv 0 \mod U^\epsilon(g)^{(m-1)} \oplus U^\epsilon(g)^{(m-1)}$, we have

$$\sigma_m(D)(\text{Ad}(g) n_\emptyset) = 0 \quad (\forall g \in G)$$

and $\sigma_m(D) \in I^e_\emptyset(0)$. Hence it follows from (2.44) and (2.45) that there exist homogeneous elements $p^I_{J,J} \in S(g)$ satisfying $\sigma_m(D) = \sum p^I_{J,J} \sigma_{# I-j} r^I_{J,J}$. Here $r^I_{J,J}$ are generators of $I^e_\emptyset(\lambda)$ appeared in (2.17) and $\text{deg}(p^I_{J,J}) + # I - j = m$ if $p^I_{J,J} \neq 0$. Let $P^I_{J,J} \in U^\epsilon(g)^{(m-# I+j)}$ with $\sigma_{# I-j}(P^I_{J,J}) = p^I_{J,J}$ and put $D' = \sum P^I_{J,J} D^I_{J,J}$. Then $D' \in I^e_\emptyset(\lambda)$ and $D - D' \in R^\epsilon(\lambda)^{(m-1)}$ and therefore we have $D - D' \in I^e_\emptyset(\lambda)$ by the hypothesis of the induction. Thus we have completed the proof of Theorem 2.8.

**Remark 2.16.** The procedure to deform $\lambda$ to 0 under the classical limit $\epsilon = 0$ is studied by [BK].

In the proof of Theorem 2.8 we have shown the following, which is proved by [BB] together with the fact that it is not valid for a generalized Verma module of a general semisimple Lie algebra induced from a character of a parabolic subalgebra.

**Corollary 2.17.** The graded ring $\text{gr}(\text{Ann}_G(M^e_\emptyset(\lambda))) = \bigoplus_{m=0}^{\infty} (\text{Ann}_G(M^e_\emptyset(\lambda)) \cap U^\epsilon(g)^{(m)}) / (\text{Ann}_G(M^e_\emptyset(\lambda)) \cap U^\epsilon(g)^{(m-1)})$ equals the defining ideal of the closure of the nilpotent conjugacy class of the generic element $A_{\epsilon,0}$ of the form (2.4). In particular it is a prime ideal and does not depend on $(\lambda, \epsilon)$. 
Corollary 2.18. The following two conditions are equivalent.

\begin{align}
(2.47) & \quad \text{Ann}_G(M^\epsilon_\Theta(\lambda)) \supset \text{Ann}_G(M^\epsilon_\Theta(\lambda')). \\
(2.48) & \quad d^*_m(x; \Theta, \lambda) \in \mathbb{C}[x]d^*_m(x; \Theta', \lambda') \quad \text{for } m = 1, \ldots, n.
\end{align}

**Proof.** It is obvious that the latter condition implies the former. Hence suppose the first condition. Let \( f_m(x) \) be the least common multiple of \( d^*_m(x; \Theta, \lambda) \) and \( d^*_m(x; \Theta', \lambda') \). Then if \( \# I = \# J = m \), \( \det^\epsilon(x; E_{IJ}) \in U^\epsilon(\mathfrak{g})f_m(x) \mod \mathbb{C}[x] \otimes \text{Ann}_G(M^\epsilon_\Theta(\lambda)) \). Applying \( \sigma_m \) to this relation as in the proof of Theorem 2.8, we have \( \det^0(x; E_{IJ}) \in S(\mathfrak{g})x^{\deg(f_m)} \mod \mathbb{C}[x] \otimes \text{Ann}_G(M^0_\Theta(0)) \) because of the homogeneity with respect to \((\mathfrak{g}, \epsilon, \lambda)\). Let \( A_{\Theta, 0} \) be the generic element of the form \( (2.4) \) and let \( J_{\Theta} \) be the maximal ideal of \( S(\mathfrak{g}) \) corresponding to \( A_{\Theta, 0} \). Considering under modulo \( J_{\Theta} \), we can conclude that all the \( m \)-minors of the matrix \((x-A_{\Theta, 0})\) are in \( \mathbb{C}[x]x^{\deg(f_m)} \). On the other hand, \( x^{d_m(\Theta)} \) is the greatest common devisors of \( m \)-minors of \((x-A_{\Theta, 0})\) and therefore \( \deg f_m(x) \leq d_m(\Theta) = \deg d^*_m(x; \Theta, \lambda) \) and we have the latter condition. \( \square \)

**Remark 2.19.** If \( \epsilon = 0 \), Corollary 2.18 gives the closure relation in the conjugacy classes of the matrices.

**Remark 2.20.** The following theorem is a part of a conjecture proposed by [O1] for the general symmetric pair. The case in this note corresponds to the pair \((GL(n, \mathbb{C}), U(n))\). In the case of the classical limit \( \epsilon = \lambda = 0 \), the following theorem is obtained by [DP] and [Ta].

**Theorem 2.21.** Let \( W_\Theta \) be the Weyl group of \( m_\Theta \) and let \( W = W(\Theta)W_\Theta \) be the decomposition of \( W = S_n \) so that \( W(\Theta) \) be the set of the representatives of \( W/W_\Theta \) with the minimal length. Then the common zeros of \( \omega(\text{Ann}_G(M^\epsilon_\Theta(\lambda))) \) coincides with the set \( \{ w.L; w \in W(\Theta) \} \) counting their multiplicities.

In particular, the space \( S(\mathfrak{a})/\omega(\text{Ann}_G(M^\epsilon_\Theta(\lambda))) \) is naturally a representation space of \( W \) which is isomorphic to \( \text{Ind}_{W_\Theta}^W \text{id} \).

**Proof.** Under the notation \( (2.35) \)

\[ \lambda_\nu = \lambda_{\Theta} + \frac{n-1}{2} + (\nu - 1) \quad \text{for } \nu = 1, \ldots, n. \]

and

\[ \overline{\omega}(D^\epsilon_{IJ})(\lambda_k + n_{k-1}) = \prod_{\mu=1}^{m} (E_{i_\mu} - \lambda_k + (\frac{n-1}{2} - n_{k-1} + \mu - i_\mu)\epsilon). \]

Fix \( k \) with \( 1 \leq k \leq L \) and \( w \in W(\Theta) \). Put \( m = n + 1 - n' \), \( K = \{n_k-1+1, \ldots, n_k\} \), \( K^c = \{1, \ldots, n\} \setminus K \) and \( J = w(K^c) \). For \( I = \{i_1, \ldots, i_m\} \) with \( 1 \leq i_1 < \cdots < i_m \leq n \), choose \( \mu \) as in Lemma 2.15 and put \( \ell = w^{-1}(i_\mu) \). Then \( \ell \in K \) and \( \{ \nu \in K^c; w(\nu) < i_\mu \} = \mu - 1 \), which implies \( \# \{ \nu \in K; w(\nu) < i_\mu \} = i_\mu - \mu \). On the other hand, since the condition \( n_{k-1} < \nu < \nu' \leq n_k \) means \( w(\nu') < w(\nu') \),
we have \( \{ \nu \in K; w(\nu) < i_\mu \} = \{ n_{k-1} + 1, n_{k-1} + 2, \ldots, \ell - 1 \} \) and therefore \( \ell - n_{k-1} - 1 = i_\mu - \mu \) and
\[
\lambda_\ell - \lambda_k + (\frac{n_{k-1}}{2} - n_{k-1} + \mu - i_\mu)\epsilon = (\ell - n_{k-1} + \mu - i_\mu)\epsilon = 0.
\]

Since \( \lambda_\ell \) is the \( i_\mu \)-th component of \( (\lambda_{w(1)}, \ldots, \lambda_{w(n)}) \), we can conclude that \( \omega(D_{II})(\lambda_k + n_{k-1}\epsilon) \) vanishes at \( w(\lambda_\Theta + \rho^\epsilon) \), which is equivalent to the condition that \( \omega(D_{II})(\lambda_k + n_{k-1}\epsilon) \) vanishes at \( w.\lambda_\Theta \). Hence if \( \lambda \) is generic, \( \omega(I^\epsilon_g(\lambda)) \) vanishes at \( w.\lambda_\Theta \) for \( w \in W(\Theta) \) and therefore for any \( \lambda \in \mathbb{C}^L \) because of the continuity. In particular, \( \dim S(a)/\omega(I^\epsilon_g(\lambda)) \geq \#W(\Theta) \) for generic \( \lambda \) and therefore for any \( \lambda \) by the same reason.

Since \( \omega(I^\epsilon_g(\lambda)) \) are generated by homogeneous polynomials of \((a, \lambda, \epsilon)\) and \([Ta, \text{Theorem 1}]\) shows \( \dim S(a)/\omega(I^\epsilon_g(0)) = \#W(\Theta) \), we have \( \dim S(a)/\omega(I^\epsilon_g(\lambda)) \leq \#W(\Theta) \). Thus we can conclude \( \dim S(a)/\omega(I^\epsilon_g(\lambda)) = \#W(\Theta) \) and the theorem follows from this. In fact, the last claim is clear because \( I^\epsilon_g(\lambda) \) is \( W \)-invariant. \( \Box \)

3. Generalized Verma modules

In this section we study the necessary and sufficient condition on \( \lambda \in \mathbb{C}^L \) so that
\[
J^\epsilon_g(\lambda) = \mathrm{Ann}_G (M^\epsilon_g(\lambda)) + J^*(\lambda_\Theta).
\]
Note that it is clear by the definition that \( J^\epsilon_g(\lambda) \supset \mathrm{Ann}_G (M^\epsilon_g(\lambda)) + J^*(\lambda_\Theta) \) and
\[
\mathrm{Ann}_G (M^\epsilon_g(\lambda)) = \mathrm{Ann}_G (U^\epsilon(g)/(\mathrm{Ann}_G (M^\epsilon_g(\lambda)) + J^*(\lambda_\Theta))).
\]

In general it is proved by \([BG]\) and \([Jo]\) that for \( \mu \in a^* \) the map
\[
(3.3) \quad \{ I; \ I \text{ is the two sided ideal of } U(g) \text{ with } I \supset \mathrm{Ann} (M(\mu)) \}_{\exists} \ni I \mapsto I + J(\mu) \in \{ J; \ J \text{ is the left ideal of } U(g) \text{ with } J \supset J(\mu) \}
\]
is injective if \( \mu \) is dominant:
\[
(3.4) \quad 2 \frac{\langle \mu + \rho, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \{-1, -2, \ldots\} \quad \text{for any positive root } \alpha \text{ for the pair } (n, a).
\]
Moreover the map is surjective if \( \mu \) is regular, that is,
\[
(3.5) \quad \langle \mu + \rho, \alpha \rangle \neq 0 \quad \text{for any root } \alpha \text{ for the pair } (n, a)
\]
and dominant. Hence in our case with \( \epsilon \neq 0 \), (3.1) is valid if \( \lambda_\Theta + \rho^\epsilon \) is regular and dominant:
\[
(3.6) \quad \lambda_j - \lambda_i \notin \{0, -\epsilon, -2\epsilon, \ldots\} \quad \text{for } 1 \leq i < j \leq n.
\]

For \( \mu \in a^* \) and \( D \in U^\epsilon(g) \) let \( \gamma(\mu; D) \) denote the unique element in \( U^\epsilon(\tilde{n}) \) with \( D \equiv \gamma(\mu; D) \mod J^* (\mu) \). For a basis \( \{ R_j \} \) of an \( \text{ad}(g) \)-invariant subspace \( V \) of \( U^\epsilon(g) \) we note that
\[
(3.7) \quad \gamma(\mu; \sum P_j R_j) \in \sum U^\epsilon(\tilde{n}) \gamma(\mu; R_j) \quad \text{for } P_j \in U^\epsilon(g).
\]
Let $R_-$ denote the set of weights of $U^\epsilon(\widehat{n})$ with respect to $\alpha$. Then

$$R_- = \{ \sum_{i=1}^{n} m_i e_i; m_i \in \mathbb{Z}, \sum m_i = 0 \text{ and } m_1 \geq m_2 \geq \cdots \geq m_n \} \setminus \{0\}. $$

Suppose $R_j \in U^\epsilon(\mathfrak{g})$ are weight vectors and $U^\epsilon(\mathfrak{g})V + J^\epsilon(\mu) \neq U^\epsilon(\mathfrak{g})$. Since $\gamma(\mu; R_j)$ has the weight which equals that of $R_j$, $\gamma(\mu; R_j) = 0$ if the weight of $R_j$ is not in $R_-$. Moreover since $E_{ii+1}$ has a maximal weight $e_i - e_{i+1}$ in $R_-$ for any integer $i$ with $1 \leq i < n$,

$$(3.8) \quad E_{ii+1} \in U^\epsilon(\mathfrak{g})V + J^\epsilon(\lambda) \Leftrightarrow \mathbb{C}E_{ii+1} = \sum_{\text{the weight of } R_j = e_i - e_{i+1}} \mathbb{C}\gamma(\mu; R_j).$$

The key to studying the condition for (3.1) is the following argument used in [O2, proof of Theorem 5.1].

Fix positive integers $k$, $\overline{i}$ and $\overline{j}$ satisfying $1 \leq k \leq L$ and $n_{k-1} < \overline{i} < \overline{j} \leq n_k$. Let $I = \{i_m, \ldots, i_1\}$ and $J = \{j_m, \ldots, j_1\}$ be a set of positive numbers such that

$$1 \leq i_1 < i_2 < \cdots < i_m \leq n,$$

$$i_\nu = j_\nu \quad \text{if} \ \nu \neq \ell,$$

$$i_\ell = \overline{i} < j_\ell = \overline{j} < i_{\ell+1}$$

with a suitable $1 \leq \ell \leq m$. Define non-negative integers

$$\begin{cases}
  m' = n - m, \\
  a'_j = n'_j - \#\{\nu; n_{j-1} < i_\nu \leq n_j\}, \\
  a_j = n_j - \#\{\nu; i_\nu \leq n_j\} = a'_1 + \cdots + a'_j, \quad a_0 = 0, \\
  b = \#\{\nu; n_{k-1} < i_\nu < \overline{i}\}, \\
  b' = \#\{\nu; \overline{j} < i_\nu \leq n_k\}.
\end{cases}$$

Then

$$1 \leq a_L = m' \leq n - 2, \quad 1 \leq a'_k = n'_k - b - b' - 1,$$

$$0 \leq a'_j \leq n'_j - \delta_{kj}, \quad 0 \leq b \leq \overline{i} - n_{k-1} + 1, \quad 0 \leq b' \leq n_k - \overline{j}$$

and we have

$$\det^\epsilon(x; E_{IJ}) \equiv \prod_{\nu=\ell+1}^{m} (x - E_{i_\nu} - (\nu - 1)\epsilon) \cdot E_{\overline{i}\overline{j}}$$

$$\equiv \prod_{\nu=1}^{\ell-1} (x - E_{i_\nu} - (\nu - 1)\epsilon) \mod U^\epsilon(\mathfrak{g})\overline{n}$$

$$(3.12) \quad \prod_{\nu=1}^{\ell-1} (x - E_{i_\nu} - (\nu - 1)\epsilon) \mod U^\epsilon(\mathfrak{g})\overline{n}$$

$$\equiv \frac{\prod_{j=1}^{L} p_{IJ}(x)}{s_{IJ}(x)} E_{\overline{i}\overline{j}} \mod J^\epsilon(\lambda_{\ominus})$$
by putting

\[(3.13) \quad \begin{cases} p^j_{IJ}(x) = (x - \lambda_j - (n_{j-1} - a_{j-1})\epsilon)^{(n_j' - a_j')}, \\ s^j_{IJ}(x) = x - \lambda_k - (n_{k-1} - a_{k-1} + b)\epsilon. \end{cases} \]

Hence it follows from (2.17) that

\[(3.14) \quad \sum_{i=0}^{d_m-1} \mathbb{C}r^{i} \equiv \begin{cases} C E_{\overline{i}\overline{j}} \mod J^\epsilon(\lambda) & \text{if} \ \prod_{j=1}^{L} p^j_{IJ}(x) \not\in \mathbb{C}[x]s_{IJ}(x)d_{m}\epsilon, \\ 0 \mod J^\epsilon(\lambda) & \text{otherwise}. \end{cases} \]

Since \((n_j'-a_j'-a_{j-1})-(n_j'-m') = m'-a_j \geq m'-a_L \geq 0\), we can define polynomials

\[\bar{p}^j_{IJ}(x) = \frac{p^j_{IJ}(x)}{(x - \lambda_j - n_{j-1}\epsilon)^{(n_j'-m')}}.\]

Then the condition \(\prod_{j=1}^{L} p^j_{IJ}(x) \in \mathbb{C}[x]s_{IJ}(x)d_{m}\epsilon\) is equivalent to the existence of \(j\) with

\[(3.15) \quad \bar{p}^j_{IJ}(x) \in \mathbb{C}[x]s_{IJ}(x). \]

If \(\epsilon \neq 0\), the condition (3.15) is equivalent to the condition that \(\nu\) is an integer satisfying

\[(3.16) \quad 0 \leq \nu \leq n_j' - a_j' - 1 \quad \text{and} \quad (\nu < a_{j-1} \\text{or} \ \nu \geq a_{j-1} + n_j' - m') \]

by denoting

\[(3.17) \quad \lambda_{k} + (n_{k-1} - a_{k-1} + b)\epsilon = \lambda_{j} + (n_{j-1} - a_{j-1} + \nu)\epsilon.\]

If \(\epsilon = 0\), it is equivalent to

\[(3.18) \quad \lambda_j = \lambda_k \quad \text{and} \quad a_j' < m'.\]

Put \(I = \{n, n-1, \ldots, n_k+1, n_{k-1}, n_{k-1}-1, \ldots, 1\}\) and \(J = \{n, n-1, \ldots, n_k+1, n_{k-1}, n_{k-1}-1, \ldots, 1\}\). Then

\[m' = n_k' - 1, \quad b = b' = 0, \quad a_k' = n_k' - 1, \quad a_j' = 0 \quad \text{and} \quad n_j' - a_j' - 1 = n_j' - 1 \quad \text{if} \ j \neq k.\]

Suppose (3.15) holds. Then \(j \neq k\) because \(p^k_{IJ}(x) = 1\). Since

\[\begin{cases} a_{j-1} - 1 = -1 < 0 \quad \text{and} \quad a_{j-1} + n_j' - m' = n_j' - n_k' + 1 & \text{if} \ j < k, \\ a_{j-1} - 1 = n_k' - 2 \quad \text{and} \quad a_{j-1} + n_j' - m' = n_j' > n_j' - a_j' - 1 & \text{if} \ j > k, \end{cases} \]

the condition (3.16) is equivalent to

\[
\begin{cases}
\max\{0, n_j' - n_k' + 1\} \leq \nu' \leq n_j' - 1 & \text{if} \ j < k, \\
1 - n_k' \leq \nu' \leq \min\{n_j' - n_k', -1\} & \text{if} \ k < j
\end{cases}
\]

with

\[\nu' = (\nu - a_{j-1}) - (b - a_{k-1}) = \begin{cases} \nu & \text{if} \ j < k, \\ \nu - n_k' + 1 & \text{if} \ k < j. \end{cases} \]
Hence (3.15) is equivalent to the condition (cf. Remark 2.14)

\[(3.19)\]

\[\Lambda_k \cap \Lambda_j \neq \emptyset, \Lambda_k \not\subset \Lambda_j \text{ and } \left( \mu \in \Lambda_j, \mu' \in \Lambda_k \setminus \Lambda_j \Rightarrow (\mu' - \mu)(k - j) > 0 \right)\]

with \(\Lambda_i = \{ \lambda_\nu; n_{i-1} < \nu \leq n_i \} = \{ \lambda_i + ((\nu - 1) - \frac{n_{i-1}}{2})\epsilon; n_{i-1} < \nu \leq n_i \}\)

if \(\epsilon \neq 0,\)

\[\lambda_j = \lambda_k \text{ and } n'_k > 1 \quad \text{if } \epsilon = 0.\]

Thus we have the following theorem.

**THEOREM 3.1.** i) Fix \(k \) with \(1 \leq k \leq L.\) Recall \(m^k_{\Theta} = \sum_{n_{k-1} < i \leq n_k} CE_{ij}^k.\)

Then

\[(3.20)\]

\[\text{Ann}_{\mathcal{G}} \left( M^k_{\Theta}(\lambda) \right) + J^\epsilon(\lambda_{\Theta}) \supset m^k_{\Theta} \cap \mathfrak{n}\]

if and only if (3.19) does not hold for \(j = 1, \ldots, L.\)

ii) The equality (3.1) is valid if and only if (3.19) does not hold for \(j = 1, \ldots, L\) and \(k = 1, \ldots, L,\) which is equivalent to the condition

\[(3.21)\]

\[\begin{cases} 
\min A_i > \min \tilde{\lambda}_j \text{ or } \max A_i > \max \tilde{\lambda}_j \text{ or } \Lambda_i \cap \Lambda_j = \emptyset \text{ or } A_i = \Lambda_j & \text{if } \epsilon \neq 0, \\
\lambda_i \neq \lambda_j \text{ or } n'_i = n'_j = 1 & \text{if } \epsilon = 0,
\end{cases}\]

for \(1 \leq i < j \leq L.\)

Here \(\tilde{\lambda}_i = \{ \text{Re} \mu; \mu \in A_i \} \) etc. In particular (3.1) is valid if the infinitesimal character of \(M^k_{\Theta}(\lambda)\) is regular.

**Proof.** We have only to prove that (3.20) is not valid if (3.19) holds for a suitable \(j.\) Suppose there exists \(j = j_o\) which satisfies (3.19). Fix such \(j_o\) and continue the argument just before the theorem. Put \(j = \hat{i} + 1\) and suppose (3.15) does not valid for \(j = k.\) Then if \(\epsilon \neq 0, \nu = b \) in (3.17) and since \(0 \leq b \leq n'_k - a'_k - 1\)

and (3.16) is not valid with \(j = k,\) we have

\[(3.22)\]

\[a_{k-1} \leq b < a_k - 1 + n'_k - m' \text{ and } m' > n'_k \quad \text{if } \epsilon \neq 0.\]

On the other hand, if \(\epsilon = 0,\) we have \(a'_k = m'\) because \(a'_k \leq a_L = m'.\)

First consider the case when \(j_o < k.\) Put \(\ell = \lambda_k + n_{k-1} - \lambda_{j_o} - n_{j_o-1}, \hat{i} = n_{k-1} + 1\) and \(\hat{j} = \hat{i} + 1.\) Then \(b = 0.\) If \(\epsilon \neq 0, a_{k-1} = a_{j_o} = 0\) because of (3.22) and it follows from (3.19) that

\[0 \leq \ell < n'_{j_o} \text{ and } \ell + n'_k > n'_{j_o}.\]

In this case putting \(j = j_o\) in (3.17) we have \(\nu = \ell\) and then \(0 \leq \nu, n'_j - n'_0 + 1 \leq \nu\) and \(\nu \leq n'_j - 1\) in (3.16), which implies \(\overline{p}^o_{IJ}(x) \in \mathbb{C}[x](x)\). We have this relation also in the case when \(\epsilon = 0\) because \(\deg \overline{n}^o_{IJ}(x) = n'_j - a'_j > (n'_j - m') = m' - a'_j > 0.\) Thus we can conclude \(n^o_{IJ} \equiv 0 \mod J^\epsilon(\lambda_{\Theta})\) if the weight...
of $r_{IJ}^{j}$ is $e_{i} - e_{i+1}$. Note that the weight of $r_{\{i_{1}, \ldots, i_{m}\}\{j_{1}, \ldots, j_{m}\}}$ is $\sum_{\nu=1}^{m} e_{i\nu} - e_{j\nu}$. Hence $E_{ii+1}^{\epsilon} \notin \text{Ann}_{G}(M_{\ominus}^{\epsilon}(\lambda)) + J^{\epsilon}(\lambda_{\ominus})$ because of (3.8).

Lastly consider the case when $k < j_{0}$. If $\epsilon = 0$, the same argument as in the case when $j_{0} < k$ works and therefore we may assume $\epsilon \neq 0$. Put $\ell = \lambda_{j_{0}} + n_{j_{0}-1} - \lambda_{k} - n_{k-1}$, $i = n_{k} - 1$ and $j = n_{k}$. Then similarly we have

$$1 \leq \ell < n'_{k}, \ n'_{k} \leq \ell + n'_{j_{0}}, \ b' = 0, \ a'_{k} = n'_{k} - b - 1$$

and $a_{k} = a'_{k} + a_{k-1} > (n'_{k} - b - 1) + (b - n'_{k} + m') = m' - 1$ by (3.22). Since $a_{k} \leq a_{L} = m'$, we have $a_{k} = a_{j_{0}} = a_{j_{0}-1} = m'$ and $a'_{j_{0}} = 0$. Putting $j = j_{0}$ in (3.17), we have $\nu = -\ell - a_{k-1} + b + a_{j_{0}-1} = a'_{k} - \ell + b = n'_{k} - \ell - 1$ and therefore $0 \leq \nu$ and $\nu < n'_{j_{0}} - 1 = n'_{j_{0}} - a'_{j_{0}} - 1$ and $\nu < n'_{k} - 1 \leq m' = a_{j_{0}-1}$ in (3.16). Hence $p_{ij}^{j_{0}}(x) \in \mathbb{C}[x]s_{IJ}(x)$ and thus $E_{ii+1}^{\epsilon} \notin \text{Ann}_{G}(M_{\ominus}^{\epsilon}(\lambda)) + J^{\epsilon}(\lambda_{\ominus})$ as in the previous case.

EXAMPLE 3.2. Suppose $n = 3$, $\Theta = \{2, 3\}$ and $\lambda = (\lambda_{1}, \lambda_{2})$. Then

$$d_{1}^{\epsilon}(x) = 1, \ d_{2}^{\epsilon}(x) = x - \lambda_{1}, \ d_{3}^{\epsilon}(x) = (x - \lambda_{1})(x - \lambda_{1} - \epsilon)(x - \lambda_{2} - 2\epsilon),$$

$$J^{\epsilon}(\lambda_{\ominus}) = \sum_{3 \geq i > j \geq 1} U(\mathfrak{g})E_{ij} + U(\mathfrak{g})(E_{1} - \lambda_{1}) + U(\mathfrak{g})(E_{2} - \lambda_{1}) + U(\mathfrak{g})(E_{3} - \lambda_{2}),$$

$$J^{\epsilon}_{\lambda}(\lambda) = J^{\epsilon}(\lambda_{\ominus}) + U^{\epsilon}(\mathfrak{g})E_{12}.$$  

Since

$$D_{ij}^{\epsilon}(x) = (E_{i_{1}j_{1}} - (x - \epsilon)\delta_{i_{1}j_{1}})(E_{i_{2}j_{2}} - x\delta_{i_{2}j_{2}}) - (E_{i_{2}j_{1}} - (x - \epsilon)\delta_{i_{2}j_{1}})(E_{i_{1}j_{2}} - x\delta_{i_{1}j_{2}})$$

for $I = \{i_{1} > i_{2}\}$ and $J = \{j_{1} > j_{2}\}$, we have

$$ \begin{cases} 
D_{\{21\}\{21\}}^{\epsilon}(\lambda_{1}) = (E_{2} - \lambda_{1} + \epsilon)(E_{1} - \lambda_{1}) - E_{12}E_{21} \equiv 0, \\
D_{\{32\}\{32\}}^{\epsilon}(\lambda_{1}) = (E_{3} - \lambda_{1} + \epsilon)(E_{2} - \lambda_{1}) - E_{23}E_{32} \equiv 0, \\
D_{\{31\}\{31\}}^{\epsilon}(\lambda_{1}) = (E_{3} - \lambda_{1} + \epsilon)(E_{1} - \lambda_{1}) - E_{13}E_{31} \equiv 0, \\
D_{\{21\}\{32\}}^{\epsilon}(\lambda_{1}) = E_{23}E_{12} - E_{13}(E_{2} - \lambda_{1}) \equiv E_{23}E_{12}, \\
D_{\{21\}\{31\}}^{\epsilon}(\lambda_{1}) = E_{23}(E_{1} - \lambda_{1}) - E_{13}E_{21} \equiv 0, \\
D_{\{32\}\{21\}}^{\epsilon}(\lambda_{1}) = E_{32}E_{21} - (E_{2} - \lambda_{1} + \epsilon)E_{31} \equiv 0, \\
D_{\{32\}\{31\}}^{\epsilon}(\lambda_{1}) = (E_{3} - \lambda_{1} + \epsilon)E_{21} - E_{23}E_{31} \equiv 0, \\
D_{\{31\}\{21\}}^{\epsilon}(\lambda_{1}) = E_{32}(E_{1} - \lambda_{1}) - E_{12}E_{31} \equiv 0, \\
D_{\{31\}\{32\}}^{\epsilon}(\lambda_{1}) = (E_{3} - \lambda_{1} + \epsilon)E_{12} - E_{13}E_{32} \equiv (\lambda_{2} - \lambda_{1} + \epsilon)E_{12}. 
\end{cases}$$  

\( \square \)
Here the above \( \equiv \) is considered under modulo \( J^\epsilon(\lambda_\Theta) \). Note that

\[
(3.24) \quad \text{Ann}_G(M(\Theta^\epsilon(\lambda)) = \sum_{3 \geq i_1 > i_2 \geq 1} U^\epsilon(\mathfrak{g})D_{\{i_1,i_2\}\{j_1,j_2\}}(\lambda_1) + \sum_{D \in U^\epsilon(\mathfrak{g})} U^\epsilon(\mathfrak{g})(D - \omega(D)(\lambda_\Theta)).
\]

Hence if \( \lambda_1 \neq \lambda_2 + \epsilon \) which is equivalent to (3.21), we have (3.1).

Suppose \( \lambda_1 = \lambda_2 + \epsilon \). Then since \( \text{ad}(\mathfrak{p})(E_{32}E_{12}) \subset J^\epsilon(\lambda_\Theta) \), we have

\[
(3.25) \quad J^\epsilon_\Theta(\lambda) = U^\epsilon(\mathfrak{n})E_{12} \oplus J^\epsilon(\lambda_\Theta) \quad \text{and} \quad J^\epsilon_\Theta(\lambda) \supset \text{Ann}_G(M_\Theta(\lambda)) + J^\epsilon(\lambda_\Theta) = U^\epsilon(\mathfrak{n})E_{23}E_{12} \oplus J^\epsilon(\lambda_\Theta) \supset \text{Ann}_G(M_\Theta(\lambda)).
\]

If \( \epsilon \neq 0 \), the above inclusion relation gives a Jordan-Hölder sequence of \( M^\epsilon(\lambda_\Theta) \) and

\[
J^\epsilon_\Theta(\lambda)/(\text{Ann}_G(M_\Theta(\lambda)) + J^\epsilon(\lambda_\Theta)) \simeq M^\epsilon_\Theta(\lambda')
\]

with \( \Theta' = \{1,3\} \) and \( \lambda' = (\lambda_1 + \epsilon, \lambda_1 - \epsilon) \). Note that \( \rho^\epsilon = (-\epsilon,0,\epsilon) \), \( \lambda_\Theta + \rho^\epsilon = (\lambda_1 - \epsilon, \lambda_1, 1) \), \( \lambda'_\Theta - \lambda_\Theta = \epsilon(e_1 - e_2) \), \( (1,2,\lambda_\Theta) = \lambda'_\Theta \), and \( \text{Ann}_G(M_\Theta(\lambda)) = \text{Ann}_G(M_\Theta(\lambda')) \) under the notation in Remark 2.14. Here \( \text{Ann}_G(M_\Theta(\lambda)) \) is the unique two-sided proper ideal of \( U(\mathfrak{g}) \) which is larger than \( U(\mathfrak{g})(J(\lambda_\Theta) \cap U(\mathfrak{g})^G) \).

References


