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Quantum Logical Gate Based on Fock Space

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Abstracts:
In usual computer, there exists a restriction of computational speed because of irreversibility of logical gate. In order to avoid this demerit, Fredkin and Toffoli [3] proposed a conservative logical gate. Based on their work, Milburn [4] introduced a physical model of reversible quantum logical gate using beam splittings and a Kerr medium. This model is called FTM (Fredkin - Toffoli - Milburn gate). FTM gate was described by the quantum channel and the efficiency of information transmission of the FTM gate was discussed in [10]. FTM gate is using a photon number state as an input state for control gate. The photon number state might be difficult to realize physically. In this paper, we introduced a new unitary operator related to the Kerr device on symmetric Fock space in order to avoid this difficulty.

Key words: quantum logical gate, channels, beam splittings, FTM gate, Fock space

1. Quantum channels

Let \( (\mathcal{B}(\mathcal{H}_1), \mathfrak{S}(\mathcal{H}_1)) \) and \( (\mathcal{B}(\mathcal{H}_2), \mathfrak{S}(\mathcal{H}_2)) \) be input and output systems, respectively, where \( \mathcal{B}(\mathcal{H}_k) \) is the set of all bounded linear operators on a separable Hilbert space \( \mathcal{H}_k \) and \( \mathfrak{S}(\mathcal{H}_k) \) is the set of all density operators on \( \mathcal{H}_k \) \((k = 1, 2)\). Quantum channel \( \Lambda^* \) is a mapping from \( \mathfrak{S}(\mathcal{H}_1) \) to \( \mathfrak{S}(\mathcal{H}_2) \). \( \Lambda^* \) is linear if \( \Lambda^*(\lambda\rho_1 + (1 - \lambda)\rho_2) = \lambda\Lambda^*(\rho_1) + (1 - \lambda)\Lambda^*(\rho_2) \) holds for any \( \rho_1, \rho_2 \in \mathfrak{S}(\mathcal{H}_1) \).
and any $\lambda \in [0,1]$. $\Lambda^*$ is completely positive (C.P.) if $\Lambda^*$ is linear and its dual $\Lambda : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1)$ satisfies
\[
\sum_{i,j=1}^{n} A_i^* \Lambda(A_i^* A_j) A_j \geq 0
\]
for any $n \in \mathbb{N}$, any $\{ \overline{A}_i \} \subset \mathcal{B}(\mathcal{H}_2)$ and any $\{ A_i \} \subset \mathcal{B}(\mathcal{H}_1)$, where the dual map $\Lambda$ of $\Lambda^*$ is defined by
\[
\mathrm{tr} \Lambda^*(\rho) B = \mathrm{tr} \rho \Lambda(B), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}_1), \quad \forall B \in \mathcal{B}(\mathcal{H}_2). \tag{1.1}
\]

Almost all physical transformation can be described by the CP channel [5], [7], [8]

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Hilbert spaces expressing noise and loss systems, respectively. Quantum communication process including the influence of noise and loss is denoted by the following scheme [6]: Let $\rho$ be an input state in $\mathfrak{S}(\mathcal{H}_1)$, $\xi$ be a noise state in $\mathfrak{S}(\mathcal{K}_1)$.
\[
\begin{array}{ccc}
\mathfrak{S}(\mathcal{H}_1) & \xrightarrow{\Lambda^*} & \mathfrak{S}(\mathcal{H}_2) \\
\gamma^* \downarrow & & \uparrow a^* \\
\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1) & \xrightarrow{\Pi^*} & \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)
\end{array}
\]
The above maps $\gamma^*$, $a^*$ are given as
\[
\gamma^*(\rho) = \rho \otimes \xi, \quad \rho \in \mathfrak{S}(\mathcal{H}_1), \tag{1.2}
\]
\[
a^*(\sigma) = \mathrm{tr}_{\mathcal{K}_2} \sigma, \quad \sigma \in \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2). \tag{1.3}
\]
The map $\Pi^*$ is a channel from $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$ to $\mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$ determined by physical properties of the device transmitting information. Hence the channel for the above process is given by
\[
\Lambda^*(\rho) \equiv \mathrm{tr}_{\mathcal{K}_2} \Pi^* (\rho \otimes \xi) = (a^* \circ \Pi^* \circ \gamma^*) (\rho) \tag{1.4}
\]
for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$. Based on this scheme, the noisy quantum channel [9] are constructed as follows:

Noisy quantum channel $\Lambda^*$ with a noise state $\xi$ is defined by
\[
\Lambda^*(\rho) \equiv \mathrm{tr}_{\mathcal{K}_2} \Pi^*(\rho \otimes \xi) = \mathrm{tr}_{\mathcal{K}_2} V (\rho \otimes \xi) V^*, \tag{1.5}
\]
where $\xi = |m_1\rangle\langle m_1|$ is the $m_1$ photon number state in $\mathfrak{S}(K_1)$ and $V$ is a mapping from $H_1 \otimes K_1$ to $H_2 \otimes K_2$ denoted by

$$V(|n_1\rangle \otimes |m_1\rangle) = \sum_j^{n_1+m_1} C_j^{n_1,m_1} |j\rangle \otimes |n_1 + m_1 - j\rangle,$$

$(1.6)$

$C_j^{n_1,m_1} = \sum_{r=L}^{K} (-1)^{n_1+j-r} \frac{\sqrt{n_1!m_1!j!(n_1+m_1-j)!}}{r!(n_1-j)!(j-r)!(m_1-j+r)!} \alpha^{m_1-j+2r} (-\overline{\beta})^{n_1+j-2r}$

$K$ and $L$ are constants given by $K = \min\{n_1, j\}, L = \max\{m_1-j, 0\}$.

In particular for the coherent input state $\rho = |\theta\rangle \langle \theta| \otimes |\kappa\rangle \langle \kappa| \in \mathfrak{S}(H_1 \otimes K_1)$, we obtain the output state of $\Pi^*$ by

$$\Pi^* (|\theta\rangle \langle \theta| \otimes |\kappa\rangle \langle \kappa|) = |\alpha\theta + \beta\kappa\rangle \langle \alpha\theta + \beta\kappa| \otimes | -\overline{\beta}\theta + \alpha\kappa\rangle \langle -\overline{\beta}\theta + \alpha\kappa|,$$

where $\Pi^*$ is called a generalized beam splitting. When the noise $\xi_0 = |0\rangle\langle 0|$ is given by the vacuum state, $\Lambda_0^*$ is called an attenuation channel [5] and $E_0^*$ (or $\Pi_0^*$) is called a beam splitting. Based on liftings, the beam splitting was studied by Accardi - Ohya [1] and Fichtner - Freudenberg - Libsher [2].

**2. Quantum logical gate on symmetric Fock space**

Recently, we reformulate a quantum channel for the FTM gate and we rigorously study the conservation of information for FTM gate [10]. However, it might be difficult to realize the photon number state $|n\rangle \langle n|$ for the input of the Kerr medium physically.

In this section, we reformulate beam splittings on symmetric Fock space and we introduce a new operator on this space instead of the Kerr medium. We discuss the mathematical formulation of quantum logical gate by means of beam splittings and the new operator.

Let $G$ be a complete separable metric space and $\mathcal{G}$ be a Borel $\sigma$-algebra of $G$. $\nu$ is called a locally finite diffuse measure on the measurable space $(G, \mathcal{G})$ if $\nu$ satisfies the conditions (1) $\nu(K) < \infty$ for bounded $K \in \mathcal{G}$ and (2) $\nu(\{x\}) = 0$ for any $x \in G$. We denote the set of all finite integer-valued measures $\varphi$ on $(G, \mathcal{G})$ by $M$. For a set $K \in \mathcal{G}$ and a natural number $n \in \mathbb{N}$, we put the set of $\varphi$ satisfying $\varphi(K) = n$ as

$$M_{K,n} \equiv \{\varphi \in M; \varphi(K) = n\}.$$
Let \( \mathfrak{M} \) be a \( \sigma \)-algebra generated by \( M_{K,n} \). \( F \) is the \( \sigma \)-finite measure on \( (M, \mathfrak{M}) \) defined by

\[
F(Y) \equiv 1_Y(\varphi_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_M 1_Y(\sum_{j=1}^{n} \delta_{x_j}) v^n(dx_1 \cdots dx_n),
\]

where \( 1_Y \) is the characteristic function of a set \( Y \), \( \varphi_0 \) is an empty configuration in \( M \) and \( \delta_{x_j} \) is a Dirac measure in \( x_j \). \( \mathcal{M} \equiv L^2(M, \mathfrak{M}, F) \) is called a (symmetric) Fock space. We define an exponential vector \( \exp_g \): 

\[
\exp_g(\varphi) \equiv \begin{cases} 1 & (\varphi = \varphi_0), \\ \prod_{x \in \varphi} g(x) & (\varphi \neq \varphi_0), \\ (\varphi \in M) \end{cases}
\]

2.1. Generalized beam splittings on Fock space

Let \( \alpha, \beta \) be measurable mappings from \( G \) to \( \mathbb{C} \) satisfying \( \overline{\alpha} |\alpha(x)|^2 + |\beta(x)|^2 = 1 \), \( x \in G \).

We introduce an unitary operator \( V_{\alpha,\beta} \): \( \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{M} \) defined by

\[
(V_{\alpha,\beta}\Phi)(\varphi_1, \varphi_2) \equiv \sum_{\hat{\varphi}_1 \leq \varphi_1} \sum_{\hat{\varphi}_2 \leq \varphi_2} \exp_{\alpha}(\hat{\varphi}_1) \exp_{\beta}(\varphi_1 - \hat{\varphi}_1) \exp_{-\beta}(\hat{\varphi}_2) \exp_{\overline{\alpha}}(\varphi_2 - \hat{\varphi}_2) \times \Phi(\hat{\varphi}_1 + \hat{\varphi}_2, \varphi_1 + \varphi_2 - \hat{\varphi}_1 - \hat{\varphi}_2)
\]

for \( \Phi \in \mathcal{M} \otimes \mathcal{M} \) and \( \varphi_1, \varphi_2 \in M \). Let \( \mathcal{A} \equiv \mathcal{B}(\mathcal{H}) \) be the set of all bounded operators on \( \mathcal{M} \) and \( \mathfrak{S}(\mathcal{A}) \) be the set of all normal states on \( \mathcal{A} \). \( \mathcal{E}_{\alpha,\beta} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \) is the lifting in the sense of Accardi and Ohya [1] and the dual map \( \mathcal{E}^{*}_{\alpha,\beta} \) of \( \mathcal{E}_{\alpha,\beta} \) given by

\[
\mathcal{E}^{*}_{\alpha,\beta}(\omega)(\bullet) \equiv \omega(\mathcal{E}_{\alpha,\beta}(\bullet)), \quad \forall \omega \in \mathfrak{S}(\mathcal{A} \otimes \mathcal{A})
\]

is the CP channel from \( \mathfrak{S}(\mathcal{A} \otimes \mathcal{A}) \) to \( \mathfrak{S}(\mathcal{A} \otimes \mathcal{A}) \). Using the exponential vectors, one can denote a coherent state \( \theta^f \) by

\[
\theta^f(A) \equiv \langle \exp_f, A \exp_f \rangle e^{-\|f\|^2}, \quad \forall f \in L^2(G, \nu), \quad \forall A \in \mathcal{A}.
\]
In particular, for the input coherent states \( \eta_0 \otimes \omega_0 = \theta^f \otimes \theta^g \), two output states \( \omega_1 (\bullet) = \eta_0 \otimes \omega_0 (\mathcal{E}_{\alpha, \beta}((\bullet) \otimes I)) \) and \( \eta_1 (\bullet) = \eta_0 \otimes \omega_0 (\mathcal{E}_{\alpha, \beta}(I \otimes (\bullet))) \) are obtained by

\[
\omega_1 = \theta^{\alpha f + \beta g}, \quad \eta_1 = \dot{\theta}^{-\overline{\beta}f + \overline{\alpha}g}.
\]

\( \mathcal{E}_{\alpha, \beta}^* \) is called a generalized beam splitting on Fock space because it also hold the same properties satisfied by the generated beam splitting \( \Pi^* \) in Section 1.

Now we introduce a self-adjoint unitary operator \( \tilde{U} \), which denotes a new device instead of the Kerr medium, defined by

\[
\tilde{U} (\Phi)(\varphi_1, \varphi_2) \equiv (-1)^{||\varphi_1|||\varphi_2||} \Phi(\varphi_1, \varphi_2)
\]

for any \( \Phi \in \mathcal{M} \otimes \mathcal{M} \) and \( \varphi_1, \varphi_2 \in \mathcal{A} \), where \( |\varphi_k| \equiv \varphi_k(G) \) \( (k=1,2) \). For the input state \( \omega_1 \otimes \kappa = \theta^f \otimes \frac{1}{||\psi||^2} \langle \psi, \bullet \psi \rangle \), the output state \( \omega_2 \) of new device is

\[
\omega_2(A) = \lambda_1 \theta^{-f}(A) + \lambda_2 \theta^f(A) \quad \forall A \in \mathcal{A},
\]

where \( \lambda_1 \) and \( \lambda_2 \) are given by

\[
\begin{cases}
\lambda_1 = \frac{1}{||\psi||^2} \int_{M_{1}} F(d\varphi) |\psi(\varphi)|^2, \\
\lambda_2 = \frac{1}{||\psi||^2} \int_{M_{2}} F(d\varphi) |\psi(\varphi)|^2.
\end{cases}
\]

Two output states \( \omega_3 (\bullet) = \omega_2 \otimes \eta_2 (\mathcal{E}_{\alpha_k, \beta_k}((\bullet) \otimes I)) \) and \( \eta_3 (\bullet) = \omega_2 \otimes \eta_2 (\mathcal{E}_{\alpha_k, \beta_k}(I \otimes (\bullet))) \) of the total logical gate including two beam splittings \( \mathcal{E}_{\alpha_k, \beta_k}^* \) with \( (|\alpha_k|^2 + |\beta_k|^2 = 1) \) \( (k=1,2) \) and the new device instead of Kerr medium are obtained by

\[
\begin{align*}
\omega_3 &= \lambda_1 \theta^{\alpha_2((-\alpha_1 f + \beta_1 g) + \beta_2(\beta_1 f + \alpha_1 g))} + \lambda_2 \theta^{\alpha_2(\alpha_1 f + \beta_1 g) + \beta_2(\beta_1 f + \alpha_1 g)}, \\
\eta_3 &= \lambda_1 \theta^{-\beta_2((-\alpha_1 f + \beta_1 g) + \beta_2(\beta_1 f + \alpha_1 g)) + \alpha_2(-\beta_1 f + \alpha_1 g))} + \lambda_2 \theta^{-\beta_2(\alpha_1 f + \beta_1 g) + \alpha_2(-\beta_1 f + \alpha_1 g))}.
\end{align*}
\]
where $\omega_2 = \lambda_1 \theta^{-(\alpha_1 f + \beta_1 g)} + \lambda_2 \theta^{\alpha_1 f + \beta_1 g}$ and $\eta_2 = \eta_1 = \theta^{-\overline{\beta}_1 f + \overline{\alpha}_1 g}$.

Based on the above settings, we could show that new logical gate performs the complete truth table. The further development of our study will be appear in [11].

References


