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Kyoto University
Quasi M-convex Functions
and Minimization Algorithms

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Abstract: We introduce a class of discrete quasiconvex functions, called quasi M-convex functions, by generalizing the concept of M-convexity due to Murota (1996). We investigate the structure of quasi M-convex functions with respect to level sets, and show that various greedy algorithms work for the minimization of quasi M-convex functions.

Keywords: quasiconvex function, discrete optimization, matroid, base polyhedron.

1 Introduction

The concept of convexity for sets and functions plays a central role in continuous optimization (or nonlinear programming with continuous variable), and has various applications in the areas of mathematical economics, engineering, operations research, etc. \([2, 12, 15]\). The importance of convexity relies on the fact that a local minimum of a convex function is also a global minimum. Due to this property, we can find a global minimum of a convex function by iteratively moving in descent directions, i.e., so-called descent algorithms work for the convex function minimization. Therefore, convexity for a function is a sufficient condition for the success of descent methods. Most of descent methods, however, work for a fairly larger class of functions called quasiconvex functions.

Let \(f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) be defined over a nonempty convex set, i.e., \(\text{dom } f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}\) is a nonempty convex set. A function \(f\) is said to be quasiconvex if it satisfies

\[f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}\]

for all \(x, y \in \text{dom } f\) and \(0 < \alpha < 1\), and semistrictly quasiconvex if it satisfies

\[f(\alpha x + (1-\alpha)y) < \max\{f(x), f(y)\}\]

for all \(x, y \in \text{dom } f\) with \(f(x) \neq f(y)\) and \(0 < \alpha < 1\). It is easy to see that convexity implies semistrict quasiconvexity, and semistrict quasiconvexity implies quasiconvexity under the assumption of lower semicontinuity. Although (semistrict) quasiconvexity is a weaker property than convexity, it still has nice properties as follows:

\begin{itemize}
    \item strict local minimality leads to global minimality for quasiconvex functions,
    \item local minimality leads to global minimality for semistrictly quasiconvex functions,
    \item level sets of quasiconvex functions are convex sets.
\end{itemize}

Due to these properties, quasiconvexity also plays an important role in continuous optimization. See \([1]\) for more accounts on quasiconvexity.

In the area of discrete optimization, on the other hand, discrete analogues of convexity, or "discrete convexity" for short, have been considered, with a view to identifying the discrete structure that guarantees the success of descent methods, i.e., so-called "greedy algorithms." Examples of discrete convexity are "discretely-convex functions" by Miller \([7]\), "integrally-convex functions" by Favati–Tardella \([3]\), and "M-convex/L-convex functions" by Murota \([8, 9, 10]\).

A function \(f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}\) is called M-convex if \(\text{dom } f \neq \emptyset\) and \(f\) satisfies the following property:

\((\text{M-EXC}) \ \forall x, y \in \text{dom } f, \ \forall u \in \text{supp}^+(x - y), \)

\[f(x + u) < \min\{f(x), f(x + u)\}\]
\[ \exists v \in \text{supp}^-(x - y):
\]
\[ f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v), \]

where

\[
\begin{align*}
\text{dom } f &= \{ x \in \mathbb{Z}^V \mid f(x) < +\infty \}, \\
\text{supp}^+(x - y) &= \{ w \in V \mid x(w) > y(w) \}, \\
\text{supp}^-(x - y) &= \{ w \in V \mid x(w) < y(w) \},
\end{align*}
\]

and \( \chi_w \in \{0,1\}^V \) is the characteristic vector of \( w \in V \). M-convex functions have various desirable properties as discrete convexity:

(i) local minimality leads to global minimality for M-convex functions,

(ii) M-convex functions can be extended to ordinary convex functions,

(iii) various duality theorems hold,

(iv) M-convex functions are conjugate to L-convex functions.

In particular, the property (i) shows that greedy algorithms work for the M-convex function minimization. However, we see from results in continuous optimization that strong properties such as M-convexity are not required for the success of greedy algorithms, and that some property like "quasi M-convexity" will suffice.

The main aim of this paper is to introduce the concept of quasi M-convex functions by generalizing the concept of M-convexity. To define quasi M-convexity, we use the following weaker properties than (M-EXC):

(QM) \( \forall x, y \in \text{dom } f, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):
\]
\[ f(x - \chi_u + \chi_v) \leq f(x) \quad \text{or} \quad f(y + \chi_u - \chi_v) \leq f(y). \]

(SSQM) \( \forall x, y \in \text{dom } f, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):
\]

(i) \( f(x - \chi_u + \chi_v) \geq f(x) \)
\[ \Rightarrow \quad f(y + \chi_u - \chi_v) \leq f(y), \quad \text{and} \]

(ii) \( f(y + \chi_u - \chi_v) \geq f(y) \)
\[ \Rightarrow \quad f(x - \chi_u + \chi_v) \leq f(x). \]

We define a quasi M-convex (resp. semistrictly quasi M-convex) function as a function \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) with \( \text{dom } f \neq \emptyset \) satisfying (QM) (resp. (SSQM)). We show that various nice properties hold for (semistrictly) quasi M-convex functions, which justifies the definitions of quasi M-convexity above.

We first review some fundamental results on M-convex functions in Section 2. Then, we show some properties for level sets of quasi M-convex functions, and prove that the class of quasi M-convex functions is closed under various fundamental operations in Section 3. Finally, we show that greedy algorithms work for the minimization of (semistrictly) quasi M-convex functions in Section 4. We also show a proximity theorem on (semistrictly) quasi M-convex functions, which guarantee that the so-called "scaling technique" is applicable to the quasi M-convex function minimization.

2 Review of Fundamental Results on M-convex Functions

We denote by \( \mathbb{R} \) the set of reals, and by \( \mathbb{Z} \) the set of integers. Also, we denote by \( \mathbb{R}_{++} \) the set of positive reals. Throughout this paper, we assume that \( V \) is a nonempty finite set of cardinality \( n \) (> 0). For \( w \in V \), we denote by \( \chi_w \in \{0,1\}^V \) the characteristic vector of \( w \).

Let \( x \in \mathbb{R}^V \). For \( S \subseteq V \), we define \( x(S) = \sum_{v \in S} x(v) \). We also define
\[
\begin{align*}
\text{supp}^+(x) &= \{ w \in V \mid x(w) > 0 \}, \\
\text{supp}^-(x) &= \{ w \in V \mid x(w) < 0 \}, \\
||x||_1 &= \sum_{w \in V} |x(w)|.
\end{align*}
\]

Let \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \). The effective domain \( \text{dom } f \) of \( f \) is defined by
\[
\text{dom } f = \{ x \in \mathbb{Z}^V \mid f(x) < +\infty \}.
\]

We denote by \( \arg \min f \) the set of the minimizers of \( f \), i.e.,
\[
\arg \min f = \{ x \in \mathbb{Z}^V \mid f(x) \leq f(y) \quad (\forall y \in \mathbb{Z}^V) \}.
\]

For any \( \alpha \in \mathbb{R} \cup \{+\infty\} \), the level set \( \operatorname{L}(f, \alpha) \) is defined by
\[
\operatorname{L}(f, \alpha) = \{ x \in \mathbb{Z}^V \mid f(x) \leq \alpha \}.
\]

Note that \( \arg \min f = \operatorname{L}(f, \inf f) \) and \( \text{dom } f = \operatorname{L}(f, +\infty) \) are special cases of level sets. For any \( x \in \text{dom } f \) and \( u, v \in V \), we denote the directional difference of \( f \) at \( x \) w.r.t. \( u \) and \( v \) by
\[
\Delta f(x; u, v) = f(x + \chi_u - \chi_v) - f(x).
\]
For a set $S \subseteq \mathbb{Z}^V$, the function $\delta_S : \mathbb{Z}^V \to \{0, +\infty\}$ given by

$$\delta_S(z) = \begin{cases} 0 & (z \in S), \\ +\infty & (z \notin S) \end{cases}$$

is called the **indicator function** of $S$.

Let $\varphi : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$. A function $\varphi$ is called **quasiconvex** if it satisfies

$$\varphi(\beta) \leq \max\{\varphi(\alpha_1), \varphi(\alpha_2)\}$$

$(\forall \alpha_1, \alpha_2, \beta \in \mathbb{Z}$ with $\alpha_1 < \beta < \alpha_2)$. Similarly, $\varphi$ is called **semistrictly quasiconvex** if it is a quasiconvex function and satisfies

$$\varphi(\beta) < \max\{\varphi(\alpha_1), \varphi(\alpha_2)\}$$

$(\forall \alpha_1, \alpha_2, \beta \in \mathbb{Z}$ with $\alpha_1 < \beta < \alpha_2, \varphi(\alpha_1) \neq \varphi(\alpha_2))$.  \hfill (2.1)

**Remark 2.1.** For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, semistrictly quasiconvexity implies quasi convexity under the assumption of lower semicontinuity [1, 2]. For a function $\varphi : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$, on the other hand, the property (2.1) alone does not imply the quasiconvexity in general. For convenience, we assume quasi convexity in the definition of semistrict quasiconvexity for $\varphi$. \hfill $\square$

**Theorem 2.2.** Let $\varphi : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$.

(i) $\varphi$ is quasiconvex if and only if for all $\alpha_1, \alpha_2 \in \text{dom} \varphi$ with $\alpha_1 < \alpha_2$ we have $\varphi(\alpha_1 + 1) \leq \varphi(\alpha_1)$ or $\varphi(\alpha_1 - 1) \leq \varphi(\alpha_2)$.

(ii) $\varphi$ is semistrictly quasiconvex if and only if for all $\alpha_1, \alpha_2 \in \text{dom} \varphi$ with $\alpha_1 < \alpha_2$ we have both

$$\varphi(\alpha_1 + 1) \geq \varphi(\alpha_1) \Rightarrow \varphi(\alpha_2 - 1) \leq \varphi(\alpha_2)$$

and

$$\varphi(\alpha_1 - 1) \geq \varphi(\alpha_2) \Rightarrow \varphi(\alpha_1 + 1) \leq \varphi(\alpha_1).$$

A function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is said to be nondecreasing if $\varphi(\alpha) \leq \varphi(\beta)$ holds for all $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and strictly increasing if for all $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ we have either $\varphi(\alpha) < \varphi(\beta)$ or $\varphi(\alpha) = \varphi(\beta) = +\infty$.

A function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ is called **$M$-convex** if dom $f \neq \emptyset$ and $f$ satisfies the following property:

**($\text{M-EXC}_w$)** $\forall x, y \in \text{dom} f$, $\forall u \in \text{supp}^+ (x-y)$, $\exists v \in \text{supp}^- (x-y)$:

$$f(x) + f(y) \geq f(x - \chi u + \chi v) + f(y + \chi u - \chi v).$$

Note that the inequality (2.2) can be rewritten as follows in terms of directional differences:

$$\Delta f(x; v, u) + \Delta f(y; u, v) \leq 0.$$  \hfill (2.3)

$M$-convex functions can be characterized by the following (seemingly) weaker property:

**($\text{M-EXC}'$)** $\forall x, y \in \text{dom} f$ with $x \neq y$, $\exists u \in \text{supp}^+(x-y)$, $\exists v \in \text{supp}^-(x-y)$ satisfying (2.2).

**Theorem 2.3 ([9, Th. 3.1]).** For $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$, we have (M-EXC) $\iff$ ($\text{M-EXC}_w$).

We also define the set version of $M$-convexity as follows. A set $B \subseteq \mathbb{Z}^V$ is called **$M$-convex** if $B \neq \emptyset$ and it satisfies

**($\text{B-EXC}$)** $\forall x, y \in B$, $\forall u \in \text{supp}^+ (x-y)$, $\exists v \in \text{supp}^- (x-y)$:

$$x - \chi u + \chi v \in B \quad \text{and} \quad y + \chi u - \chi v \in B.$$  

Note that an $M$-convex set is nothing but (the set of integral vectors in) an integral base polyhedron [4]. For $x \in B$ and $u, v \in V$, the exchange capacity associated with $x, v$ and $u$ is defined as

$$\tilde{\delta}_B(x, v, u) = \max\{\alpha \in \mathbb{R} \mid x + \alpha(\chi v - \chi u) \in B\}.$$  

$M$-convex sets can be characterized also by the following (seemingly) weaker property:

**($\text{B-EXC}_w$)** $\forall x, y \in B$ with $x \neq y$, $\exists u \in \text{supp}^+(x-y)$, $\exists v \in \text{supp}^- (x-y)$:

$$x - \chi u + \chi v \in B \quad \text{and} \quad y + \chi u - \chi v \in B.$$  

**Theorem 2.4 ([16]).** For $B \subseteq \mathbb{Z}^V$, we have (B-EXC) $\iff$ (B-EXC)$_w$.

## 3 Quasi M-convex Functions

### 3.1 Definitions

To extend the concept of $M$-convexity to quasi $M$-convexity, we relax the condition (2.3) while keeping the possible sign patterns of values $\Delta f(x; v, u)$ and $\Delta f(y; u, v)$ in mind. Table 1 shows the possible sign patterns of those values.

Let $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ be a function. Then, we call $f$ a **quasi $M$-convex function** if dom $f \neq \emptyset$ and it satisfies the following property:

**($\text{QM}$)** $\forall x, y \in \text{dom} f$, $\forall u \in \text{supp}^+ (x-y)$, $\exists v \in \text{supp}^- (x-y)$:

$$\Delta f(x; v, u) \leq 0 \quad \text{or} \quad \Delta f(y; u, v) \leq 0.$$
Similarly, we call $f$ a semistrictly quasi $M$-convex function if $\text{dom} \ f \neq \emptyset$ and it satisfies the following property:

$$(\text{SSQM}) \forall x, y \in \text{dom} \ f$, $\forall u \in \text{supp}^+(x - y)$, $\exists v \in \text{supp}^-(x - y)$:

(i) $\Delta f(x; v, u) \geq 0 \Rightarrow \Delta f(y; u, v) \leq 0$, and

(ii) $\Delta f(y; u, v) \geq 0 \Rightarrow \Delta f(x; v, u) \leq 0$.

Note that (SSQM) can be rewritten as follows:

$\forall x, y \in \text{dom} \ f$, $\forall u \in \text{supp}^+(x - y)$, $\exists v \in \text{supp}^-(x - y)$ satisfying at least one of the following:

(i) $\Delta f(x; v, u) < 0$,

(ii) $\Delta f(y; u, v) < 0$,

(iii) $\Delta f(x; v, u) = \Delta f(y; u, v) = 0$.

We also consider weaker properties than (QM) and (SSQM):

$$(\text{QM}_w) \forall x, y \in \text{dom} \ f $ with $ x \neq y$, $\exists u \in \text{supp}^+(x - y)$, $\exists v \in \text{supp}^-(x - y)$:

$\Delta f(x; v, u) \leq 0 \text{ or } \Delta f(y; u, v) \leq 0$.

$$(\text{SSQM}_w) \forall x, y \in \text{dom} \ f$ with $ x \neq y$, $\exists u \in \text{supp}^+(x - y)$, $\exists v \in \text{supp}^-(x - y)$:

(i) $\Delta f(x; v, u) \geq 0 \Rightarrow \Delta f(y; u, v) \leq 0$, and

(ii) $\Delta f(y; u, v) \geq 0 \Rightarrow \Delta f(x; v, u) \leq 0$.

The set version of quasi $M$-convexity can be obtained by translating the properties (QM) and (QM$_w$) for the indicator function $\delta_B: Z^V \rightarrow \{0, +\infty\}$ of a set $B \subseteq Z^V$ in terms of $B$.

$$(\text{Q-EXC}) \forall x, y \in B$, $\forall u \in \text{supp}^+(x - y)$, $\exists v \in \text{supp}^-(x - y)$:

$z - x_u + x_v \in B$ or $y + x_u - x_v \in B$.

$$(\text{Q-EXC}_w) \forall x, y \in B$ with $ x \neq y$, $\exists u \in \text{supp}^+(x - y)$, $\exists v \in \text{supp}^-(x - y)$:

$z - x_u + x_v \in B$ or $y + x_u - x_v \in B$.

Note that the properties (Q-EXC) and (Q-EXC$_w$) are the same as (EXC) and (EXC$_w$) discussed in [14], respectively.

**Theorem 3.1.** Let $B \subseteq Z^V$.

(i) $(\text{Q-EXC})$ for $B \iff (\text{QM})$ for $\delta_B$.

(ii) $(\text{Q-EXC}_w)$ for $B \iff (\text{QM}_w)$ for $\delta_B$.

(iii) $(\text{B-EXC})$ for $B \iff (\text{SSQM})$ for $\delta_B \iff (\text{SSQM}_w)$ for $\delta_B$.

We show some examples of quasi $M$-convex functions below.

**Example 3.2.** Let $\psi : Z \rightarrow R \cup \{+\infty\}$. We define $f: Z^2 \rightarrow R \cup \{+\infty\}$ by

$$f(x_1, x_2) = \left\{ \begin{array}{ll} \psi(x_1) & (x_1 + x_2 = 0), \\ +\infty & (x_1 + x_2 \neq 0). \end{array} \right. \quad (3.1)$$

By Theorem 2.2, $f$ satisfies $(\text{QM})$ (or $(\text{QM}_w)$) if and only if $\psi$ is quasiconvex, and $f$ satisfies $(\text{SSQM})$ (or $(\text{SSQM}_w)$) if and only if $\psi$ is semistrictly quasiconvex.

**Example 3.3.** Let $f: Z^V \rightarrow R \cup \{+\infty\}$ be an $M$-convex function, and $\varphi: R \rightarrow R \cup \{+\infty\}$ be a nondecreasing function. We define a function $\tilde{f}: Z^V \rightarrow R \cup \{+\infty\}$ by

$$\tilde{f}(x) = \left\{ \begin{array}{ll} \varphi(f(x)) & (x \in \text{dom} f), \\ +\infty & (x \notin \text{dom} f). \end{array} \right. \quad (3.2)$$

Then, $\tilde{f}$ satisfies $(\text{QM})$. Furthermore, if $\varphi$ is strictly increasing, then $\tilde{f}$ satisfies $(\text{SSQM})$.

**Example 3.4.** Let $B \subseteq Z^V$ be an $M$-convex set, $u \in R^V$, and $\alpha \in R$. Then, the set $S = \{x \in B \mid \langle p, x \rangle \leq \alpha \}$ satisfies $(\text{Q-EXC})$. Moreover, the function $f: S \rightarrow R$ defined by $f(x) = \langle p, x \rangle (x \in S)$ satisfies $(\text{SSQM})$.

**Remark 3.5.** The concept of (semistrict) quasi $M$-convexity can be naturally extended to functions $f: S \rightarrow T$ with $S \subseteq Z^V$ and a totally ordered set $T$ with total order $\prec$. For example, the property (SSQM) is rewritten for such functions as follows:

$\forall x, y \in S$, $\forall u \in \text{supp}^+(x - y)$, $\exists v \in \text{supp}^-(x - y)$:

(i) if either $x - x_u + x_v \notin S$, or $x - x_u + x_v \in S$ and $f(x - x_u + x_v) \geq f(x)$, then $y + x_u - x_v \in S$ and $f(y + x_u - x_v) \leq f(y)$, and

(ii) if either $y + x_u - x_v \notin S$, or $y + x_u - x_v \in S$ and $f(y + x_u - x_v) \geq f(y)$, then $x - x_u + x_v \in S$ and $f(x - x_u + x_v) \leq f(x)$,

where for $p, q \in T$ the notation $p \preceq q$ means $p < q$ or $p = q$. It is easy to see that the properties
of (semistrictly) quasi M-convex functions shown in this paper still holds true. For simplicity and convenience, we assume, in this paper, that the codomain of a function is $\mathbb{R} \cup \{+\infty\}$.

**Example 3.6.** Suppose that $V = \{1, 2, \cdots, n\}$ ($n \geq 1$). Let $a : V \rightarrow \mathbb{Z} \cup \{-\infty\}$, $b : V \rightarrow \mathbb{Z} \cup \{+\infty\}$, and $\alpha \in \mathbb{Z}$ satisfy $a(v) \leq b(v)$ ($v \in V$) and $\sum_{i \in V} a(i) \leq \alpha \leq \sum_{i \in V} b(i)$. For $i \in V$, let $f_i : [a(i), b(i)) \rightarrow \mathbb{R}$ be a semistrictly quasiconvex function. We define $B \subseteq \mathbb{Z}^V$ and $f : B \rightarrow \mathbb{R}^V$ by

$$B = \{ x \in \mathbb{Z}^V | x(V) = \alpha, a \leq x \leq b \},$$

$$f(x) = (f_i(x(i)) | i \in V) \ (x \in B),$$

where the total order $<$ on the codomain $\mathbb{R}^V$ of $f$ is given by the lexicographic order, i.e., for each $p, q \in \mathbb{R}^V$, $p < q$ holds if there exists some $k$ ($1 \leq k \leq n$) such that $p_i = q_i$ for $i = 1, \cdots, k-1$ and $p_k < q_k$. Then, $f$ satisfies (SSQM) in the extended sense (see Remark 3.5).

**Proof.** Let $x, y \in B$ be distinct vectors. Also, let $u \in \supp^+(x - y)$, $v \in \supp^-(x - y)$ be any elements, and w.l.o.g. assume that $u < v$. Then, we have $x - x_u + x_v \in B$ and $y - x_u - x_v \in B$. If $f_u(x(u) - 1) < f_u(x(u))$ or $f_u(y(u) - 1) < f_u(y(u))$ holds, then we have $f(x - x_u + x_v) < f(x)$ or $f(y + x_u - x_v) < f(y)$. Otherwise, we have $f_u(x(u) - 1) = f_u(x(u))$ and $f_u(y(u) - 1) = f_u(y(u))$ by Theorem 2.2. If $f_v(x(v) - 1) < f_v(x(v))$ or $f_v(y(v) - 1) < f_v(y(v))$ holds, then we have $f(x - x_u + x_v) < f(x)$ or $f(y + x_u - x_v) < f(y)$. Otherwise, we have $f_u(x(u) - 1) = f_u(x(u))$ and $f_u(y(v) - 1) = f_u(y(v))$, from which follows $f(x - x_u + x_v) = f(x)$ and $f(y + x_u - x_v) = f(y)$.

The relationship among various properties for sets and functions is summarized as follows. Note that the claim (i) of Theorem 3.7 is already shown in [14, Remark 11].

**Theorem 3.7.** (i) For $S \subseteq \mathbb{Z}^V$, we have

$$(B\text{-EXC}) \quad \Rightarrow \quad (Q\text{-EXC})$$

$$(B\text{-EXC}_w) \quad \Rightarrow \quad (Q\text{-EXC}_w).$$

(ii) For $f : \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{+\infty\}$, we have

$$(M\text{-EXC}) \quad \Rightarrow \quad (SSQM) \quad \Rightarrow \quad (QM)$$

$$(M\text{-EXC}_w) \quad \Rightarrow \quad (SSQM_w) \quad \Rightarrow \quad (QM_w).$$

### 3.2 Level Sets

We show various properties for level sets of quasi M-convex functions.

The following two theorems claim that level sets of quasi M-convex functions have quasi M-convexity. Furthermore, the weaker version of quasi M-convexity (QM$_w$) for functions can be characterized by quasi M-convexity (Q-EXC$_w$) of level sets.

**Lemma 3.8 ([14]).** Let $B \subseteq \mathbb{Z}^V$.

(i) If $B$ satisfies (Q-EXC$_w$), then $x(V) = y(V)$ for all $x, y \in \text{dom } f$.

(ii) (Q-EXC$_w$) $\iff$ (Q-EXC$_{w+}$): $B \subseteq \mathbb{Z}^V$.

**Theorem 3.9.** A function $f : \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfies (QM$_w$) if and only if the level set $L(f, \alpha)$ satisfies (Q-EXC$_w$) for all $\alpha \in \mathbb{R} \cup \{+\infty\}$. In particular, if $f$ satisfies (QM$_w$), then $\text{dom } f$ and $\arg \min f$ satisfy (Q-EXC$_w$).

**Proof.** $[\Rightarrow]$ Let $\alpha \in \mathbb{R} \cup \{+\infty\}$, and $x, y \in L(f, \alpha)$ be vectors with $x \neq y$. Applying (QM$_w$) to $x$ and $y$, we have $\Delta f(x; u, v) \leq 0$ or $\Delta f(y; u, v) \leq 0$ for some $u \in \supp^+(x - y)$ and $v \in \supp^-(x - y)$. Therefore, we have $x - x_u + x_v \in L(f, \alpha)$ or $y + x_u - x_v \in L(f, \alpha)$. $[\Leftarrow]$ Let $x, y \in \text{dom } f$, and we may assume that $f(x) \geq f(y)$. By Lemma 3.8 (ii), the level set $L(f, f(x))$ satisfies (Q-EXC$_{w+}$), from which follows $x - x_u + x_v \in L(f, f(x))$ for some $u \in \supp^+(x - y)$ and $v \in \supp^-(x - y)$. This implies that $f(x - x_u + x_v) \leq f(x)$, which yields (QM$_w$) for $L(f, f(x))$.

**Theorem 3.10.** Let $f : \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function satisfying (QM). Then, the level set $L(f, \alpha)$ satisfies (Q-EXC) for all $\alpha \in \mathbb{R} \cup \{+\infty\}$. In particular, $\text{dom } f$ and $\arg \min f$ satisfy (Q-EXC).

**Proof.** The proof is similar to that for the "only if" part of Theorem 3.9.

**Theorem 3.11.** Suppose $f : \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfies (SSQM$_w$). Then $\arg \min f$ satisfies (B-EXC), i.e., $\arg \min f$ is an M-convex set if it is nonempty.
An M-convex function can be characterized also by quasi M-convexity for level sets of a function perturbed by linear functions. For any function \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) and any vector \( p \in \mathbb{R}^V \), the function \( f[p] : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) is given by
\[
f[p](x) = f(x) + \sum_{v \in V} p(v)x(v) \quad (x \in \mathbb{Z}^V).
\]

**Theorem 3.12** ([14, Th. 1]). A function \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) satisfies (M-EXC) if and only if \( L(f[p], \alpha) \) satisfies (Q-EXC) for all \( p \in \mathbb{R}^V \) and \( \alpha \in \mathbb{R} \cup \{+\infty\} \).

Combining Theorems 3.9 and 3.12, we have the following property.

**Corollary 3.13.** A function \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) satisfies (M-EXC) if and only if \( f[p] \) satisfies (QM) for all \( p \in \mathbb{R}^V \).

### 3.3 Operations

The classes of (semistrictly) quasi M-convex functions are closed under several fundamental operations.

Let \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \). For any subset \( U \subseteq V \), define \( f_U : \mathbb{Z}^U \to \mathbb{R} \cup \{+\infty\} \) by
\[
f_U(y) = f(y, 0_{V \backslash U}) \quad (y \in \mathbb{Z}^U),
\]
where \( 0_{V \backslash U} \in \mathbb{R}^{V \backslash U} \) denotes the vector with each component equal to zero. For any functions \( a : V \to \mathbb{Z} \cup \{-\infty\} \) and \( b : V \to \mathbb{Z} \cup \{+\infty\} \), define \( f^b_a : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) by
\[
f^b_a(x) = \begin{cases} f(x) & (a \leq x \leq b), \\ +\infty & \text{(otherwise).} \end{cases}
\]

**Theorem 3.14.** Let \((\ast\text{QM}_*)\) denote one of (QM), (QM*), (SSQM), or (SSQM*), and \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) be a function with \((\ast\text{QM}_*)\) as functions in \( x \).

(i) For any \( a \in \mathbb{Z}^V \) and \( \nu > 0 \), the functions \( \nu \cdot f(a-x) \) and \( \nu \cdot f(a+x) \) satisfy \((\ast\text{QM}_*)\) as functions in \( x \).

(ii) For any \( U \subseteq V \), the function \( f_U : \mathbb{Z}^U \to \mathbb{R} \cup \{+\infty\} \) satisfies \((\ast\text{QM}_*)\).

(iii) For any \( a : V \to \mathbb{Z} \cup \{-\infty\} \) and \( b : V \to \mathbb{Z} \cup \{+\infty\} \) with \( a \leq b \), the function \( f^b_a : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) satisfies \((\ast\text{QM}_*)\).

(iv) Let \( f_i : \mathbb{Z}^{V_i} \to \mathbb{R}^{+\infty} \cup \{+\infty\} \) \((i = 1, 2)\) be functions with \((\ast\text{QM}_*)\). Then, the function \( f : \mathbb{Z}^{V_1} \times \mathbb{Z}^{V_2} \to \mathbb{R}^{+\infty} \cup \{+\infty\} \) defined by
\[
f(z_1, z_2) = f_1(z_1)f_2(z_2) \quad ((z_1, z_2) \in \mathbb{Z}^{V_1} \times \mathbb{Z}^{V_2})
\]
satisfies \((\ast\text{QM}_*)\).

**Proof.** We prove (iv) only. We consider the case when \((\ast\text{QM}_*) = (\text{SSQM})\). Let \( x = (x_1, x_2), y = (y_1, y_2) \in \text{dom } f_1 \times \text{dom } f_2 \), and let \( u \in \text{supp}^+(x - y) \), where \( u \in \text{supp}^+(x_1 - y_1) \) w.l.o.g. Then, there exists \( v \in \text{supp}^+(x_1 - y_1) \) such that
\[
\Delta f_1(x_1; v, u) \geq 0 \implies \Delta f_1(y_1; u, v) \leq 0,
\]
\[
\Delta f_1(y_1; u, v) \geq 0 \implies \Delta f_1(x_1; v, u) \leq 0.
\]
This implies that
\[
\Delta f(x; u, v) \geq 0 \implies \Delta f(y; u, v) \leq 0,
\]
\[
\Delta f(y; u, v) \geq 0 \implies \Delta f(x; v, u) \leq 0.
\]
Hence, (SSQM) holds for \( f \). \( \square \)

**Remark 3.15.** The class of (semistrictly) quasi M-convex functions is not closed under addition; in particular, it is not closed under addition of a linear function.

**Theorem 3.16.** For \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) and \( \varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \), define \( \tilde{f} : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) by (3.2).

(i) If \( f \) satisfies (QM) (resp. (QM*)) and \( \varphi \) is nondecreasing, then \( \tilde{f} \) satisfies (QM) (resp. (QM*)).

(ii) If \( f \) satisfies (SSQM) (resp. (SSQM*)) and \( \varphi \) is strictly increasing, then \( \tilde{f} \) satisfies (SSQM) (resp. (SSQM*)).

**Remark 3.17.** A quasi M-convex function \( \tilde{f} : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) is not necessarily given as the form (3.2). As an example, let \( \tilde{f} : \mathbb{Z}^3 \to \mathbb{R} \cup \{+\infty\} \) be a function given by
\[
dom \tilde{f} = \{(0,0,0), (1,0,1), (2,0,-2), (2,1,-3), (2,2,-4)\},
\]
\[
\tilde{f}(x_1, x_2, x_3) = -x_1 + x_2 \quad (x \in \text{dom } \tilde{f}).
\]
Although \( \tilde{f} \) satisfies (SSQM), it cannot be represented in the form (3.2) with an M-convex function \( f : \mathbb{Z}^3 \to \mathbb{R} \cup \{+\infty\} \) and a nondecreasing function \( \varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \). \( \square \)

**Theorem 3.18.** Let \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) and \( g : \mathbb{Z}^V \to \mathbb{R} \cup \{-\infty\} \) be functions such that \( g(x) > 0 \) for all \( x \in \text{dom } f \). Suppose that the function \( f(-a)g(-x) \) satisfies (QM) for all \( a \in \mathbb{R} \cup \{+\infty\} \).

Then, the function \( r : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) given by
\[
r(x) = \begin{cases} f(x)/g(x) & (x \in \text{dom } f), \\ +\infty & (x \notin \text{dom } f), \end{cases}
\]
satisfies (QM*).
Proof. Clear from Theorem 3.9. □

Remark 3.19. The statement of Theorem 3.18 cannot be strengthened by replacing \((QM_w)\) with \((QM)\), even if \(f\) and \(g\) are linear functions. □

3.4 Characterization by Local Exchange Properties

An M-convex function is characterized by a localized version of the property \((M-E\text{XC})\):

\[(M-E\text{XC-loc}) \forall x, y \in \text{dom} \hspace{0.1cm} f \text{ with } ||x - y||_1 = 4, \exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y) \text{ satisfying } (2.2).\]

Theorem 3.20 ([9, Th. 3.1], [14, Th. 2]).

Let \(f : Z^V \rightarrow R \cup \{+\infty\}\) be a function such that dom \(f\) is a nonempty set with \((Q-E\text{XC}_w)\). Then, \((M-E\text{XC}) \iff (M-E\text{XC-loc}).\)

We show that (semistrict) quasi M-convexity can be characterized also by the localized version of \((SSQM)\) and \((QM)\):

\[(SSQM_{w-loc}) \forall x, y \in \text{dom} \hspace{0.1cm} f \text{ with } ||x - y||_1 = 4, \exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):\]

(i) \(\Delta f(x; v, u) \geq 0 \Rightarrow \Delta f(y; u, v) \leq 0\),

(ii) \(\Delta f(y; u, v) \geq 0 \Rightarrow \Delta f(x; v, u) \leq 0\).

\[(SSQM_{w}\text{-loc}) \forall x, y \in \text{dom} \hspace{0.1cm} f \text{ with } ||x - y||_1 = 4, \exists u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y):\]

(i) \(\Delta f(x; v, u) \geq 0 \Rightarrow \Delta f(y; u, v) \leq 0\),

(ii) \(\Delta f(y; u, v) \geq 0 \Rightarrow \Delta f(x; v, u) \leq 0\).

Theorem 3.21. Let \(f : Z^V \rightarrow R \cup \{+\infty\}\) be a function such that dom \(f\) satisfies \((Q-E\text{XC}_w)\).

Then,

(i) \((SSQM) \iff (SSQM\text{-loc}).\)

(ii) \((SSQM_w) \iff (SSQM_{w-loc}).\)


Remark 3.22. The localized version of \((QM)\) does not characterize \((QM)\) in general. Let \(f : Z^2 \rightarrow Z \cup \{+\infty\}\) be a function such that

\[\text{dom} \hspace{0.1cm} f = \{ (0,0), (1,-1), (2,-2), (3,-3) \}, \hspace{0.1cm} f(0,0) = f(3,-3) = 0, \hspace{0.1cm} f(1,-1) = f(2,-2) = 1.\]

Then, dom \(f\) satisfies \((Q-E\text{XC})\), and \((QM)\) holds for any \(x, y \in \text{dom} \hspace{0.1cm} f\) with \(||x - y||_1 = 4\). However, \((QM)\) does not hold for \(x = (0,0)\) and \(y = (3,-3)\). □

4 Minimization of Quasi M-convex Functions

4.1 Theorems

Global minimality of quasi M-convex functions is characterized by local minimality.

Theorem 4.1. Let \(f : Z^V \rightarrow R \cup \{+\infty\}\) and \(x \in \text{dom} \hspace{0.1cm} f\).

(i) Assume \((QM_w)\) for \(f\). Then, \(\Delta f(x; v, u) > 0\) \((\forall u, v \in V, u \neq v) \iff f(x) < f(y) (\forall y \in Z^V \setminus \{x\})\).

(ii) Assume \((SSQM_w)\) for \(f\). Then, \(\Delta f(x; v, u) \geq 0\) \((\forall u, v \in V) \iff f(x) \leq f(y) (\forall y \in Z^V)\).

Proof. We show the sufficiency of (ii) only. Assume, to the contrary, that there exists some \(y \in \text{dom} \hspace{0.1cm} f\) such that \(f(y) < f(x)\). We further assume that \(y\) minimizes the value \(||y - z||_1\) among all such vectors. By \((SSQM_w)\), there exist some \(v' \in \text{supp}^+(x - y)\) and \(v'' \in \text{supp}^-(x - y)\) such that \(\Delta f(x; v', u'') = f(y) < f(x)\) and \(||y + \chi_{v'} - \chi_{v''}\|_1 < ||y - z||_1\) a contradiction to the choice of \(y\). □

If \(f\) satisfies \((SSQM)\), then any vector in \(\text{dom} \hspace{0.1cm} f\) can be easily separated from some minimizer of \(f\) (cf. [13, Th. 2.2, Cor. 2.3]).

Theorem 4.2. Let \(f : Z^V \rightarrow R \cup \{+\infty\}\) be a function with \((SSQM)\). Assume \(\arg \min f \neq \emptyset\).

(i) For \(x \in \text{dom} \hspace{0.1cm} f\) and \(v \in V\), let \(u \in V\) satisfy

\[f(x - \chi_u + \chi_v) = \min_{z \in V} f(x - \chi_z + \chi_v).\]

Then, there exists \(z_* \in \arg \min f \text{ with } z_*(u) \leq x(u) - 1 + \chi_u(u)\).

(ii) For \(x \in \text{dom} \hspace{0.1cm} f\) and \(u \in V\), let \(v \in V\) satisfy

\[f(x - \chi_u + \chi_v) = \min_{t \in V} f(x - \chi_u + \chi_t).\]

Then, there exists \(z_* \in \arg \min f \text{ with } z_*(v) \geq x(v) - \chi_u(v) + 1.\)

Proof. We prove (i) only. Put \(z' = x - \chi_u + \chi_v\). Assume, to the contrary, that there is no \(z \in \arg \min f \text{ with } x(u) \leq z'(u)\). Let \(z_* \in \arg \min f \text{ with minimum } z_*(u)\). Then, we have \(z_*(u) > z'(u)\). By applying \((SSQM)\) to \(z_*, z'\), and \(u\), we have some \(w \in \text{supp}^-(z_* - z')\) such that if
$\Delta f(z_{*}; w, u) > 0$ then $\Delta f(x'; u, w) < 0$. Due to the choice of $z_{*}$, we have $\Delta f(x_{*}; w, u) > 0$. Hence, it holds that

$$f(x') > f(x' + \chi_{u} - \chi_{w}) = f(x - \chi_{w} + \chi_{v}),$$

a contradiction to the definition of $u \in V$. □

Corollary 4.3. Let $f : \mathbb{Z}^{V} \to \mathbb{R} \cup \{+\infty\}$ be a function with (SSQM). Also, let $x \in \text{dom} f \setminus \arg \min f$, and $u, v \in V$ satisfy

$$f(x - \chi_{u} + \chi_{v}) = \min_{s, t \in V} f(x - \chi_{s} + \chi_{t}).$$

Then, there exists $z_{*} \in \arg \min f$ with $z_{*}(u) \leq x(u) - 1$ and $z_{*}(v) \geq x(v) + 1$.

Remark 4.4. The statements in Theorem 4.2 do not hold even if $f$ satisfies the property (SSQM) (and not (SSQM)).

The following theorem shows that a global minimizer of a semistrictly quasi M-convex function exists in the neighborhood of a $\Delta$-local minimum. This generalizes [6, Th. 4.1].

Theorem 4.5. Let $f : \mathbb{Z}^{V} \to \mathbb{R} \cup \{+\infty\}$ be a function with (SSQM), and $\Delta$ be any positive integer. Suppose that $x_{\Delta} \in \text{dom} f$ satisfies $f(x_{\Delta}) \leq f(x_{\Delta} + \Delta(\chi_{v} - \chi_{u}))$ for all $u, v \in V$. Then, there exists some $z_{*} \in \arg \min f$ such that

$$|z_{\Delta}(v) - z_{*}(v)| \leq (n - 1)(\Delta - 1) \quad (v \in V).$$

(4.1)

Proof. It suffices to show that for all $\varepsilon > 0$ there exists some $z_{*} \in \text{dom} f$ satisfying $f(z_{*}) \leq \inf f + \varepsilon$ and (4.1).

Let $z_{*} \in \text{dom} f$ satisfy $f(z_{*}) \leq \inf f + \varepsilon$, and suppose that $z_{*}$ minimizes the value $\|x_{*} - \Delta_{*}\|$ among all such vectors. In the following, we fix $v \in V$ and prove $x_{\Delta}(v) - z_{*}(v) \leq (n - 1)(\Delta - 1)$. The inequality $x_{*}(v) - z_{*}(v) \leq (n - 1)(\Delta - 1)$ can be shown similarly.

We may assume $x_{\Delta}(v) > z_{*}(v)$. We first prove the following two claims.

Claim 1 There exist $w_{1}, w_{2}, \ldots, w_{k} \in V \setminus \{v\}$ and $y_{0}(= x_{\Delta}), y_{1}, \ldots, y_{k} \in \text{dom} f$ with $k = x_{\Delta}(v) - z_{*}(v)$ such that

$$y_{i} = y_{i-1} - \chi_{v} + \chi_{w_{i}}, \quad f(y_{i}) < f(y_{i-1}) \quad (i = 1, \ldots, k).$$

[Proof of Claim 1] We show the claim by induction on $i$. If $i - 1 < k$, then $v \in \text{supp}^{+}(y_{i-1} - z_{*})$. By (SSQM) applied to $y_{i-1}, z_{*}$, and $v$, we have some $w_{i} \in \text{supp}^{+}(y_{i-1} - z_{*}) \subseteq \text{supp}^{-}(z_{*} - z_{*}) \subseteq V \setminus \{v\}$ such that if $\Delta f(x_{*}; v, w_{i}) > 0$ then $\Delta f(y_{i-1}; w_{i}, v) < 0$. By the choice of $z_{*}$, we have $\Delta f(x_{*}; v, w_{i}) > 0$ since $\|x_{*} + \chi_{v} - \chi_{w_{i}}\| - \Delta_{*} |_{1} < \|x_{*} - \Delta_{*} |_{1}$. Therefore, $f(y_{i}) = f(y_{i-1} - \chi_{v} + \chi_{w_{i}}) < f(y_{i-1})$.

[End of Proof for Claim 1]

Claim 2 For any $w \in V \setminus \{v\}$ with $y_{k}(w) > x_{\Delta}(w)$ and $\alpha \in [0, y_{k}(w) - x_{\Delta}(w) - 1]$, we have

$$f(x_{\Delta} - (\alpha + 1)(\chi_{v} - \chi_{w})) < f(x_{\Delta} - \alpha(\chi_{v} - \chi_{w})).$$

(4.2)

[Proof of Claim 2] We prove (4.2) by induction on $\alpha$. Put $x' = x_{\Delta} - \alpha(\chi_{v} - \chi_{w})$ for $\alpha \in [0, y_{k}(w) - x_{\Delta}(w) - 1]$, and suppose $x' \in \text{dom} f$.

Let $j_{*}$ ($1 \leq j_{*} \leq k$) be the largest index such that $w_{j_{*}} = w$. Then, $y_{j_{*}}(w) = x_{j_{*}}(w) > x'(w)$ and $\text{supp}^{-}(y_{j_{*}} - x') \subseteq \{v\}$ (SSQM) implies that if $\Delta f(y_{j_{*}}; v, w) > 0$ then $\Delta f(x'; v, w) < 0$. By Claim 1, we have $\Delta f(y_{j_{*}}; v, w) > 0$. Hence, (4.2) follows.

[End of Proof for Claim 2]

The $\Delta$-local minimality of $x_{\Delta}$ implies $f(x_{\Delta} - \Delta(\chi_{v} - \chi_{w})) \geq f(x_{\Delta})$, which, combined with Claim 2, implies $y_{k}(w) - x_{\Delta}(w) \leq \Delta - 1$. Thus,

$$x_{\Delta}(v) - z_{*}(v) = x_{\Delta}(v) - y_{k}(v) = \sum_{w \in V \setminus \{v\}} \{y_{k}(w) - x_{\Delta}(w)\} \leq (n - 1)(\Delta - 1),$$

where the second equality is by Lemma 3.8 (i). □

4.2 Algorithms

Let $f : \mathbb{Z}^{V} \to \mathbb{R} \cup \{+\infty\}$ be a function such that $\text{dom} f$ is a nonempty bounded set, and put

$$L = \max \{|x - y|_{\infty} | x, y \in \text{dom} f\}.$$

Assume (SSQM) for $f$. Then, Theorem 4.1 immediately leads to the following algorithm.

Algorithm DESCENT

Step 0: Let $z$ be any vector in $\text{dom} f$.

Step 1: If $f(z) = \min_{x \in \text{dom} f} f(x - \chi_{s} + \chi_{t})$ then stop. [$z$ is a minimizer of $f$.]
Step 2: Find \( u, v \in V \) with \( f(x - \chi_u + \chi_v) < f(x) \).

Step 3: Set \( x := x - \chi_u + \chi_v \). Go to Step 1. \( \square \)

Algorithm **Descent** terminates in at most \(|\text{dom } f| \leq (L + 1)^{n-1}\) iterations since it generates distinct \( x \) in each iteration.

To the end of this section we assume (SSQM) for \( f \). Based on Theorem 4.5, we apply the scaling technique to Algorithm **Descent** to obtain a faster algorithm.

**Algorithm Scaling-Descent**

Step 0: Let \( x \) be any vector in \( \text{dom } f \). Put \( \Delta := [L/4n], B := \text{dom } f \).

Step 1: [\( \Delta \)-scaling phase]

Step 1-1: If

\[
f(x) = \min \{ f(x - \Delta(x_s - x_t)) \mid s, t \in V, \ x - \Delta(x_s - x_t) \in B \}
\]

then go to Step 2.

Step 1-2: Find \( u, v \in V \) with \( x - \Delta(x_u - x_v) \in B \) satisfying \( f(x - \Delta(x_u - x_v)) < f(x) \).

Step 1-3: Set \( x := x - \Delta(x_u - x_v) \). Go to Step 1-1.

Step 2: If \( \Delta = 1 \) then stop. [\( x \) is a minimizer of \( f \)].

Step 3: Put

\[
B := B \cap \{ y \in Z^V \mid |y(v) - x(v)| \leq (n - 1)(\Delta - 1) \ (v \in V) \}
\]

and \( \Delta := \lceil \Delta/2 \rceil \). Go to Step 1. \( \square \)

The number of scaling phases is \( \lceil \log L \rceil \), and each scaling phase terminates in \((4n)^{n-1}\) iterations. Therefore, Algorithm **Scaling-Descent** runs in \((4n)^{n-1}\lceil \log L \rceil \) iterations.

We then propose another elaboration of Algorithm **Descent** by exploiting Corollary 4.3

**Algorithm Steepest-Descent**

Step 0: Let \( x \) be any vector in \( \text{dom } f \). Set \( B := \text{dom } f \).

Step 1: If \( f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t) \) then stop. [\( x \) is a minimizer of \( f \)].

Step 2: Find \( u, v \in V \) with \( x - \chi_u + \chi_v \in B \) satisfying

\[
f(x - \chi_u + \chi_v) = \min \{ f(x - \chi_s + \chi_t) \mid s, t \in V, \ x - \chi_s + \chi_t \in B \}. \tag{4.3}
\]

Step 3: Set \( x := x - \chi_u + \chi_v \) and

\[
B := B \cap \{ y \in Z^V \mid |y(u) - x(u)| \leq 1, \ y(v) \geq x(v) + 1 \}. \tag{4.4}
\]

Go to Step 1.

By Corollary 4.3, the set \( B \) always contains a minimizer of \( f \). Hence, Algorithm **Steepest-Descent** finds a minimizer of \( f \). To analyze the number of iterations, we consider the value

\[
\sum_{w \in V} \{ \max_{y \in B} y(w) - \min_{y \in B} y(w) \}
\]

This value is bounded by \( nL \) and decreases at least by two in each iteration. Therefore, **Steepest-Descent** terminates in \( O(nL) \) iterations. In particular, if \( \text{dom } f \subseteq \{0, 1\}^V \) then the number of iterations is \( O(n^2) \).

It is shown in [13] that the minimization of an M-convex function can be done in polynomial time by the domain reduction method explained below. We show that the domain reduction method also works for the minimization of a function with (SSQM) if its effective domain is a bounded M-convex set.

Given a bounded M-convex set \( B \), we define the set \( N_B \subseteq B \) as follows. For \( w \in V \), define

\[
l_B(w) = \min_{y \in B} y(w), \quad u_B(w) = \max_{y \in B} y(w),
\]

\[
l'_B(w) = \left[ \left( 1 - \frac{1}{n} \right) l_B(w) + \frac{1}{n} u_B(w) \right],
\]

\[
u'_B(w) = \left[ \frac{1}{n} u_B(w) + \left( 1 - \frac{1}{n} \right) u_B(w) \right].
\]

Then, \( N_B \) is defined as

\[ N_B = \{ y \in B \mid l'_B \leq y \leq u'_B \}. \]

**Theorem 4.6 ([13, Th. 2.4]).** \( N_B \) is a (nonempty) M-convex set.

The next algorithm maintains a set \( B \) which is an M-convex set containing a minimizer of \( f \). It reduces \( B \) iteratively by exploiting Corollary 4.3 and finally finds a minimizer.

**Algorithm Domain-Reduction**

Step 0: Set \( B := \text{dom } f \).

Step 1: Find a vector \( x \in N_B \).

Step 2: If \( f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t) \) then stop. [\( x \) is a minimizer of \( f \)].

Step 3: Find \( u, v \in V \) with \( x - \chi_u + \chi_v \in B \) satisfying (4.3).

Step 4: Set \( B \) by (4.4). Go to Step 1. \( \square \)
We analyze the number of iterations of Domain Reduction. Denote by $B_i$ the set $B$ in the $i$-th iteration, and let $l_i(w) = l_{B_i}(w)$, $u_i(w) = u_{B_i}(w)$ ($w \in V$). It is clear that $u_i(w) - l_i(w)$ is nonincreasing w.r.t. $i$. Furthermore, we have the following property:

**Lemma 4.7 ([13, Lemma 3.1]).**

$$u_{i+1}(w) - l_{i+1}(w) < (1 - 1/n)\{u_i(w) - l_i(w)\}$$

for $w \in \{u, v\}$, where $u, v \in V$ are the elements found in Step 3.

This lemma implies that Algorithm Domain Reduction terminates in $O(n^2 \log L)$ iterations.

We then consider the time complexity of each step. Steps 2, 3, and 4 can be done in $O(n^2)$ time. In Step 1, we use the exchange capacity to compute the values $l_B(w)$ and $u_B(w)$ and to find a vector in $N_B$. For any $w \in V$, the values $l_B(w)$ and $u_B(w)$ can be computed by evaluating the exchange capacity at most $n$ times, provided that a vector in $B$ is given [4, Th. 3.27]. A vector in $N_B$ can be found by evaluating the exchange capacity at most $n^2$ times, provided that a vector in $B$ is given [13, Th. 2.5]. The exchange capacity can be computed in $O(\log L)$ time by binary search. Hence, Step 1 requires $O(n^2 \log L)$ time.

**Theorem 4.8.** Suppose that $f : Z^V \rightarrow R \cup \{+\infty\}$ satisfies (SSQM) and that dom $f$ is a bounded M-convex set. If a vector in dom $f$ is given, Algorithm Domain Reduction finds a minimizer of $f$ in $O(n^4 (\log L)^2)$ time.

References


