<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
</table>
| Title | 不規則外乱の影響を考慮した相転移現象のモデリング  
|       | 函数解析学の応用としての情報数理の研究 |
| Author(s) | 石川 昌明 宮島 啓一 |
| Citation | 数理解析研究所講究録 2001, 1186: 194-204 |
| Issue Date | 2001-01 |
| URL | http://hdl.handle.net/2433/64650 |
| Type | Departmental Bulletin Paper |

京都大学
On James and Schäffer constants for Banach spaces

Gohshik Gohshik (Yasuki Takahashi) 九州工大・工  加藤幸雄 (Mikio Kato)
山形大・工 高橋厚原 (Sin-Ei Takahashi)

We introduce James and Schäffer type constants for Banach spaces \( X \), and investigate the relation between these constants and some geometrical properties of Banach spaces.

Let \( X \) be a Banach space with \( \dim X \geq 2 \). Then, geometrical properties of \( X \) are determined by its unit ball \( B_X = \{ x \in X : ||x|| \leq 1 \} \) or its unit sphere \( S_X = \{ x \in X : ||x|| = 1 \} \). The modulus of convexity of \( X \) is a function \( \delta_X : [0,2] \rightarrow [0,1] \) defined by

\[
\delta_X(\epsilon) = \inf \{ 1 - \frac{||x+y||}{2} : x, y \in S_X, ||x-y|| = \epsilon \}
\]

In the above definition, it is well-known that \( S_X \) may be replaced by \( B_X \). The space \( X \) is called uniformly convex (Clarkson [1]) if \( \delta_X(\epsilon) > 0 \) for all \( 0 < \epsilon < 2 \), and called uniform non-square (James [5]) if \( \delta_X(\epsilon) > 0 \) for some \( 0 < \epsilon < 2 \).

James and Schäffer constants:

James constant of \( X \) is defined by

\[
J(X) = \sup \{ \min( ||x+y||, ||x-y||) : x, y \in S_X \}
\]

and Schäffer constant of \( X \) is defined by

\[
S(X) = \inf \{ \max( ||x+y||, ||x-y||) : x, y \in S_X \}.
\]

Known Facts (cf. [3], [4], [7]):

1. In the definition of \( J(X) \), \( S_X \) may be replaced by \( B_X \).
2. \( J(X)S(X) = 2 \)
3. \( X : \text{unif. non-square} \iff J(X) < 2 \iff S(X) > 1 \)
4. Let \( 1 \leq p \leq \infty, 1/p + 1/p' = 1, t = \min\{p,p'\} \) and \( s = \max\{p,p'\} \).
   Then, \( J(L_p) = 2^{1/t} \) and \( S(L_p) = 2^{1/s} \).
5. \( \sqrt{2} \leq J(X) \leq 2 \) and \( 1 \leq S(X) \leq \sqrt{2} \) for any Banach space \( X \).
6. If \( X \) is a Hilbert space, then \( J(X) = \sqrt{2} \), but the converse is not true.
(7) There is a Banach space $X$ such that $J(X) \neq J(X^*)$ ($S(X) \neq S(X^*)$), where $X^*$ is a dual space of $X$.

(8) $2J(X) - 2 \leq J(X^*) \leq J(X)/2 + 1$ for any Banach space $X$.

New constants of James and Schäffer type:

We denote by $M_t(a,b)$ the power means of order $t$ of the positive real numbers $a$ and $b$, that is,

$$M_t(a,b) = \{(a^t+b^t)/2\}^{1/t} (t \neq 0) \quad \text{and} \quad M_0(a,b) = (ab)^{1/2}$$

Remark. (1) $M_t(a,b)$ is defined for $a, b \geq 0$ ($M_t(a,b) = 0$ if $t<0$, $ab=0$).

(2) If $t \to -\infty$ ($t \to +\infty$), then $M_t(a,b) \to \min\{a,b\}$ ($M_t(a,b) \to \max\{a,b\}$).

James type constants:

$$J_t(X) = \sup \{ M_t(\|x+y\|, \|x-y\|) : x,y \in S_X \}, \quad -\infty < t < +\infty$$

Schäffer type constants:

$$S_t(X) = \inf \{ M_t(\|x+y\|, \|x-y\|) : x,y \in S_X \}, \quad -\infty < t < +\infty$$

Remark. In the definition of $J_t(X)$, $S_X$ may be replaced by $B_X$.

Proposition 1. (1) $\sqrt{2} \leq J(X) \leq J_t(X) \leq 2$ for all $t \in (-\infty, +\infty)$, and if $t \geq 2$, then $J_t(X) \geq 2^{1-1/t}$.

(2) $J_t(X)$ is non-decreasing on $(-\infty, +\infty)$, $J_t(X) \to 2$ if $t \to +\infty$, and $J_t(X) \to J(X)$ if $t \to -\infty$.

(3) $S_t(X) = 0$ if $t \leq 0$, $S_t(X) = 2^{1-1/t}$ if $0 < t \leq 1$, $S_t(X) \leq 2^{1-1/t}$ for all $t < \infty$, and $1 \leq S_t(X) \leq S(X) \leq \sqrt{2}$ for all $t \in (1, +\infty)$.

(4) $S_t(X)$ is non-decreasing on $(-\infty, +\infty)$, $S_t(X) \to 1$ if $t \to 1+0$, and $S_t(X) \to S(X)$ if $t \to +\infty$.

Theorem 2. The following assertions are equivalent:

(1) $X$ is uniformly non-square.

(2) $J_t(X) < 2$ for all $t$ (some $t$).

(3) $J(X) < J_t(X)$ for some $t$.

(4) There exists $t_0$ such that $J_t(X)$ is strictly increasing on $[t_0, +\infty)$.

(5) $S_t(X) > 1$ for all $t > 1$ (some $t > 1$).

(6) $S(X) > S_t(X)$ for some $t > 1$. 

$\tau_M(a,b) = \{(a^t+b^t)/2\}^{1/t} (t \neq 0)$
Let \( 1 \leq p \leq 2 \) and \( 1/p + 1/p' = 1 \). We say that the \((p,p')\)-Clarkson inequality holds in a Banach space \( X \) if for any \( x, y \in X \), the inequality

\[
(CI_p) \quad (\|x + y\|^{p'} + \|x - y\|^{p'})^{1/p'} \leq 2^{1/p'} (\|x\|^{p} + \|y\|^{p})^{1/p}
\]

holds.

Remark. Let \( 1 \leq p \leq 2 \).

(1) \((CI_p)\) holds in \( L_p \) and \( L_{p'} \) (Clarkson [1]).

(2) \((CI_p)\) holds in \( X \) if and only if it holds in \( X^* \); if \((CI_p)\) holds in \( X \), then \((CI_t)\) holds in \( X \) for any \( t \in [1,p] \); and if \((CI_p)\) holds in \( X \), then \((CI_t)\) holds in \( L_r(X) \), where \( 1 \leq r \leq \infty \) and \( t = \min\{p,r,r'\} \) (Takahashi and Kato [9]).

A Banach space \( Y \) is said to be finitely representable (f.r.) in a Banach space \( X \) if for any \( \lambda > 1 \) and for any finite dimensional subspace \( F \) of \( Y \) there is a finite dimensional subspace \( E \) of \( X \) with \( \dim E = \dim F \) such that the Banach-Mazur distance \( d(E,F) \leq \lambda \).

Proposition 3. If \( Y \) is f.r. in \( X \), then \( J_t(Y) \leq J_t(X) \) and \( S_t(Y) \geq S_t(X) \) for any \( t \).

Theorem 4. Let \( 1 < p \leq 2 \) and suppose that the \((p,p')\)-Clarkson inequality holds in \( X \).

(1) \( J_t(X) = 2^{1-1/t} \) for \( t \geq p' \), and \( S_t(X) = 2^{1-1/t} \) for \( 0 < t \leq p \).

(2) If \( \mathcal{E}_p \) (or \( \mathcal{E}_{p'} \)) is finitely representable (f.r.) in \( X \), then \( J_t(X) = 2^{1/p} \) for \( t \leq p' \), and \( S_t(X) = 2^{1/p'} \) for \( t \geq p \).

Corollary 1. (1) \( J_t(H) = \sqrt{2} \) if \( t \leq 2 \), \( J_t(H) = 2^{1-1/t} \) if \( t \geq 2 \), \( S_t(X) = 2^{1-1/t} \) if \( 0 < t \leq 2 \), and \( S_t(X) = \sqrt{2} \) if \( t \geq 2 \), where \( H \) is a Hilbert space.

(2) \( J_t(L_p) = 2^{1/r} \) if \( t \leq r' \), \( J_t(L_p) = 2^{1-1/t} \) if \( t \geq r' \), \( S_t(L_p) = 2^{1-1/t} \) if \( 0 < t \leq r 
and \( S_t(L_p) = 2^{1/r'} \) if \( t \geq r 
where \( r = \min\{p,p'\} \).

(3) Let \( X = L_p(L_q) \), and \( r = \min\{p,p',q,q'\} \). Then \( J_t(X) = 2^{1/r} \) if \( t \leq r' \), \( J_t(X) = 2^{1-1/t} \) if \( t > r' \), \( S_t(X) = 2^{1-1/t} \), and \( S_t(X) = 2^{1/r'} \) if \( t \geq r \).
Corollary 2. Let \( X = \ell_p(\ell_q) \), \( 1 < p, q < \infty \). Then, \( J(X) = 2^{1/r} \) and \( S(X) = 2^{1/r'} \), where \( r = \min\{p, p', q, q'\} \).

Remark. As already mentioned, for any Banach space \( X \), it holds \( J(X) S(X) = 2 \). \( J_t(X) \to J(X) \) if \( t \to -\infty \), and \( S_t(X) \to S(X) \) if \( t \to +\infty \). By Corollary 1, we know that for various Banach spaces \( X \), \( J_t(X) S_t(X) = 2 \), where \( 1 < t < \infty \) and \( 1/t + 1/t' = 1 \).

Now we give a characterization of a Hilbert space. As mentioned before, if \( X \) is a Hilbert space, then \( J(X) = \sqrt{2} \); but the converse is not true.

Theorem 5. A Banach space \( X \) is isometric to a Hilbert space if and only if \( J_2(X) = \sqrt{2} \).

Remark. Let \( C_{NJ}(X) \) denote the von Neumann-Jordan constant of \( X \) (Clarkson [2]). Then it is easy to see that \( \sqrt{2} \leq J_2(X) \leq \sqrt{2C_{NJ}(X)} \) for any Banach space \( X \). Hence, Theorem 5 generalizes a result of Jordan and von Neumann [6], which asserts that \( X \) is a Hilbert space if and only if \( C_{NJ}(X) = 1 \).

Proposition 6. Let \( X \) be a Banach space. If there is \( t \in [2, \infty) \) such that \( J_t(X) = 2^{1-1/t} \), then \( X \) is uniformly convex.

Remark. For any Banach space \( X \), we have \( J_t(X) \geq 2^{1-1/t} \) for all \( t \geq 2 \) (see, Proposition 1). It can be shown that for any \( \epsilon > 0 \), there is a Banach space \( X \) which is not uniformly convex such that \( J_t(X) < 2^{1-1/t} + \epsilon \).

Theorem 7. (1) For any Banach space \( X \), \( J_1(X) = J_1(X^*) \).

(2) For any \( t < 1 \), there is a Banach space \( X \) such that \( J_t(X) \neq J_t(X^*) \).

Corollary 3. \( X \) : uniformly non-square \( \iff X^*: \) uniformly non-square
References