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Quantum Logical Gate Based on Fock Space
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Abstracts:
In usual computer, there exists a restriction of computational speed because of irreversibility of logical gate. In order to avoid this demerit, Fredkin and Toffoli [3] proposed a conservative logical gate. Based on their work, Milburn [4] introduced a physical model of reversible quantum logical gate using beam splittings and a Kerr medium. This model is called FTM (Fredkin - Toffoli - Milburn gate). FTM gate was described by the quantum channel and the efficiency of information transmission of the FTM gate was discussed in [10]. FTM gate is using a photon number state as an input state for control gate. The photon number state might be difficult to realize physically. In this paper, we introduced a new unitary operator related to the Kerr device on symmetric Fock space in order to avoid this difficulty.

Key words: quantum logical gate, channels, beam splittings, FTM gate, Fock space

1. Quantum channels
Let \((B(H_1), \mathfrak{S}(H_1))\) and \((B(H_2), \mathfrak{S}(H_2))\) be input and output systems, respectively, where \(B(H_k)\) is the set of all bounded linear operators on a separable Hilbert space \(H_k\) and \(\mathfrak{S}(H_k)\) is the set of all density operators on \(H_k\) \((k = 1, 2)\). Quantum channel \(\Lambda^*\) is a mapping from \(\mathfrak{S}(H_1)\) to \(\mathfrak{S}(H_2)\). \(\Lambda^*\) is linear if \(\Lambda^*(\lambda \rho_1 + (1 - \lambda) \rho_2) = \lambda \Lambda^*(\rho_1) + (1 - \lambda) \Lambda^*(\rho_2)\) holds for any \(\rho_1, \rho_2 \in \mathfrak{S}(H_1)\)
and any \( \lambda \in [0,1] \). \( \Lambda^* \) is completely positive (C.P.) if \( \Lambda^* \) is linear and its dual \( \Lambda : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1) \) satisfies

\[
\sum_{i,j=1}^{n} A_i^* \Lambda(A_i^* A_j) A_j \geq 0
\]

for any \( n \in \mathbb{N} \), any \( \{ \overline{A}_i \} \subset \mathcal{B}(\mathcal{H}_2) \) and any \( \{ A_i \} \subset \mathcal{B}(\mathcal{H}_1) \), where the dual map \( \Lambda \) of \( \Lambda^* \) is defined by

\[
\text{tr} \Lambda^*(\rho) B = \text{tr} \rho \Lambda(B), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}_1), \quad \forall B \in \mathcal{B}(\mathcal{H}_2).
\]  

Almost all physical transformation can be described by the CP channel [5], [7], [8]

Let \( K_1 \) and \( K_2 \) be two Hilbert spaces expressing noise and loss systems, respectively. Quantum communication process including the influence of noise and loss is denoted by the following scheme [6]: Let \( \rho \) be an input state in \( \mathfrak{S}(\mathcal{H}_1) \), \( \xi \) be a noise state in \( \mathfrak{S}(K_1) \).

\[
\begin{array}{ccc}
\mathfrak{S}(\mathcal{H}_1) & \xrightarrow{\Lambda^*} & \mathfrak{S}(\mathcal{H}_2) \\
\gamma^* \downarrow & & \uparrow a^* \\
\mathfrak{S}(\mathcal{H}_1 \otimes K_1) & \xrightarrow{\Pi^*} & \mathfrak{S}(\mathcal{H}_2 \otimes K_2)
\end{array}
\]

The above maps \( \gamma^* \), \( a^* \) are given as

\[
\begin{align*}
\gamma^* (\rho) &= \rho \otimes \xi, \quad \rho \in \mathfrak{S}(\mathcal{H}_1), \\
a^* (\sigma) &= \text{tr}_{K_2} \sigma, \quad \sigma \in \mathfrak{S}(\mathcal{H}_2 \otimes K_2).
\end{align*}
\]

The map \( \Pi^* \) is a channel from \( \mathfrak{S}(\mathcal{H}_1 \otimes K_1) \) to \( \mathfrak{S}(\mathcal{H}_2 \otimes K_2) \) determined by physical properties of the device transmitting information. Hence the channel for the above process is given by

\[
\Lambda^*(\rho) \equiv \text{tr}_{K_2} \Pi^* (\rho \otimes \xi) = (a^* \circ \Pi^* \circ \gamma^*) (\rho)
\]

for any \( \rho \in \mathfrak{S}(H_1) \). Based on this scheme, the noisy quantum channel [9] are constructed as follows:

Noisy quantum channel \( \Lambda^* \) with a noise state \( \xi \) is defined by

\[
\Lambda^*(\rho) \equiv \text{tr}_{K_2} \Pi^*(\rho \otimes \xi) = \text{tr}_{K_2} V (\rho \otimes \xi) V^*,
\]
where $\xi = |m_1\rangle\langle m_1|$ is the $m_1$ photon number state in $\mathcal{S}(\mathcal{K}_1)$ and $V$ is a mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ denoted by

$$V(|n_1\rangle \otimes |m_1\rangle) = \sum_j^{n_1+m_1} C_j^{n_1,m_1} |j\rangle \otimes |n_1 + m_1 - j\rangle,$$

where $C_j^{n_1,m_1} = \sum_{r=L}^{K} (-1)^{n_1+j-r} \frac{\sqrt{n_1!m_1!j!(n_1+m_1-j)!}}{r!(n_1-j)!(j-r)!(m_1-j+r)!} \alpha^{m_1-j+2r} (-\overline{\beta})^{n_1+j-2r}$, $K$ and $L$ are constants given by $K = \min\{n_1, j\}$, $L = \max\{m_1-j, 0\}$.

In particular for the coherent input state $\rho = |\theta\rangle \langle \theta| \otimes |\kappa\rangle \langle \kappa| \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$, we obtain the output state of $\Pi^*$ by

$$\Pi^* (|\theta\rangle \langle \theta| \otimes |\kappa\rangle \langle \kappa|) = |\alpha \theta + \beta \kappa\rangle \langle \alpha \theta + \beta \kappa| \otimes | -\overline{\beta} \theta + \alpha \kappa\rangle \langle -\overline{\beta} \theta + \alpha \kappa|,$$

where $\Pi^*$ is called a generalized beam splitting. When the noise $\xi_0 = |0\rangle\langle 0|$ is given by the vacuum state, $\Lambda_0^*$ is called an attenuation channel [5] and $\mathcal{E}_0^*$ (or $\Pi_0^*$) is called a beam splitting. Based on liftings, the beam splitting was studied by Accardi - Ohya [1] and Fichtner - Freudenberg - Libsher [2].

### 2. Quantum logical gate on symmetric Fock space

Recently, we reformulate a quantum channel for the FTM gate and we rigorously study the conservation of information for FTM gate [10]. However, it might be difficult to realize the photon number state $|n\rangle \langle n|$ for the input of the Kerr medium physically.

In this section, we reformulate beam splittings on symmetric Fock space and we introduce a new operator on this space instead of the Kerr medium. We discuss the mathematical formulation of quantum logical gate by means of beam splittings and the new operator.

Let $G$ be a complete separable metric space and $\mathcal{G}$ be a Borel $\sigma$-algebra of $G$. $\nu$ is called a locally finite diffuse measure on the measurable space $(G, \mathcal{G})$ if $\nu$ satisfies the conditions (1) $\nu(K) < \infty$ for bounded $K \in \mathcal{G}$ and (2) $\nu\{x\} = 0$ for any $x \in G$. We denote the set of all finite integer - valued measures $\varphi$ on $(G, \mathcal{G})$ by $M$. For a set $K \in \mathcal{G}$ and a natural number $n \in \mathbb{N}$, we put the set of $\varphi$ satisfying $\varphi(K) = n$ as

$$M_{K,n} \equiv \{\varphi \in M; \varphi(K) = n\}.$$
Let $\mathfrak{M}$ be a $\sigma$-algebra generated by $M_{K,n}$. $F$ is the $\sigma$-finite measure on $(M, \mathfrak{M})$ defined by

$$F(Y) \equiv 1_Y(\varphi_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_M 1_Y(\sum_{j=1}^{n} \delta_{x_j}) v^n(dx_1 \cdots dx_n),$$

where $1_Y$ is the characteristic function of a set $Y$, $\varphi_0$ is an empty configuration in $M$ and $\delta_{x_j}$ is a Dirac measure in $x_j$. $\mathcal{M} \equiv L^2(M, \mathfrak{M}, F)$ is called a (symmetric) Fock space. We define an exponential vector $\exp_g : M \rightarrow \mathbb{C}$ generated by a given function $g : G \rightarrow \mathbb{C}$ such that

$$\exp_g(\varphi) \equiv \begin{cases} 1 & (\varphi = \varphi_0), \\ \prod_{x \in \varphi} g(x) & (\varphi \neq \varphi_0), \\ (\varphi \in M). \end{cases}$$

2.1. Generalized beam splittings on Fock space

Let $\alpha, \beta$ be measurable mappings from $G$ to $\mathbb{C}$ satisfying $\alpha(x) \overline{\beta(x)} + |\alpha(x)|^2 + |\beta(x)|^2 = 1$, $x \in G$. We introduce an unitary operator $V_{\alpha,\beta} : \mathcal{M} \otimes \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{M}$ defined by

$$\Phi \mapsto (V_{\alpha,\beta} \Phi)(\varphi_1, \varphi_2) \equiv \sum_{\hat{\varphi}_1 \leq \varphi_1} \sum_{\hat{\varphi}_2 \leq \varphi_2} \exp_{\alpha}(\hat{\varphi}_1) \exp_{\beta}(\varphi_1 - \hat{\varphi}_1) \exp_{-\overline{\beta}}(\hat{\varphi}_2) \exp_{\overline{\alpha}}(\varphi_2 - \hat{\varphi}_2) \Phi(\hat{\varphi}_1 + \hat{\varphi}_2, \varphi_1 + \varphi_2 - \hat{\varphi}_1 - \hat{\varphi}_2)$$

for $\Phi \in \mathcal{M} \otimes \mathcal{M}$ and $\varphi_1, \varphi_2 \in M$. Let $A \equiv \mathcal{B}(\mathcal{H})$ be the set of all bounded operators on $\mathcal{M}$ and $\mathcal{S}(A)$ be the set of all normal states on $A$. $E_{\alpha,\beta} : A \otimes A \rightarrow A \otimes A$ is the lifting in the sense of Accardi and Ohya [1] and the dual map $E_{\alpha,\beta}^*$ of $E_{\alpha,\beta}$ given by

$$E_{\alpha,\beta}^*(\omega)(\bullet) \equiv \omega(\mathcal{E}_{\alpha,\beta}(\bullet)), \quad \forall \omega \in \mathcal{S}(A \otimes A)$$

is the CP channel from $\mathcal{S}(A \otimes A)$ to $\mathcal{S}(A \otimes A)$. Using the exponential vectors, one can denote a coherent state $\theta^f$ by

$$\theta^f(A) \equiv \langle \exp_f, A \exp_f \rangle e^{-\|f\|^2}, \quad \forall f \in L^2(G, \nu), \forall A \in A.$$
In particular, for the input coherent states $\eta_0 \otimes \omega_0 = \theta^f \otimes \theta^g$, two output states $\omega_1 (\bullet) \equiv \eta_0 \otimes \omega_0 (\mathcal{E}_{\alpha,\beta} ((\bullet) \otimes I))$ and $\eta_1 (\bullet) \equiv \eta_0 \otimes \omega_0 (\mathcal{E}_{\alpha,\beta} (I \otimes (\bullet)))$ are obtained by
\[
\omega_1 = \theta^{\alpha f + \beta g}, \quad \eta_1 = \dot{\theta}^{-\overline{\beta} f + \overline{\alpha} g}.
\]
$\mathcal{E}_{\alpha,\beta}^{*}$ is called a generalized beam splitting on Fock space because it also hold the same properties satisfied by the generated beam splitting $\Pi^{*}$ in Section 1.

Now we introduce a self-adjoint unitary operator $\tilde{U}$, which denotes a new device instead of the Kerr medium, defined by
\[
\tilde{U} (\Phi) (\varphi_1, \varphi_2) \equiv (-1)^{|\varphi_1||\varphi_2|} \Phi (\varphi_1, \varphi_2)
\]
for $\Phi \in \mathcal{M} \otimes \mathcal{M}$ and $\varphi_1, \varphi_2 \in G$, where $|\varphi_k| \equiv \varphi_k (G)$ ($k = 1, 2$). For the input state $\omega_1 \otimes \kappa \equiv \theta^f \otimes \frac{1}{||\psi||^2} \langle \psi, \bullet \psi \rangle$, the output state $\omega_2$ of new device is
\[
\omega_2 (A) \equiv \omega_1 \otimes \kappa \left( \tilde{U} (A \otimes I) \tilde{U} \right) = \frac{1}{||\psi||^2} \int_{M} F (d\varphi) |\psi (\varphi)|^2 \theta^{(-1)^{|\varphi|^2} f} (A)
\]
for any $A \in \mathcal{A}$, $\psi \in \mathcal{M}$ ($\psi \neq 0$) and $f \in L^2 (G, \nu)$. If $\kappa$ is given by the vacuum state $\theta^0$, then the output state $\omega_2$ is equals to $\omega_1$ and if $\kappa$ is given by one particle state, that is, $\kappa = \frac{1}{||\psi||^2} \langle \psi, \bullet \psi \rangle$ with $\psi \mid M_1$ (where $M_1$ is the set of one-particle states), then $\omega_2$ is obtained by $\theta^{-f}$. Let $M_o$ (resp. $M_e$) be the set of $\varphi \in M$ which satisfies that $|\varphi|$ is odd (resp. even) and $M$ be the union of $M_o$ and $M_e$. The output states $\omega_2$ of the new device is written by
\[
\omega_2 (A) = \lambda_1 \theta^{-f} (A) + \lambda_2 \theta^f (A) \quad \forall A \in \mathcal{A},
\]
where $\lambda_1$ and $\lambda_2$ are given by
\[
\begin{cases}
\lambda_1 = \frac{1}{||\psi||^2} \int_{M_o} F (d\varphi) |\psi (\varphi)|^2, \\
\lambda_2 = \frac{1}{||\psi||^2} \int_{M_e} F (d\varphi) |\psi (\varphi)|^2.
\end{cases}
\]
Two output states $\omega_3 (\bullet) \equiv \omega_2 \otimes \eta_2 (\mathcal{E}_{\alpha_2,\beta_2} ((\bullet) \otimes I))$ and $\eta_3 (\bullet) \equiv \omega_2 \otimes \eta_2 (\mathcal{E}_{\alpha_2,\beta_2} (I \otimes (\bullet)))$ of the total logical gate including two beam splittings $\mathcal{E}_{\alpha_k,\beta_k}^*$ with $(|\alpha_k|^2 + |\beta_k|^2 = 1)$ ($k = 1, 2$) and the new device instead of Kerr medium are obtained by
\[
\begin{align*}
\omega_3 &= \lambda_1 \theta^{\alpha_2 (- (\alpha_1 f + \beta_1 g)) + \beta_2 (- \beta_1 f + \alpha_1 g)} + \lambda_2 \theta^{\alpha_2 (\alpha_1 f + \beta_1 g) + \beta_2 (- \beta_1 f + \alpha_1 g)}, \\
\eta_3 &= \lambda_1 \theta^{- \beta_2 (- (\alpha_1 f + \beta_1 g)) + \alpha_2 (- \beta_1 f + \alpha_1 g)} + \lambda_2 \theta^{- \beta_2 (\alpha_1 f + \beta_1 g) + \alpha_2 (- \beta_1 f + \alpha_1 g)},
\end{align*}
\]
where $\omega_2 = \lambda_1 \theta^{-(\alpha_1 f + \beta_1 g)} + \lambda_2 \theta^{\alpha_1 f + \beta_1 g}$ and $\eta_2 = \eta_1 = \theta^{-\overline{\beta}_1 f + \overline{\alpha}_1 g}$.

Based on the above settings, we could show that new logical gate performs the complete truth table. The further development of our study will be appear in [11].

References


