Nonstandard Representations of Unbounded Self-Adjoint Operators

爱知学院大学教養部 山下秀康 (Hideyasu Yamashita)
名古屋大学情報文化学部 小澤正直 (Masanao Ozawa)

1. Introduction

In nonstandard analysis, *standardizations* of internal (or nonstandard) objects have been studied for constructing standard mathematical objects; e.g. an internal measure space is converted into a measure space in the standard sense, called Loeb space ([1][2][3][4]). The standardization of an internal Hilbert space $\mathcal{H}$ is called the *nonstandard hull* of $\mathcal{H}$, written as $\hat{\mathcal{H}}$ (Henson and Moore [5]). Then the standardization of an internal operator $A$ on $\mathcal{H}$ with finite norm is naturally defined on $\mathcal{H}$, In this paper, the standardization of $A$ shall be called the *standard part* of $A$, written as $\hat{A}$. A prominent work of Moore [6] was focused on the case where $\mathcal{H}$ is hyperfinite-dimensional, and studied hyperfinite-dimensional extension of bounded operators on $\hat{\mathcal{H}}$. On the other hand, in the case where the norm of $A$ is not finite, it is not straightforward to give an adequate definition of the standard part of $A$. Albeverio et al. [4] defined $\hat{A}$ only when $\mathcal{H}$ is hyperfinite-dimensional real Hilbert space and $A$ is an internal positive symmetric operator on $\mathcal{H}$.

In this paper, we give a definition of $\hat{A}$ for any internal complex Hilbert space $\mathcal{H}$ and for any internal S-bonded self-adjoint operator $A$ on $\mathcal{H}$, as well as a general consideration on $\hat{A}$ so defined, which suggests the adequacy of the definition.

2. Preliminaries

We work in a $\aleph_1$-saturated nonstandard universe [7]. Note that every nonstandard universe constructed by a bounded ultrapower is $\aleph_1$-saturated.

Let $(V, || \cdot ||)$ be an internal normed linear space. Define the subspaces $\mu(V, || \cdot ||)$ and $\text{fin}(V, || \cdot ||)$ of $V$ by

$$
\mu(V, || \cdot ||) = \{ \xi \in V \mid ||\xi|| \approx 0 \}, \quad \text{fin}(V, || \cdot ||) = \{ \xi \in V \mid ||\xi|| < \infty \}. \tag{1}
$$

We often abbreviate them as $\mu(V)$ and $\text{fin}(V)$. Let $\hat{\xi} = \xi + \mu(V)$ and $\hat{V} = \text{fin}(V)/\mu(V)$, the quotient space. We can naturally define the usual norm $\| \cdot \|$ on $\hat{V}$ by $\|\hat{\xi}\| = \circ \|\xi\|$. A countably infinite sequence $\{\xi_i\}_{i \in \mathbb{N}}$, where $\xi_i \in \text{fin}(V, || \cdot ||)$, *approximately converges to* $\xi \in V$ in the norm $\| \cdot \|$ if

$$
\forall \varepsilon \in \mathbb{R}^+ \exists n \in \mathbb{N} \forall k \in \mathbb{N} \quad [k > n \Rightarrow \|\xi - \xi_i\| < \varepsilon]. \tag{2}
$$
A sequence $\{\xi_i\}_{i\in\mathbb{N}}$ approximately converges to $\xi \in V$ if and only if $\{\hat{\xi}_i\}_{i\in\mathbb{N}}$ converges to $\hat{\xi} \in \hat{V}$. A sequence $\{\xi_i\}_{i\in\mathbb{N}}$, where $\xi_i \in \text{fin}(V, ||\cdot||)$, is $S$-$||\cdot||$-Cauchy if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n \in \mathbb{N} \forall k, l \in \mathbb{N} \ [k, l > n \Rightarrow ||\xi_k - \xi_l|| < \varepsilon].$$

(3)

A sequence $\{\xi_i\}_{i\in\mathbb{N}}$ is $S$-$||\cdot||$-Cauchy if and only if the sequence $\{\hat{\xi}_i\}_{i\in\mathbb{N}}$ is Cauchy.

A subset $X \subset \text{fin}(V, ||\cdot||)$ is $S$-$||\cdot||$-complete if for any $S$-$||\cdot||$-Cauchy sequence $\{\xi_i\}_{i\in\mathbb{N}}$, there exists $\xi \in X$ such that $\{\xi_i\}$ approximately converges to $\xi$ in the norm $||\cdot||$. The subset $X$ is $S$-$||\cdot||$-complete if and only if $\hat{X}$ is complete in $\hat{V}$, where $\hat{X} = \{\hat{\xi} | \xi \in X\}$.

The following results, called the hull completeness theorem, is a fundamental property of an internal normed space $(V, ||\cdot||)$. See Hurd and Loeb [3] for detail.

**Theorem 2.1.** The subspace $\text{fin}(V)$ is $S$-complete in $||\cdot||$.

**Corollary 2.2.** (The Hull Completeness Theorem) $\hat{V}$ is a Banach space.

Let $\mathcal{H}$ be an internal Hilbert space, and $T : \mathcal{H} \rightarrow \mathcal{H}$ an internal bounded linear operator such that the bound $||T||$ is finite. The bounded operator $\hat{T} : \mathcal{H} \rightarrow \mathcal{H}$, called the standard part of $T$, is defined by the relation $\hat{T} \hat{x} = \hat{Tx}$ for any $x \in \text{fin}(\mathcal{H})$.

For further information on nonstandard real analysis, we refer to Stroyan and Luxemburg [3] and Hurd and Loeb [2].

### 3. Several definitions of standard parts

We give several equivalent definitions of the standard part of an internal bounded self-adjoint operator which is not $S$-bounded.

The following lemma, which is a basic property for self-adjointness, is used to give the first definition of standard parts (see [8]).

**Lemma 3.1.** Let $A$ be a symmetric operator on a Hilbert space $\mathcal{H}$. Then, $A$ is self-adjoint if and only if $\text{Rng}(A \pm i) = \mathcal{H}$.

Let $\mathcal{H}$ be an internal Hilbert space, and $A$ an internal bounded self-adjoint operator on $\mathcal{H}$. Let $\hat{\mathcal{K}} = \text{Ker}([(A+i)^{-1}]^\perp)$. Using the unitarity of $(A+i)(A-i)^{-1}$, we can easily check that $\text{Ker}([(A-i)^{-1}]^\perp) = \hat{\mathcal{K}}$.

**Proposition 3.2.** There exists the unique (possibly unbounded) self-adjoint operator $S$ on $\hat{\mathcal{K}}$ satisfying

$$\begin{equation}
(S + i)^{-1} = [(A + i)^{-1}]^\perp |\hat{\mathcal{K}}. \\
\end{equation}$$

(4)

**Proof.** We see $||(A + i)^{-1}|| < \infty$, and $[(A + i)^{-1}]^\perp$ is an bounded normal operator on $\mathcal{H}$. The operator $T := [(A + i)^{-1}]^\perp |\hat{\mathcal{K}}$ is a bijection from $\hat{\mathcal{K}}$ to $[(A + i)^{-1}]^\perp \hat{\mathcal{K}}$. Hence the inverse $T^{-1}$ from $[(A + i)^{-1}]^\perp \hat{\mathcal{K}}$ to $\hat{\mathcal{K}}$ is defined. Clearly the operator $S = T^{-1} - i$ satisfies the equation (4).

We will show that $S$ is symmetric. Let $x_1, x_2 \in \text{Dom}(S)$ $(= [(A + i)^{-1}]^\perp \hat{\mathcal{K})$. Then, we can show that there exist $\xi_i \in x_i$ such that $A\xi_i \in Sx_i$ $(i = 1, 2)$ as follows. There
are \( y_i \in \hat{\mathcal{K}} \) and \( \eta_i \in \mathcal{H} \) such that \( (S + i)^{-1} y_i = [(A + i)^{-1}]^\perp y_i = x_i \) and \( \eta_i \in y_i \). Let \( \zeta_i = (A + i)^{-1} \eta_i \). Then \( \zeta_i \in x_i \) and \( (A + i) \zeta_i = \eta_i \in y_i = (S + i) x_i \). Hence \( A \zeta_i \in S x_i \). Thus, \( \langle x_1, S x_2 \rangle = \langle \xi_1, \zeta_2 \rangle = \langle \zeta_1, \xi_2 \rangle = \langle s_{x_1}, x_2 \rangle \). Therefore, \( S \) is symmetric.

To prove the self-adjointness, it is sufficient to show \( \text{Rng}(S + i) = \text{Rng}(S - i) = \hat{\mathcal{K}} \) by Lemma 3.1. Clearly \( \text{Rng}(S + i) = \text{Rng}(T^{-1}) = \hat{\mathcal{K}} \). Let \( x \in \text{Dom}(S) \), \( \xi \in x \) and \( A \xi \in S x \). Then we have

\[
\frac{A - i}{A + i} (S + i) x = \left( \frac{A - i}{A + i} (A + i) \xi \right)^\perp = (S - i) x.
\]

Thus, by the equation (4) with \( \text{Ker}([(A - i)^{-1}]^\perp) = \hat{\mathcal{K}} \), we have

\[
(S - i)^{-1} = [(A - i)^{-1}]^\perp |\hat{\mathcal{K}}.
\]

Therefore, we can show \( \text{Rng}(S - i) = \hat{\mathcal{K}} \) in the similar way to the proof of \( \text{Rng}(S + i) = \hat{\mathcal{K}} \). The uniqueness of \( S \) is clear. \( \text{QED} \)

**Definition 3.3.** Under the condition of Proposition 3.2, define the self-adjoint operator \( \text{st}_1(A) \) on \( \hat{\mathcal{K}} \) by \( (\text{st}_1(A) + i)^{-1} = [(A + i)^{-1}]^\perp |\hat{\mathcal{K}} \).

The operator \( \text{st}_1(A) \) is called the *standard part* of \( A \). We see that \( \text{st}_1(A) = \hat{A} \) when \( A \) is S-bounded.

**Definition 3.4.** Let \( A \) be an internal bounded operator on \( \mathcal{H} \), an internal Hilbert space. Define \( \text{fin}(A) \subseteq \mathcal{H} \) by

\[
\text{fin}(A) = \{ \xi \in \text{fin} \mathcal{H} | \text{fin} A \xi \in \text{fin} \mathcal{H} \}.
\]

**Definition 3.5.** Let \( A \) be an internal bounded self-adjoint operator on \( \mathcal{H} \). Let \( \hat{\mathcal{K}} \) be the closure of the subspace \( \{ \text{fin}(A)^\perp | (A) \} \) of \( \mathcal{H} \). Define the self-adjoint operator \( \text{st}_2(A) \) on \( \hat{\mathcal{K}} \) by

\[
e^{it \text{st}_2(A)} = e^{it \hat{A}}|\hat{\mathcal{K}}. \quad t \in \mathbb{R}.
\]

We see that \( \{ e^{it \hat{A}} | \hat{\mathcal{K}} \} \) is one-parameter unitary group, since \( \hat{\mathcal{K}} \) is invariant under \( e^{it \hat{A}} \) for all \( t \in \mathbb{R} \). We also see that it is strongly continuous as follows. Let \( \xi \in \text{fin}(A) \). Then, we have \( \| (\star d/dt) e^{it \hat{A}} \xi \| = \| i e^{it \hat{A}} A \xi \| < \infty \), where \( \star d/dt \) is the internal differentiation. This implies that \( e^{it \hat{A}} \xi \) is continuous with respect to \( t \in \mathbb{R} \). Thus, \( e^{it \hat{A}} \) is strongly continuous on \( \text{fin}(A)^\perp \). Hence by Stone's theorem, \( \text{st}_2(A) \) is uniquely defined.

If \( A \) is S-bounded, \( \text{st}_2(A) \) coincides with \( \hat{A} \) defined in Section 2. This is seen from the following:

**Proposition 3.6.** Let \( A \) be an internal S-bounded self-adjoint operator. Then,

\[
e^{it \hat{A}} = e^{it \hat{A}}, \quad t \in \mathbb{R}.
\]
Proof. For any infinitesimal $\epsilon \in \mathbb{R}_0^+$,
\[ \epsilon^{-1}(e^{i\epsilon A} - I) \approx iA, \]  
holds, because
\[ \|\epsilon^{-1}(e^{i\epsilon A} - I) - iA\| = \|\epsilon^{-1} \sum_{\nu=2}^{\infty} (i\epsilon A)^\nu / \nu! \| \leq \epsilon^{-1} \sum_{\nu=2}^{\infty} (\epsilon\|A\|)^\nu / \nu! \]
\[ = \epsilon^{-1}(e^{\epsilon\|A\|} - 1) - \|A\| \approx 0. \]
Thus, by the permanence principle,
\[ \forall \delta \in \mathbb{R}_+, \exists \epsilon \in \mathbb{R}_+, |t| < \epsilon \Rightarrow \|t^{-1}(e^{itA} - I) - iA\| < \delta. \]  
Hence, we have
\[ \lim_{\epsilon \to 0} \|\epsilon^{-1}(e^{i\epsilon A} - I) - iA\| = 0. \]  
Thus we have $(d/dt)e^{i\epsilon A}|_{t=0} = i\hat{A}$, where $d/dt$ is the usual differentiation. Because $(e^{i\epsilon A})_{\epsilon \in \mathbb{R}}$ is one-parameter unitary group, it follows that $e^{i\epsilon A} = e^{i\hat{A}}$. QED

Let $E(\cdot)$ be an internal projection-valued measure on $\mathbb{R}$, i.e., for each internal Borel set $\Omega \subseteq \mathbb{R}$, $E(\Omega)$ is an orthogonal projection on $\mathcal{H}$ such that
1. $E(\phi) = 0, E(\mathbb{R}) = I$
2. If $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \phi$ if $n \neq m$, then $E(\Omega) = \lim_{N \to \infty} \sum_{n=1}^{N} E(\Omega_n)$
3. $E(\Omega_1)E(\Omega_2) = E(\Omega_1 \cap \Omega_2)$

For $r \in \mathbb{R}$, let $\mathcal{H}_r = \text{Rng}(E(-r, r))$, the range of $E((-r, r))$. Let $D(E) = \bigcup_{r \in \mathbb{R}^+} \mathcal{H}_r \cap \text{fin}\mathcal{H}$. $D(E)$ is called the standardization domain of $E(\cdot)$. Clearly, $\overline{D(E)}^{\perp\perp} = (\bigcup_{r \in \mathbb{R}^+} \mathcal{H}_r)^{\perp\perp}$.

For $a \in \mathbb{R}$, define the orthogonal projection $\hat{E}_{st}(-\infty, a]$ by
\[ \hat{E}_{st}(-\infty, a] = \sup \{ \hat{E}(-K, a + \epsilon) | K, \epsilon \in \mathbb{R}^+ \} \]
\[ = \lim_{n \to \infty} \hat{E}(-n, a + \frac{1}{n}) | \hat{D}(E)^{\perp\perp}. \]

Then we see
\[ \lim_{a \to -\infty} \hat{E}_{st}(-\infty, a] = 0 \]
\[ \lim_{\epsilon \to 0} \hat{E}_{st}(-\infty, a + \epsilon] = \hat{E}_{st}(-\infty, a] \]
\[ \quad a < b \Rightarrow \hat{E}_{st}(-\infty, a] \leq \hat{E}_{st}(-\infty, b]. \]
Hence, $\hat{E}_{st}(-\infty, \cdot]$ defines a projection-valued measure on $\mathbb{R}$.

Definition 3.7. For any internal bounded self-adjoint operator $A$, define the self-adjoint operator $\text{st}_3(A)$ on $\overline{D(E)}^{\perp\perp}$ by
\[ \text{st}_3(A) = \int \lambda d\hat{E}_{st}(\lambda). \]
Proposition 3.8. Let $A$ be an internal bounded self-adjoint operator, and $E(\cdot)$ the internal projection-valued measure associated with the spectral decomposition of $A$. Then

$$\hat{D}(E)^{\perp\perp} = \mathrm{fin}(A)^{\perp\perp}.$$  \hfill (19)

Proof. $\hat{D}(E)^{\perp\perp} \subseteq \mathrm{fin}(A)^{\perp\perp}$ is clear. To prove $\hat{D}(E)^{\perp\perp} \supseteq \mathrm{fin}(A)^{\perp\perp}$, it is sufficient to show that for any $\hat{x} \in \mathrm{fin}(A)^{\sim}$ there is a sequence $\hat{x}_n \in \hat{D}(E)$ $(n \in \mathbb{N})$ such that $\hat{x}_n \to \hat{x}$. Let $x_n = E(-n, n)x$ $(n \in \mathbb{N})$. Notice that $\|A(x - x_n)\| \geq n\|x - x_n\|$. Suppose $\|x - x_n\| > \epsilon$ for all $n \in \mathbb{N}$. By the permanence principle, there is $N \in \mathbb{N}_\infty$ such that $\|x - x_N\| > \epsilon$. Hence, $\|A(x - x_n)\| \geq N\|x - x_N\| > N\epsilon \sim \infty$. This contradicts $\|A(x - x_N)\| \leq \|Ax\| < \infty$. \textit{QED}

Theorem 3.9. Let $A$ be an internal bounded self-adjoint operator. Then,

$$\mathrm{st}_2(A) = \int \lambda d\hat{E}_{\mathrm{st}}(\lambda),$$ \hfill (20)

and hence $\mathrm{st}_2(A) = \mathrm{st}_3(A)$.

Proof. It is sufficient to show

$$\langle \hat{x}, \exp(it\mathrm{st}_2(A))\hat{x} \rangle = \int e^{it\lambda}\langle \hat{x}, d\hat{E}_{\mathrm{st}}(\lambda)\hat{x} \rangle$$ \hfill (21)

for all $\hat{x} \in \mathrm{fin}(A)^{\sim \perp \perp}$. Define the internal Borel measure $\mu$ by $\mu(d\lambda) = \langle x, E(d\lambda)\rangle$. Let $L\mu$ denote the Loeb measure of $\mu$, and $L'\mu$ the Borel measure on $\mathbb{R}$ defined by $L'\mu(\Omega) = L\mu(\mathrm{st}^{-1}[\Omega])$. We can check that $L'\mu$ is well-defined (i.e., $\mathrm{st}^{-1}[\Omega]$ is $L\mu$-measurable for any Borel set $\Omega \subseteq \mathbb{R}$). We also see that $L\mu$ is supported by $\mathrm{fin}^\star \mathbb{R}$, since $L\mu(\mathbb{R} \setminus \mathrm{fin}^\star \mathbb{R}) \leq \circ\langle x, E(\mathbb{R} \setminus (-n, n))\rangle = \circ\|1 - E(-n, n)x\|^2 \leq (1/n^2)^\circ\|Ax\|^2$ for all $n \in \mathbb{N}$. Therefore

$$\langle \hat{x}, \exp(it\mathrm{st}_2(A))\hat{x} \rangle = \langle \hat{x}, e^{itA}\hat{x} \rangle = \circ\langle x, e^{itA}x \rangle = \circ\int_{\mathbb{R}} e^{it\lambda}d\mu(\lambda) = \int_{\mathbb{R}} e^{it\lambda}dL\mu(\lambda) = \int_{\mathbb{R}} e^{it\lambda}dL'\mu(\lambda).$$

On the other hand, for $a, b \in \mathbb{R}$ with $a < b$,

$$L'\mu(a, b) = L\mu(\bigcup_{\epsilon \in \mathbb{R}^+} (a + \epsilon, b - \epsilon)) = \lim_{\epsilon \downarrow 0} \circ\langle x, E(a + \epsilon, b - \epsilon)x \rangle = \lim_{\epsilon \downarrow 0} \langle \hat{x}, \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle = \langle \hat{x}, s\lim_{\epsilon \downarrow 0} \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle = \langle \hat{x}, \hat{E}_{\mathrm{st}}(a, b)\hat{x} \rangle.$$

Hence, $L'\mu(\Omega) = \langle \hat{x}, \hat{E}_{\mathrm{st}}(\Omega)\hat{x} \rangle$ for any Borel set $\Omega \subseteq \mathbb{R}$. \textit{QED}
Let \( C \in \mathbb{R} \) be a positive constant, and \( h \) be an internal Borel function from \(*\mathbb{R}\) to \(*\mathbb{C}\) satisfying the following properties:

\[
\begin{align*}
    h(x) &\approx h(y) \quad \text{iff} \quad x \approx y \quad \text{for all } x, y \text{ with } |x|, |y| < \infty, \\
    |h(x)| &< C \quad \text{for all } x \in *\mathbb{R}.
\end{align*}
\]

Define the function \( \hat{h} : \mathbb{R} \to \mathbb{C} \) by

\[
\hat{h}(x) = h(x)
\]

for \( x \in \mathbb{R} \). We see that \( \hat{h} \) is injective and continuous. Let \( A \) be an internal bounded self-adjoint operator. Notice that \( h(A) \) is an \( \mathcal{S} \)-bounded internal normal operator.

**Theorem 3.10.** There exists the unique self-adjoint operator \( B \) on \( \text{fin}(A)^{\perp\perp} \) such that

\[
\hat{h}(B) = h(A)|\text{fin}(A)^{\perp\perp}.
\]

Moreover, \( B \) equals to \( \text{st}_3(A) \).

**Proof.** By the argument similar to the proof of Theorem 3.9, we can show

\[
\langle \hat{x}, h(A)\hat{x} \rangle = \int_{\mathbb{R}} \hat{h}(\lambda)dL'\mu(\lambda) = \int_{\mathbb{R}} \hat{h}(\lambda)\langle \hat{x}, d\hat{E}_{\text{st}}(\lambda)\hat{x} \rangle
\]

for any \( \hat{x} \in \text{fin}(A)^{\perp\perp} \). Thus,

\[
\hat{h}(A)|\text{fin}(A)^{\perp\perp} = \int_{\mathbb{R}} \hat{h}(\lambda)d\hat{E}_{\text{st}}(\lambda).
\]

Because \( \hat{h} \) is injective, the unique self-adjoint operator \( B \) satisfying (22) is \( \text{st}_3(A) = \int_{\mathbb{R}} \lambda d\hat{E}_{\text{st}}(\lambda) \). QED

**Corollary 3.11.** Definition 3.3, 3.5 and 3.7 are equivalent, that is, \( \text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A) \).

**Proof.** Let \( h(x) = 1/(x + i) \). QED

In section 2, \( \hat{A} \) is defined only when \( A \) is an internal S-bounded self-adjoint operator. Now we can extend the definition so as to include the case where \( A \) is an internal bounded self-adjoint operator which is not S-bounded; \( \hat{A} := \text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A) \).

**Definition 3.12.** Let \( A \) be an internal linear operator on an internal Hilbert space \( \mathcal{H} \). Let \( D \) be an (external) subspace of \( \text{fin}\mathcal{H} \). \( A \) is standardizable on \( D \) if \( D \subset \text{fin}(A) \) and if for any \( x, y \in D \), \( x \approx y \) implies \( Ax \approx Ay \). In this case, define the operator \( \hat{A}_D \) with domain \( \{ \hat{x} | x \in D \} \), called the standard part of \( A \) on \( D \), by

\[
\hat{A}_D\hat{x} = \hat{A}x, \quad x \in D.
\]
Clearly, $A$ is standardizable on $D$ if and only if $D \subseteq \operatorname{fin}(A)$, and if $A\xi \approx 0$ for all $\xi \in D$ with $\xi \approx 0$.

**Lemma 3.13.** An internal bounded operator $A$ is standardizable on $\operatorname{fin}(A^*A)$.

**Proof.** First, we prove $\operatorname{fin}(A^*A) \subseteq \operatorname{fin}(A)$ as follows. Suppose that $\xi \in \operatorname{fin}(A)$. Let $E(\cdot)$ be the internal spectral-valued measure of the self-adjoint operator $A^*A$. Then, $\|A\xi\| = \langle \xi, A^*A\xi \rangle = \langle \xi, E[0,1]A^*A\xi \rangle + \langle \xi, (I - E[0,1])(A^*A)^2\xi \rangle \leq \langle \xi, E[0,1]A^*A\xi \rangle + \|A^*A\xi\|^2 < \infty$. Thus, $\xi \in \operatorname{fin}(A)$. Second, suppose $x \approx 0$ and $\|A^*Ax\| < \infty$. Then, $\|Ax\|^2 = \langle x, A^*Ax \rangle \leq \|x\|\|A^*Ax\| \approx 0$. QED

**Corollary 3.14.** If $D \subseteq \operatorname{fin}\mathcal{H}$ is invariant under $A$ and $A^*$, $A$ is standardizable on $D$.

The operator $B$ in the above proof is called a hyperfinite extension of $A$ [6]. We use the following lemma in the proof of Theorem 3.16.

**Lemma 3.15.** Let $A$ be a symmetric operator with domain $D \subseteq \mathcal{H}$, a Hilbert space. Let $D_1 \subseteq D$ be a dense linear subset of $\mathcal{H}$ and suppose that $A|D_1$ is essentially self-adjoint. Then, $A$ is essentially self-adjoint and $\overline{A} = A|D_1$.

**Theorem 3.16.** Let $A$ be an internal self-adjoint operator on $\mathcal{H}$, and $E(\cdot)$ the projector-valued spectral measure of $A$. Then,

$$\hat{A} = \overline{A}_{D(E)} = \overline{A}_{\operatorname{fin}(A^2)}$$

(24)

**Proof.** We can show that $\hat{A}_{D(E)}$ is essentially self-adjoint e.g. by Nelson’s analytic vector theorem. Hence, it has one and only one self-adjoint extension, its closure. Thus, it is sufficient to show that $\hat{A}$ is an extension of $\hat{A}_{D(E)}$. If $E(-r,r)\xi = \xi$ ($r \in \mathbb{R}^+$, $\xi \in \mathcal{H}$), then $E_{\text{st}}(-s,s)\xi = \hat{\xi}$ ($s \in \mathbb{R}^+$, $s > r$). Thus, $\hat{A}_{D}\xi = \hat{A} \xi = \{ \int_{-s}^s \lambda dE_{\text{st}}(\lambda) \}^{-\hat{\xi}} = \int_{-r}^r \lambda dE_{\text{st}}(\lambda) \hat{\xi} = \text{st}_3(A) = \hat{A}\xi$. Therefore $\hat{A} = \hat{A}_{D(E)}$. $\hat{A}_{D(E)} = \hat{A}_{\operatorname{fin}(A^2)}$ follows from $D(E) \subseteq \operatorname{fin}(A^2)$ and Lemma 3.15. QED

4. The domain of $\hat{A}$

**Definition 4.1.** For an internal bounded self-adjoint operator $A$ on $\mathcal{H}$, define $D(A)$ by

$$D(A) = \{ \xi \in \operatorname{fin}\mathcal{H} \mid \text{for all } t \in \mathbb{R}_0^+, e^{-t\|A\|}A\xi \approx A\xi \in \operatorname{fin}\mathcal{H} \}.$$  

Clearly, $D(A)$ is a subspace of $\mathcal{H}$.

**Proposition 4.2.** An internal bounded self-adjoint operator $A$ is standardizable on $D(A)$.
Proof. Let \( \xi \in D(A) \) and \( \| \xi \| \approx 0 \). We can easily check \( \| e^{-t|A|} A \xi \| < \infty \) for all \( t > 0, t \neq 0 \). Hence, \( \circ ||A\xi|| \leq \circ \| e^{-t|A|} A \xi \| + \circ \| (1 - e^{-t|A|}) A \xi \| \). By the S-boundedness of \( e^{-t|A|} A \), the first term equals 0, and by the definition of \( D(A) \), the second term equals 0. Thus we have \( \circ \| A\xi \| = 0 \). QED

The following lemmas are easily shown.

Lemma 4.3. Let \( f : *N \rightarrow *R^+ \) be internal and increasing. If \( f(M) < \infty \) for some \( M \approx \infty \), then
\[
\lim_{n \rightarrow \infty} \circ f(n) < \infty.
\]

Lemma 4.4. Under the same condition to Lemma 4.3, there is \( K \approx \infty \) such that for all \( L \approx \infty \),
\[
f(K) \approx f(L) \text{ if } L \leq K.
\]

Proposition 4.5. Let \( \xi \in \text{fin}(\mathcal{H}) \). For sufficiently large \( t \approx 0 \),
\[
e^{-t|A|} \xi \in D(A).
\]

Proof. Applying Lemma 4.4 to \( f(n) = \| e^{-|A|/n} A \xi \| \), we find that for sufficiently small \( K \approx \infty \) and \( L \approx \infty \), \( e^{-|A|/K} A \xi \approx e^{-|A|/L} A \xi \). Thus, for sufficiently large \( s \approx 0 \) and \( t \approx 0 \), \( e^{-|A|} A \xi \approx e^{-|A|} A \xi \). Hence for all \( x \approx 0, x > 0 \),
\[
e^{-|A|} A e^{-|A|} \xi = e^{-(x+t)|A|} A \xi \approx A e^{-|A|} \xi.
\]
Therefore, \( e^{-|A|} \xi \in D(A) \). QED

Theorem 4.6. Let \( E(\cdot) \) be the spectral resolution of \( A \) and \( E_K = E(-K, K) \) for \( K \in *R^+ \). For any \( \xi \in \text{fin}(A) \),
\[
\xi \in D(A) \text{ iff } A \xi \approx E_K A \xi \text{ for all } K \approx \infty.
\]

Remark. The right-hand condition is equivalent to
\[
\lim_{K \rightarrow \infty} \circ \| (I - E_K) A \xi \| = 0.
\]

Proof. Suppose that \( \xi \in \text{fin}(A) \) and \( A(I - E_K) \xi \approx 0 \) for all \( K \approx \infty \). For any \( t \approx 0 \), there exists a \( K \approx \infty \) such that \( tK \approx 0 \). Thus,
\[
\| e^{-|A|} A \xi - A \xi \|^2 \approx \| e^{-|A|} E_K A \xi - E_K A \xi \|^2
\]
\[
= \| \int_{-K}^{K} e^{-t|A|} \lambda - \lambda dE(\lambda) \xi \|^2
\]
\[
= \int_{-K}^{K} |(e^{-t|A|} - 1)\lambda|^2 \| dE(\lambda) \xi \|^2
\]
\[
\leq \sup_{|\lambda| < K} |e^{-t|A|} - 1|^2 \int_{-K}^{K} \lambda^2 \| dE(\lambda) \xi \|^2
\]
\[
= \sup_{|\lambda| < K} |e^{-t|A|} - 1|^2 \| E_K A \xi \|^2
\]
\[
\approx 0.
\]
Hence $\xi \in D(A)$.

Conversely, suppose $\xi \in D(A)$ ($\subset \text{fin}(A)$). Applying Lemma 4.4 to $f(n) = ||E_n A\xi||$, we see that for sufficiently small $K \sim \infty$ and $L \sim \infty$ ($L \leq K$),

$$||E_L A\xi|| \approx ||E_K A\xi||.$$ 

Thus, $(E_K - E_L)A\xi \approx 0$, since $||E_L A\xi - E_K A\xi||^2 = ||E_K A\xi||^2 - ||E_L A\xi||^2 \approx 0$. Let $t \in \mathbb{R}_0^+$ satisfy $tK \sim \infty$ so that

$$||E_K A\xi - e^{-t|A|} A\xi||$$

$$= ||\int_{-K}^{K} \lambda(1 - e^{-t|\lambda|}) dE(\lambda)\xi - \int_{(-\infty,-K) \cup (K, \infty)} e^{-t|\lambda|} \lambda dE(\lambda)\xi||$$

$$\leq ||\int_{-K}^{K} \lambda(1 - e^{-t|\lambda|}) dE(\lambda)\xi|| + e^{-tK} ||A\xi||$$

$$\approx ||\int_{-K}^{K} \lambda(1 - e^{-t|\lambda|}) dE(\lambda)\xi||.$$ 

Let $L \sim \infty$ satisfy $tL \approx 0$, so that the above

$$\leq ||\int_{-L}^{K} \lambda(1 - e^{-t|\lambda|}) dE(\lambda)\xi|| + ||(E_K - E_L)A\xi||$$

$$\approx 0.$$ 

Thus, for sufficiently small $K \sim \infty$ and for any $t \approx 0$ such that $tK \sim \infty$,

$$E_K A\xi \approx e^{-t|A|} A\xi \approx A\xi.$$ 

Since $||A\xi - E_K A\xi|| \geq ||A\xi - E_{K'} A\xi|| > 0$ if $K < K'$, we have $E_{K'} A\xi \approx A\xi$ holds for any $K' \sim \infty$. QED

**Proposition 4.7.** Let $\xi \in \text{fin}(A)$. Then, $E_K \xi \in D(A)$ for sufficiently small $K \sim \infty$.

**Proof.** Applying Lemma 4.4 to $f(n) = ||E_n A\xi||$, we find that for sufficiently small $K, L \sim \infty$, $E_K A\xi \approx E_L A\xi$. Thus, if $L \sim \infty$, $L \leq K$, then $||(1 - E_L)E_K A\xi|| = ||(E_K - E_L)A\xi|| \approx 0$. If $L > K$, clearly $(1 - E_L)E_K A\xi = 0$. Hence for all $L \sim \infty$, $E_K A\xi \approx E_L E_K A\xi$. Thus $E_K \xi \in D(A)$ by Theorem 4.6. QED

**Corollary 4.8.** $[\text{fin}(A)]^\sim = [D(A)]^\sim$, i.e., if $\xi \in \text{fin}(A)$, then there is $\eta \in D(A)$ such that $\eta \approx \xi$.

**Example** We have seen that the following relations hold:

$$\text{fin}(A^2) \subset D(A) \subset \text{fin}(A) \subset \text{fin}\mathcal{H},$$

$$[\text{fin}(A^2)]^\sim \subset [D(A)]^\sim = [\text{fin}(A)]^\sim \subset \mathcal{H},$$

$$\text{fin}(A^2)^\sim \subset [D(A)]^\sim = [\text{fin}(A)]^\sim \subset \mathcal{H}.$$
\[ [\text{fin}(A^2)]^{-\perp} = [D(A)]^{-\perp} = [\text{fin}(A)]^{-\perp} \subset \mathcal{H}. \]

An example of $A$ such that $\text{fin}(A) \setminus D(A) \neq \emptyset$ is given as follows. Let $\nu$ be an infinite hypernatural number, and $\mathcal{H} = \star \mathbb{C}^\nu$, $\nu$-dimensional internal Hilbert space. Define the internal self-adjoint operator $A$ on $\mathcal{H}$ by $A(x_1, x_2, \ldots, x_\nu) = (x_1, 2x_2, \ldots, \nu x_\nu)$. Let $\xi = (0, 0, \ldots, 0, \nu^{-1})$. Then we see $\xi \in \text{fin}(A) \setminus D(A)$ from Theorem 4.6.

We also find $D(A) \setminus \text{fin}(A^2) \neq \emptyset$; let $\eta = (1^{-2}, 2^{-2}, \ldots, \nu^{-2})$, then we easily see $\eta \in D(A) \setminus \text{fin}(A^2)$. Moreover we find $\hat{\eta} \in [D(A)]^\wedge \setminus [\text{fin}(A^2)]^{- \perp}$. In fact, if $\eta' \approx \eta$, then $^\circ ||A^2\eta'|| \geq \lim_{n \to \infty} n \in \mathbb{N}^\star ||A^2E_n\eta'|| = \lim_{n \to \infty} \sqrt{n} = \infty$. Thus, we have $\hat{\eta} \not\in [\text{fin}(A^2)]^{- \perp}$ by Theorem 4.6.

**Theorem 4.9.** Let $\xi \in \text{fin}(A)$, then

$$\xi \in D(A) \iff \lim_{t \to 0, t \neq 0} \left( \frac{e^{-t|A|} - 1}{t} \xi \right) = -|A|\xi. \quad (28)$$

*Proof.* Suppose that the right-hand side does not hold. In other words, suppose that

$$\exists \epsilon \in \mathbb{R}^\star \forall n \in \mathbb{N} \exists t \in \star \mathbb{R}, \ 0 < t < \frac{1}{n} \land \left\| \left( \frac{e^{-t|A|} - 1}{t} + |A|I \right) \xi \right\| > \epsilon. \quad (29)$$

By permanence,

$$\exists \epsilon \in \mathbb{R}^\star \exists N \in \mathbb{N}^\star \exists t \in \star \mathbb{R}, \ 0 < t < \frac{1}{n} \land \left\| \left( \frac{e^{-t|A|} - 1}{t} + |A| \right) \xi \right\| > \epsilon. \quad (30)$$

That is, there is positive infinitesimal $t$ such that $t^{-1}(e^{-t|A|} - 1)\xi \not\simeq -|A|\xi$.

Thus, for some $\eta \in \text{fin}(\mathcal{H})$,

$$\Re \left( \eta, \frac{e^{-t|A|} - 1}{t} \xi \right) \not\simeq \Re(\eta, -|A|\xi).$$

Let $f(t) = \Re(\eta, e^{-t|A|}\xi)$. By the mean value theorem, for some $s \in \star \mathbb{R}$ with $0 < s < t$,

$$f'(s) = \frac{f(t) - f(0)}{t} = \Re \left( \eta, \frac{e^{-t|A|} - 1}{t} \xi \right) \not\simeq \Re(\eta, -|A|\xi). \quad (31)$$

Therefore, by the definition of $D(A)$, we have $\xi \in \text{fin}(A) \setminus D(A)$.

Conversely, suppose $\xi \in \text{fin}(A) \setminus D(A)$. Then, there is positive infinitesimal $t_0$ satisfying $e^{-t_0|A|}A\xi \not\simeq |A|\xi$. Let $\eta = (|A| - e^{-t_0|A|}|A|)\xi (\in \text{fin}(\mathcal{H}))$. Then this is equivalent to

$$\langle \eta, e^{-t_0|A|}A|A|\xi \rangle \not\simeq \langle \eta, |A|\xi \rangle. \quad (31)$$

Let $f(t) = \langle \eta, e^{-t|A|}\xi \rangle (x \in \star \mathbb{R}^+)$. We see that $f'$ is increasing and $-\infty < f' < 0$, and hence $f$ is decreasing and $0 < f < \infty$. The relation (31) is equivalent to

$$f'(t_0) \not\simeq f'(0), \quad (32)$$

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We have \( f(x) \geq f'(t_0)(x - t_0) + f(t_0) \). Thus we have

\[
0 > \frac{f(x) - f(0)}{x} \geq \frac{f'(t_0)(x - t_0) + f(t_0) - f(0)}{x}.
\]  

(33)

Let \( F(x) = [f'(t_0)(x - t_0) + f(t_0) - f(0)]/x \), then for \( c \in \mathbb{R}^+ \),

\[
F(ct_0) = f'(t_0) \left( 1 - \frac{1}{c} \right) + \frac{1}{c} \frac{f(t_0) - f(0)}{t_0}.
\]  

(34)

By the mean value theorem and \(-\infty < f'(x) < 0\), we have \(|f(x) - f(0)|/x| < \infty\). Hence \( F(ct_0) \approx f'(t_0) \) for all \( c \sim \infty \). Thus, by (32) and (33),

\[
0 > \frac{f(ct_0) - f(0)}{ct_0} \geq F(ct_0) > f'(0),
\]  

(35)

for all \( c \sim \infty \). Thus there is \( \varepsilon \in \mathbb{R}^+ \) such that for sufficiently large \( x \approx 0 \), \( f(x)/x - f'(0) > \varepsilon \). By the permanence principle, for sufficiently small \( x \in \mathbb{R}^+ \), \( f(x)/x - f'(0) > \varepsilon \). We can check the relations

\[
\langle \eta, \left( e^{-x|A|} - 1 \right) \xi \rangle = \frac{f(x) - f(0)}{x}, \quad \langle \eta, |A|\xi \rangle = -f'(0), \quad \frac{e^{-x|A|} - 1}{x} > -|A|,
\]

for \( x > 0 \). Therefore, using the increasingness of \((e^{-x\|A\|^{-1}})/x, x, \) we have

\[
\lim_{x \downarrow 0, x \neq 0}^\circ \langle \eta, \left( e^{-x|A|} - 1 \right) \xi \rangle \neq \langle \eta, |A|\xi \rangle.
\]

QED

**Theorem 4.10.** Let \( A \) be an internal bounded self-adjoint operator. Then, \( \hat{A} = \hat{A}_{D(A)} \).

**Proof.** By Theorem 3.6 and Lemma 3.15, it suffices to show that \( \hat{A}_{D(A)} \) is a closed extension of \( \hat{A}_{\text{fin}(A^2)} \). If \( \xi \in \text{fin}(A^2) \), for any \( K \sim \infty \), \( \| (1 - E_K)A\xi \| \leq \frac{1}{K} \| (1 - E_K)A^2\xi \| \leq \frac{1}{K} \| A^2\xi \| \approx 0 \). Hence \( \xi \in D(A) \), and hence \( \hat{A}_{D(A)} \) is an extension of \( \hat{A}_{\text{fin}(A^2)} \).

To prove that \( \hat{A}_{D(A)} \) is closed, it suffices to show that \( \hat{D}(A) = [D(A)]^\circ \) is complete in the norm \( \| \cdot \|_A \) defined by \( \| \xi \|_A = \| \xi \| + \| A\xi \| \). Define the internal norm \( \| \cdot \|_A \) on \( \mathcal{H} \) by \( \| \xi \|_A = \| \xi \| + \| A\xi \| \). We can check \( \| \xi \|_A = \circ \| \xi \|_A \) for \( \xi \in D(A) \).

By Theorem 2.1, \( \text{fin}(A) \) is \( \text{S-}\| \cdot \|_A \)-complete. Hence, if the sequence \( \{ \xi_i \}_{i \in \mathbb{N}} \subset D(A) \) \( \subset \text{fin}(A) \) is \( \text{S-}\| \cdot \|_A \)-Cauchy, then there is \( \xi \in \text{fin}(A) \) such that \( \{ \xi_i \} \) approximately converges to \( \xi \) in the norm \( \| \cdot \|_A \). This \( \xi \) is shown to be in \( D(A) \) as follows. Regarding Theorem 4.6, and \( \xi_i \in D(A) \) \( (i < \infty) \), this relation leads to \( \circ \| (I - E_K)A\xi_i \| = \lim_{i \rightarrow \infty} \circ \| (I - E_K)A\xi_i \| = 0 \), for any \( K \sim \infty \). Therefore, from Theorem 4.6, we have \( \xi \in D(A) \) and hence any Cauchy sequence in \( \hat{D}(A) \) converges in \( \hat{D}(A) \) in the norm \( \| \cdot \|_A \). QED
Theorem 4.11. The domain $D(A)$ is maximal. That is, if $D(A) \subset S \subset \text{fin} (\mathcal{H})$ and $A$ is standardizable on $S$, then $S = D(A)$.

Proof. Suppose that $D(A) \subset S \subset \text{fin} (\mathcal{H})$ and that $A$ is standardizable on $S$. Let $\eta \in S$. By Corollary 4.8 and $\eta \in \text{fin}(A)$, there is $\xi \in D(A)$ such that $\xi \approx \eta$. By the definition of $D(A)$ and the standardizability on $S$, for all positive infinitesimal $t$, $e^{-t|A|}A\eta \approx e^{-t|A|}A\xi \approx A\xi \approx A\eta$, since $||e^{-t|A||}|| \leq 1$. Thus, $\eta \in D(A)$. QED

Proposition 4.12. Let $A$ be an internal positive operator on $\mathcal{H}$. Then, for any $\eta \in \text{fin}(A^{\frac{1}{2}})$,

$$\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \inf_{\alpha \sim \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle. \quad (36)$$

Proof. Suppose $\eta \approx \xi$. If $\alpha < \infty$, $\langle \eta, E_{\alpha}A\eta \rangle \approx \langle \xi, E_{\alpha}A\xi \rangle \leq \langle \xi, A\xi \rangle$, that is,

$$\forall \in \mathbb{R}^{+}, \forall \alpha < \infty, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \epsilon,$$

Thus, by the permanence principle,

$$\forall \in \mathbb{R}^{+}, \exists K \sim \infty, \forall \alpha \leq K, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \epsilon.$$

By saturation,

$$\exists K \sim \infty, \forall \in \mathbb{R}^{+}, \forall \alpha \leq K, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \epsilon.$$

Hence we have

$$\exists K \sim \infty, \quad \inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \leq \inf_{\alpha \sim \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle.$$

It follows that $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \geq \inf_{\alpha \sim \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle$.

On the other hand, we see that for all $\alpha \sim \infty$, $||\eta - E_{\alpha}\eta||^{2} \leq \alpha^{-1}||A^{\frac{1}{2}}(\eta - E_{\alpha}\eta)||^{2} \leq \alpha^{-1}||A^{\frac{1}{2}}\eta||^{2} \approx 0$. Hence,

$$\forall \alpha \sim \infty, \inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \leq \circ \langle E_{\alpha}\eta, AE_{\alpha}\eta \rangle = \circ \langle \eta, E_{\alpha}A\eta \rangle.$$

Thus it follows that $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \leq \inf_{\alpha \sim \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle$. QED

Proposition 4.13. Let $A$ be an internal positive operator and $\eta \in \text{fin}(A)$. Then,

$$\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \langle \hat{\eta}, \hat{A}\hat{\eta} \rangle. \quad (37)$$

Proof. From Proposition 4.12, we see $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \inf_{\alpha \sim \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle$. By Theorem 4.10 and Proposition 4.7, for sufficiently small $\alpha \sim \infty$, $\circ \langle \eta, E_{\alpha}A\eta \rangle = \circ \langle E_{\alpha}\eta, AE_{\alpha}\eta \rangle = \langle \hat{E}_{\alpha}\eta, \hat{A}E_{\alpha}\eta \rangle = \langle \hat{\eta}, \hat{A}\hat{\eta} \rangle$. QED

Definition 4.14. Let $A$ be a internal bounded positive operator, and $D \subset \text{fin}(A^{\frac{1}{2}})$. The sesquilinear form $\langle \cdot, A\cdot \rangle$ is standardizable on $D$ if $\langle \xi_{1}, A\eta_{1} \rangle \approx \langle \xi_{2}, A\eta_{2} \rangle$ for all $\xi_{1}, \xi_{2}, \eta_{1}, \eta_{2} \in D$ with $\xi_{1} \approx \xi_{2}$ and $\eta_{1} \approx \eta_{2}$.
Proposition 4.15. Let $D$ be a subspace of fin($\mathcal{H}$) and $A \geq 0$. Then, $\langle \cdot, A \cdot \rangle$ is standardizable on $D$ if and only if $A^{\frac{1}{2}}$ is standardizable on $D$.

Proof. Suppose that $A^{\frac{1}{2}}$ is standardizable on $D$. Then $A^{\frac{1}{2}}\xi \approx A^{\frac{1}{2}}\eta$ for any $\xi, \eta \in D$ with $\xi \approx \eta$. Thus, $\langle \xi, A\xi \rangle = ||A^{\frac{1}{2}}\xi||^2 \approx ||A^{\frac{1}{2}}\eta||^2 = \langle \eta, A\eta \rangle$. Conversely, suppose that $\langle \cdot, A \cdot \rangle$ is standardizable on $D$. Then for any $\xi, \eta \in D$ with $\xi \approx \eta$, $||A^{\frac{1}{2}}\xi - A^{\frac{1}{2}}\eta||^2 = ||A^{\frac{1}{2}}(\xi - \eta)||^2 = \langle \xi - \eta, A(\xi - \eta) \rangle \approx 0$. QED

Corollary 4.16. The set $D(A^{\frac{1}{2}})$ is a maximal domain of $\langle \cdot, A \cdot \rangle$, and $\circ \langle \xi, A\eta \rangle = \langle A^{\frac{1}{2}}\xi, A^{\frac{1}{2}}\eta \rangle$ for any $\xi, \eta \in D(A^{\frac{1}{2}})$.

References