<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
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</thead>
<tbody>
<tr>
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On James and Schäffer constants for Banach spaces

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We introduce James and Schäffer type constants for Banach spaces $X$, and investigate the relation between these constants and some geometrical properties of Banach spaces.

Let $X$ be a Banach space with $\dim X \geqq 2$. Then, geometrical properties of $X$ are determined by its unit ball $B_X = \{ x \in X : \| x \| \leqq 1 \}$ or its unit sphere $S_X = \{ x \in X : \| x \| = 1 \}$. The modulus of convexity of $X$ is a function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf \{ 1 - \frac{\| x+y \|}{2} : x, y \in S_X, \| x-y \| = \varepsilon \}$$

In the above definition, it is well-known that $S_X$ may be replaced by $B_X$. The space $X$ is called uniformly convex (Clarkson [1]) if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon < 2$, and called uniform non-square (James [5]) if $\delta_X(\varepsilon) > 0$ for some $0 < \varepsilon < 2$.

James and Schäffer constants:

James constant of $X$ is defined by

$$J(X) = \sup \{ \min (\| x+y \|, \| x-y \|) : x, y \in S_X \}$$

and Schäffer constant of $X$ is defined by

$$S(X) = \inf \{ \max (\| x+y \|, \| x-y \|) : x, y \in S_X \}.$$

Known Facts (cf. [3], [4], [7]):

1. In the definition of $J(X)$, $S_X$ may be replaced by $B_X$.
2. $J(X)S(X) = 2$
3. $X : \text{unif. non-square} \Leftrightarrow J(X) < 2 \Leftrightarrow S(X) > 1$
4. Let $1 \leqq p \leqq \infty$, $1/p+1/p' = 1$, $t = \min \{ p, p' \}$ and $s = \max \{ p, p' \}$. Then, $J(L_p) = 2^{1/t}$ and $S(L_p) = 2^{1/s}$.
5. $\sqrt{2} \leqq J(X) \leqq 2$ and $1 \leqq S(X) \leqq \sqrt{2}$ for any Banach space $X$.
6. If $X$ is a Hilbert space, then $J(X) = \sqrt{2}$, but the converse is not true.
(7) There is a Banach space X such that \( J(X) \neq J(X^*) \) \((S(X) \neq S(X^*))\), where \( X^* \) is a dual space of \( X \).

(8) \( 2J(X) - 2 \leq J(X^*) \leq J(X)/2 + 1 \) for any Banach space \( X \).

New constants of James and Schäffer type:

We denote by \( M_t(a,b) \) the power means of order \( t \) of the positive real numbers \( a \) and \( b \), that is,

\[
M_t(a,b) = \left( \frac{a^t + b^t}{2} \right)^{1/t} \quad (t \neq 0) \quad \text{and} \quad M_0(a,b) = \sqrt{ab}.
\]

Remark. (1) \( M_t(a,b) \) is defined for \( a, b \geq 0 \) \((M_t(a,b) = 0 \text{ if } t < 0, \ ab = 0)\).

(2) If \( t \to -\infty \) \((t \to +\infty)\), then \( M_t(a,b) \to \min\{a,b\} \) \((M_t(a,b) \to \max\{a,b\})\).

James type constants:

\[
J_t(X) = \sup \{ M_t(\|x+y\|, \|x-y\|) : x,y \in S_X \}, \quad -\infty < t < +\infty
\]

Schäffer type constants:

\[
S_t(X) = \inf \{ M_t(\|x+y\|, \|x-y\|) : x,y \in S_X \}, \quad -\infty < t < +\infty
\]

Remark. In the definition of \( J_t(X) \), \( S_X \) may be replaced by \( B_X \).

Proposition 1. (1) \( \sqrt{2} \leq J(X) \leq J_t(X) \leq 2 \) for all \( t \in (-\infty, +\infty) \), and if \( t \geq 2 \), then \( J_t(X) \geq 2^{1-1/t} \).

(2) \( J_t(X) \) is non-decreasing on \((-\infty, +\infty)\), \( J_t(X) \to 2 \) if \( t \to +\infty \), and \( J_t(X) \to J(X) \) if \( t \to -\infty \).

(3) \( S_t(X) = 0 \) if \( t \leq 0 \), \( S_t(X) = 2^{1-1/t} \) if \( 0 < t \leq 1 \), \( S_t(X) \leq 2^{1-1/t} \) for all \( t < \infty \), and \( 1 \leq S_t(X) \leq S(X) \leq \sqrt{2} \) for all \( t \in (1, +\infty) \).

(4) \( S_t(X) \) is non-decreasing on \((-\infty, +\infty)\), \( S_t(X) \to 1 \) if \( t \to 1+0 \), and \( S_t(X) \to S(X) \) if \( t \to +\infty \).

Theorem 2. The following assertions are equivalent:

(1) \( X \) is uniformly non-square.

(2) \( J_t(X) < 2 \) for all \( t \) \((\text{some } t)\).

(3) \( J(X) < J_t(X) \) for some \( t \).

(4) There exists \( t_0 \) such that \( J_t(X) \) is strictly increasing on \([t_0, +\infty)\).

(5) \( S_t(X) > 1 \) for all \( t > 1 \) \((\text{some } t>1)\).

(6) \( S(X) > S_t(X) \) for some \( t > 1 \).
Let \( 1 \leq p \leq 2 \) and \( 1/p + 1/p' = 1 \). We say that the \((p,p')\)-Clarkson inequality holds in a Banach space \( X \) if for any \( x, y \in X \), the inequality

\[
(\text{CI}_p) \quad (\|x + y\|^{p'} + \|x - y\|^{p'})^{1/p'} \leq 2^{1/p'} (\|x\|^{p'} + \|y\|^{p'})^{1/p'}
\]

holds.

Remark. Let \( 1 \leq p \leq 2 \).

1. (CI\(_p\)) holds in \( L_p \) and \( L_{p'} \) (Clarkson [1]).

2. (CI\(_p\)) holds in \( X \) if and only if it holds in \( X^* \); if (CI\(_p\)) holds in \( X \), then (CI\(_t\)) holds in \( X \) for any \( t \in [1,p] \); and if (CI\(_p\)) holds in \( X \), then (CI\(_t\)) holds in \( L_t(X) \), where \( 1 \leq r \leq \infty \) and \( t = \min\{p,r,r'\} \) (Takahashi and Kato [9]).

A Banach space \( Y \) is said to be finitely representable (f.r.) in a Banach space \( X \) if for any \( \lambda > 1 \) and for any finite dimensional subspace \( F \) of \( Y \) there is a finite dimensional subspace \( E \) of \( X \) with \( \dim E = \dim F \) such that the Banach-Mazur distance \( d(E,F) \leq \lambda \).

Proposition 3. If \( Y \) is f.r. in \( X \), then \( J_t(Y) \leq J_t(X) \) and \( S_t(Y) \geq S_t(X) \) for any \( t \).

Theorem 4. Let \( 1 < p \leq 2 \) and suppose that the \((p,p')\)-Clarkson inequality holds in \( X \).

1. \( J_t(X) = 2^{1-1/t} \) for \( t \geq p' \), and \( S_t(X) = 2^{1-1/t} \) for \( 0 < t \leq p \).

2. If \( \mathcal{E}_p \) (or \( \mathcal{E}_{p'} \)) is finitely representable (f.r.) in \( X \), then \( J_t(X) = 2^{1/p} \) for \( t \leq p' \), and \( S_t(X) = 2^{1/p'} \) for \( t \geq p \).

Corollary 1. (1) \( J_t(H) = \sqrt{2} \) if \( t \leq 2 \), \( J_t(H) = 2^{1-1/t} \) if \( t \geq 2 \), \( S_t(X) = 2^{1-1/t} \) if \( 0 < t \leq 2 \), and \( S_t(X) = \sqrt{2} \) if \( t \geq 2 \), where \( H \) is a Hilbert space.

2. \( J_t(L_p) = 2^{1/r} \) if \( t \leq r' \), \( J_t(L_p) = 2^{1-1/t} \) if \( t \geq r' \), \( S_t(L_p) = 2^{1-1/t} \) if \( 0 < t \leq r \), and \( S_t(L_p) = 2^{1/r} \) if \( t \geq r \), where \( r = \min\{p,p'\} \).

3. If \( X = L_p(L_q) \), and \( r = \min\{p,p',q,q'\} \). Then \( J_t(X) = 2^{1/r} \) if \( t \leq r' \), \( J_t(X) = 2^{1-1/t} \) if \( t > r' \), \( S_t(X) = 2^{1-1/t} \), and \( S_t(X) = 2^{1/r'} \) if \( t \geq r \).
Corollary 2. Let \( X = L_p(L_q) \), \( 1 < p, q < \infty \). Then, \( J(X) = 2^{1/r} \) and \( S(X) = 2^{1/r'} \), where \( r = \min\{p, p', q, q'\} \).

Remark. As already mentioned, for any Banach space \( X \), it holds \( J(X) S(X) = 2 \), \( J_t(X) \to J(X) \) if \( t \to -\infty \), and \( S_t(X) \to S(X) \) if \( t \to +\infty \). By Corollary 1, we know that for various Banach spaces \( X \), \( J_t(X) S_t(X) = 2 \), where \( 1 < t < \infty \) and \( 1/t + 1/t' = 1 \). Note that for any \( t \) \( (1 < t < \infty) \), there is a Banach space \( X \) such that \( J_t(X) S_t(X) \neq 2 \).

Now we give a characterization of a Hilbert space. As mentioned before, if \( X \) is a Hilbert space, then \( J(X) = \sqrt{2} \); but the converse is not true.

Theorem 5. A Banach space \( X \) is isometric to a Hilbert space if and only if \( J_2(X) = \sqrt{2} \).

Remark. Let \( C_{NJ}(X) \) denote the von Neumann-Jordan constant of \( X \) (Clarkson [2]). Then it is easy to see that \( \sqrt{2} \leq J_2(X) \leq \sqrt{2} C_{NJ}(X) \) for any Banach space \( X \). Hence, Theorem 5 generalizes a result of Jordan and von Neumann [6], which asserts that \( X \) is a Hilbert space if and only if \( C_{NJ}(X) = 1 \).

Proposition 6. Let \( X \) be a Banach space. If there is \( t \in [2, \infty) \) such that \( J_t(X) = 2^{1-1/t} \), then \( X \) is uniformly convex.

Remark. For any Banach space \( X \), we have \( J_t(X) \geq 2^{1-1/t} \) for all \( t \geq 2 \) (see, Proposition 1). It can be shown that for any \( \varepsilon > 0 \), there is a Banach space \( X \) which is not uniformly convex such that \( J_t(X) < 2^{1-1/t} + \varepsilon \).

Theorem 7. (1) For any Banach space \( X \), \( J_1(X) = J_1(X^*) \).

(2) For any \( t < 1 \), there is a Banach space \( X \) such that \( J_t(X) \neq J_t(X^*) \).

Corollary 3. \( X \) : uniformly non-square \( \Leftrightarrow X^* \) : uniformly non-square
References


