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Kyoto University
Nonstandard Representations of Unbounded Self-Adjoint Operators

1. Introduction

In nonstandard analysis, standardizations of internal (or nonstandard) objects have been studied for constructing standard mathematical objects; e.g. an internal measure space is converted into a measure space in the standard sense, called Loeb space ([1][2][3][4]). The standardization of an internal Hilbert space $\mathcal{H}$ is called the nonstandard hull of $\mathcal{H}$, written as $\hat{\mathcal{H}}$ (Henson and Moore [5]). Then the standardization of an internal operator $A$ on $\mathcal{H}$ with finite norm is naturally defined on $\hat{\mathcal{H}}$. In this paper, the standardization of $A$ shall be called the standard part of $A$, written as $\hat{A}$. A prominent work of Moore [6] was focused on the case where $\mathcal{H}$ is hyperfinite-dimensional, and studied hyperfinite-dimensional extension of bounded operators on $\hat{\mathcal{H}}$. On the other hand, in the case where the norm of $A$ is not finite, it is not straightforward to give an adequate definition of the standard part of $A$. Albeverio et al. [4] defined $\hat{A}$ only when $\mathcal{H}$ is hyperfinite-dimensional real Hilbert space and $A$ is an internal positive symmetric operator on $\mathcal{H}$.

In this paper, we give a definition of $\hat{A}$ for any internal complex Hilbert space $\mathcal{H}$ and for any internal S-bonded self-adjoint operator $A$ on $\mathcal{H}$, as well as a general consideration on $\hat{A}$ so defined, which suggests the adequacy of the definition.

2. Preliminaries

We work in a $\aleph_1$-saturated nonstandard universe [7]. Note that every nonstandard universe constructed by a bounded ultrapower is $\aleph_1$-saturated.

Let $(V,|| \cdot ||)$ be an internal normed linear space. Define the subspaces $\mu(V,|| \cdot ||)$ and $\text{fin}(V,|| \cdot ||)$ of $V$ by

$$\mu(V,|| \cdot ||) = \{ \xi \in V \mid ||\xi|| \approx 0 \}, \quad \text{fin}(V,|| \cdot ||) = \{ \xi \in V \mid ||\xi|| < \infty \}. \quad (1)$$

We often abbreviate them as $\mu(V)$ and $\text{fin}(V)$. Let $\hat{\xi} = \xi + \mu(V)$ and $\hat{V} = \text{fin}(V)/\mu(V)$, the quotient space. We can naturally define the usual norm $\| \cdot \|$ on $\hat{V}$ by $\|\hat{\xi}\| = ^0\|\xi\|$. A countably infinite sequence $\{\xi_i\}_{i \in \mathbb{N}}$, where $\xi_i \in \text{fin}(V,|| \cdot ||)$, approximately converges to $\xi \in V$ in the norm $\| \cdot \|$ if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n \in \mathbb{N} \forall k \in \mathbb{N} \ [k > n \Rightarrow \|\xi - \xi_i\| < \varepsilon]. \quad (2)$$
A sequence \(\{\xi_i\}_{i\in \mathbb{N}}\) approximately converges to \(\xi \in V\) if and only if \(\{\hat{\xi}_i\}_{i\in \mathbb{N}}\) converges to \(\hat{\xi} \in \hat{V}\). A sequence \(\{\xi_i\}_{i\in \mathbb{N}}\), where \(\xi_i \in \text{fin}(V, \| \cdot \|)\), is \(S\)-\(\| \cdot \|\)-Cauchy if
\[
\forall \epsilon \in \mathbb{R}^+ \exists n \in \mathbb{N} \forall k, l \in \mathbb{N} \ [k, l > n \Rightarrow \|\xi_k - \xi_l\| < \epsilon].
\] (3)

A sequence \(\{\xi_i\}_{i\in \mathbb{N}}\) is \(S\)-\(\| \cdot \|\)-Cauchy if and only if the sequence \(\{\hat{\xi}_i\}_{i\in \mathbb{N}}\) is Cauchy.

A subset \(X \subset \text{fin}(V, \| \cdot \|)\) is \(S\)-\(\| \cdot \|\)-complete if for any \(S\)-\(\| \cdot \|\)-Cauchy sequence \(\{\xi_i\}_{i\in \mathbb{N}}\), there exists \(\xi \in X\) such that \(\{\xi_i\}\) approximately converges to \(\xi\) in the norm \(\| \cdot \|\). The subset \(X\) is \(S\)-\(\| \cdot \|\)-complete if and only if \(\hat{X}\) is complete in \(\hat{V}\), where \(\hat{X} = \{\hat{\xi}|\xi \in X\}\).

The following results, called the hull completeness theorem, is a fundamental property of an internal normed space \((V, \| \cdot \|)\). See Hurd and Loeb [3] for detail.

**Theorem 2.1.** The subspace \(\text{fin}(V)\) is \(S\)-complete in \(\| \cdot \|\).

**Corollary 2.2.** (The Hull Completeness Theorem) \(\hat{V}\) is a Banach space.

Let \(\mathcal{H}\) be an internal Hilbert space, and \(T : \mathcal{H} \to \mathcal{H}\) an internal bounded linear operator such that the bound \(\|T\|\) is finite. The bounded operator \(\hat{T} : \mathcal{H} \to \hat{\mathcal{H}}\), called the standard part of \(T\), is defined by the relation \(\hat{T}\hat{x} = \hat{T}\hat{x}\) for any \(x \in \text{fin}(\mathcal{H})\).

For further information on nonstandard real analysis, we refer to Stroyan and Luxemburg [3] and Hurd and Loeb [2].

3. Several definitions of standard parts

We give several equivalent definitions of the standard part of an internal bounded self-adjoint operator which is not \(S\)-bounded.

The following lemma, which is a basic property for self-adjointness, is used to give the first definition of standard parts (see [8]).

**Lemma 3.1.** Let \(A\) be a symmetric operator on a Hilbert space \(\mathcal{H}\). Then, \(A\) is self-adjoint if and only if \(\text{Rng}(A \pm i) = \mathcal{H}\).

Let \(\mathcal{H}\) be an internal Hilbert space, and \(A\) an internal bounded self-adjoint operator on \(\mathcal{H}\). Let \(\hat{\mathcal{K}} = \text{Ker}((A + i)^{-1})^\perp\). Using the unitarity of \((A + i)(A - i)^{-1}\), we can easily check that \(\text{Ker}((A - i)^{-1})^{-1} = \hat{\mathcal{K}}\).

**Proposition 3.2.** There exists the unique (possibly unbounded) self-adjoint operator \(S\) on \(\hat{\mathcal{K}}\) satisfying
\[
(S + i)^{-1} = [(A + i)^{-1}]^\perp |\hat{\mathcal{K}}|.
\] (4)

**Proof.** We see \(\|(A + i)^{-1}\| < \infty\), and \([(A + i)^{-1}]^\perp\) is an bounded normal operator on \(\hat{\mathcal{H}}\). The operator \(T := [(A + i)^{-1}]^\perp |\hat{\mathcal{K}}|\) is a bijection from \(\hat{\mathcal{K}}\) to \([(A + i)^{-1}]^\perp |\hat{\mathcal{K}}|\). Hence the inverse \(T^{-1}\) from \([(A + i)^{-1}]^\perp |\hat{\mathcal{K}}|\) to \(\hat{\mathcal{K}}\) is defined. Clearly the operator \(S = T^{-1} - i\) satisfies the equation (4).

We will show that \(S\) is symmetric. Let \(x_1, x_2 \in \text{Dom}(S) = [(A + i)^{-1}]^\perp |\hat{\mathcal{K}}|\). Then, we can show that there exist \(\xi_i \in x_i\) such that \(A\xi_i \in Sx_i\) \((i = 1, 2)\) as follows. There
are $y_i \in \hat{\mathcal{K}}$ and $\eta_i \in \mathcal{H}$ such that $(S+i)^{-1}y_i = [(A+i)^{-1}]^\sim y_i = x_i$ and $\eta_i \in y_i$. Let $\xi_i = (A+i)^{-1}\eta_i$. Then $\xi_i \in x_i$ and $(A+i)^{-1}\xi_i = \eta_i \in y_i = (S+i)x_i$. Hence $A\xi_i \in Sx_i$. Thus, $(x_1, Sx_2) = \langle \xi_1, A\xi_2 \rangle = \langle A\xi_1, \xi_2 \rangle = (Sx_1, x_2)$. Therefore, $S$ is symmetric.

To prove the self-adjointness, it is sufficient to show $\text{Rng}(S+i) = \text{Rng}(S-i) = \hat{\mathcal{K}}$ by Lemma 3.1. Clearly $\text{Rng}(S+i) = \text{Rng}(T^{-1}) = \hat{\mathcal{K}}$. Let $x \in \text{Dom}(S)$, $\xi \in x$ and $A\xi \in Sx$. Then we have

$$
(A-i)(S+i)x = \left(\frac{A-i}{A+i}\right)(A+i)\xi = (S-i)x.
$$

Thus, by the equation (4) with $\text{Ker}([(A-i)^{-1}]^\perp = \hat{\mathcal{K}}$, we have

$$
(S-i)^{-1} = [(A-i)^{-1}]^\perp |\hat{\mathcal{K}}.
$$

Therefore, we can show $\text{Rng}(S-i) = \hat{\mathcal{K}}$ in the similar way to the proof of $\text{Rng}(S+i) = \hat{\mathcal{K}}$. The uniqueness of $S$ is clear. QED

**Definition 3.3.** Under the condition of Proposition 3.2, define the self-adjoint operator $\text{st}_1(A)$ on $\hat{\mathcal{K}}$ by $(\text{st}_1(A) + i)^{-1} = [(A+i)^{-1}]^\perp |\hat{\mathcal{K}}$.

The operator $\text{st}_1(A)$ is called the standard part of $A$. We see that $\text{st}_1(A) = \hat{A}$ when $A$ is $S$-bounded.

**Definition 3.4.** Let $A$ be an internal bounded operator on $\mathcal{H}$, an internal Hilbert space. Define $\text{fin}(A) \subseteq \mathcal{H}$ by

$$
\text{fin}(A) = \{\xi \in \text{fin}\mathcal{H} \mid A\xi \in \text{fin}\mathcal{H}\}.
$$

**Definition 3.5.** Let $A$ be an internal bounded self-adjoint operator on $\mathcal{H}$. Let $\hat{\mathcal{K}}$ be the closure of the subspace $[\text{fin}(A)]^\perp = \{\hat{\xi} \mid \hat{\xi} \in \text{fin}(A)\}$ of $\mathcal{H}$. Define the self-adjoint operator $\text{st}_2(A)$ on $\hat{\mathcal{K}}$ by

$$
e^{it\text{st}_2(A)} = e^{it\hat{A}}|\hat{\mathcal{K}}. \quad t \in \mathbb{R}.
$$

We see that $\{e^{it\hat{A}}|\hat{\mathcal{K}}\}_{t \in \mathbb{R}}$ is one-parameter unitary group, since $\hat{\mathcal{K}}$ is invariant under $e^{it\hat{A}}$ for all $t \in \mathbb{R}$. We also see that it is strongly continuous as follows. Let $\xi \in \text{fin}(A)$. Then, we have $\|e^{it\hat{A}}\xi\| = \|ie^{it\hat{A}}A\xi\| < \infty$, where $e^{it\hat{A}}$ is the internal differentiation. This implies that $e^{it\hat{A}}\xi$ is continuous with respect to $t \in \mathbb{R}$. Thus, $e^{it\hat{A}}$ is strongly continuous on $\text{fin}(A)^\perp$. Hence by Stone's theorem, $\text{st}_2(A)$ is uniquely defined.

If $A$ is $S$-bounded, $\text{st}_2(A)$ coincides with $\hat{A}$ defined in Section 2. This is seen from the following:

**Proposition 3.6.** Let $A$ be an internal $S$-bounded self-adjoint operator. Then,

$$
e^{it\hat{A}} = e^{itA},
$$

for all $t \in \mathbb{R}$.
Proof. For any infinitesimal $\epsilon \in \star \mathbb{R}_0^+$,
\[
\epsilon^{-1}(e^{i\epsilon A} - I) \approx iA,
\]
holds, because
\[
\|\epsilon^{-1}(e^{i\epsilon A} - I) - iA\| = \|\epsilon^{-1} \sum_{\nu=2}^{\infty} \epsilon^{\nu} / \nu! \| \leq \epsilon^{-1} \sum_{\nu=2}^{\infty} (\epsilon\|A\|)^\nu / \nu!
\]
\[
= \epsilon^{-1}(e^{\|A\|} - 1) - \|A\| \approx 0.
\]
Thus, by the permanence principle,
\[
\forall \delta \in \mathbb{R}_+, \exists \epsilon \in \mathbb{R}_+, |t| < \epsilon \Rightarrow \|t^{-1}(e^{itA} - I) - iA\| < \delta.
\]
Hence, we have
\[
\lim_{\epsilon \to 0} \|\epsilon^{-1}(e^{i\epsilon A} - I) - iA\| = 0.
\]
Thus we have $(d/dt)e^{i\epsilon A}|_{t=0} = i\hat{A}$, where $d/dt$ is the usual differentiation. Because $(e^{itA})_{t \in \mathbb{R}}$ is one-parameter unitary group, it follows that $e^{itA} = e^{it\hat{A}}$. QED

Let $E(\cdot)$ be an internal projection-valued measure on $\star \mathbb{R}$, i.e., for each internal Borel set $\Omega \subseteq \star \mathbb{R}$, $E(\Omega)$ is an orthogonal projection on $\mathcal{H}$ such that
1) $E(\phi) = 0$, $E(\star \mathbb{R}) = I$
2) If $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \phi$ if $n \neq m$, then $E(\Omega) = s\lim_{N \to \infty} \sum_{n=1}^{N} E(\Omega_n)$
3) $E(\Omega_1) E(\Omega_2) = E(\Omega_1 \cap \Omega_2)$.

For $r \in \star \mathbb{R}$, let $\mathcal{H}_r = \text{Rng}(E(-r, r))$, the range of $E((-r, r))$. Let $D(E) = \bigcup_{r \in \mathbb{R}^+} \mathcal{H}_r \cap \text{fin} \mathcal{H}$. $D(E)$ is called the standardization domain of $E(\cdot)$. Clearly, $\overline{D(E)}^\perp = (\bigcup_{r \in \mathbb{R}^+} \mathcal{H}_r)^\perp$.

For $a \in \mathbb{R}$, define the orthogonal projection $\hat{E}_{st}(\infty, a]$ by
\[
\hat{E}_{st}(\infty, a] = \sup \{ \hat{E}(-K, a + \epsilon \|\hat{D}(E)\| \perp |K, \epsilon \in \mathbb{R}^+ \}
\]
\[
= s\lim_{n \to \infty} \hat{E}(-n, a + 1/n \|\hat{D}(E)\| \perp .
\]
Then we see
\[
s\lim_{a \to -\infty} \hat{E}_{st}(\infty, a] = 0
\]
\[
s\lim_{a \to -n} \hat{E}_{st}(\infty, a + \epsilon] = \hat{E}_{st}(\infty, a]
\]
\[
a < b \Rightarrow \hat{E}_{st}(\infty, a] \leq \hat{E}_{st}(\infty, b].
\]
Hence, $\hat{E}_{st}(\infty, \cdot] \text{ defines a projection-valued measure on } \mathbb{R}$.

Definition 3.7. For any internal bounded self-adjoint operator $A$, define the self-adjoint operator $\text{st}_{3}(A)$ on $\overline{D(E)}^\perp$ by
\[
\text{st}_{3}(A) = \int \lambda d\hat{E}_{st}(\lambda).
\]
Proposition 3.8. Let $A$ be an internal bounded self-adjoint operator, and $E(\cdot)$ the internal projection-valued measure associated with the spectral decomposition of $A$. Then

$$\hat{D}(E)^{\perp\perp} = \overline{\text{fin}(A)}^{\perp\perp}. \quad (19)$$

Proof. $\hat{D}(E)^{\perp\perp} \subseteq \overline{\text{fin}(A)}^{\perp\perp}$ is clear. To prove $\hat{D}(E)^{\perp\perp} \supseteq \overline{\text{fin}(A)}^{\perp\perp}$, it is sufficient to show that for any $\hat{x} \in \overline{\text{fin}(A)} $ there is a sequence $\hat{x}_n \in \hat{D}(E)$ ($n \in \mathbb{N}$) such that $\hat{x}_n \to \hat{x}$. Let $x_n = E(-n, n)x$ ($n \in \mathbb{N}$). Notice that $\|A(x - x_n)\| \geq n\|x - x_n\|$. Suppose $\|x - x_n\| > \epsilon$ for all $n \in \mathbb{N}$. By the permanence principle, there is $N \in \mathbb{N}_\infty$ such that $\|A(x - x_N)\| > N\epsilon \sim \infty$. This contradicts $\|A(x - x_N)\| \leq \|Ax\| < \infty$. \(QED\)

Theorem 3.9. Let $A$ be an internal bounded self-adjoint operator. Then,

$$\text{st}_2(A) = \int \lambda d\hat{E}_{\text{st}}(\lambda), \quad (20)$$

and hence $\text{st}_2(A) = \text{st}_3(A)$.

Proof. It is sufficient to show

$$\langle \hat{x}, \exp(it\text{st}_2(A))\hat{x} \rangle = \int e^{it\lambda} \langle \hat{x}, d\hat{E}_{\text{st}}(\lambda)\hat{x} \rangle \quad (21)$$

for all $\hat{x} \in \overline{\text{fin}(A)}^{\perp\perp}$. Define the internal Borel measure $\mu$ by $\mu(d\lambda) = \langle x, E(d\lambda)X \rangle$. Let $L\mu$ denote the Loeb measure of $\mu$, and $L'\mu$ the Borel measure on $\mathbb{R}$ defined by $L'\mu(\Omega) = L\mu(\text{st}^{-1}[\Omega])$. We can check that $L'\mu$ is well-defined (i.e., $\text{st}^{-1}[\Omega]$ is $L\mu$-measurable for any Borel set $\Omega \subseteq \mathbb{R}$). We also see that $L\mu$ is supported by $\text{fin}^*\mathbb{R}$, since $L\mu(\mathbb{R} \setminus \text{fin}^*\mathbb{R}) \leq \circ \langle x, E(\mathbb{R} \setminus (-n, n))x \rangle = \circ \|1 - E(-n, n)\| \leq (1/n^2)^\circ \|Ax\|^2$ for all $n \in \mathbb{N}$. Therefore

$$\langle \hat{x}, \exp(it\text{st}_2(A))\hat{x} \rangle = \langle \hat{x}, e^{itA}\hat{x} \rangle = \circ \langle x, e^{itA}x \rangle$$

$$= \circ \int_{\mathbb{R}} e^{it\lambda} d\mu(\lambda)$$

$$= \int_{\mathbb{R}} e^{it\lambda} dL\mu(\lambda)$$

$$= \int_{\mathbb{R}} e^{it\lambda} dL'\mu(\lambda).$$

On the other hand, for $a, b \in \mathbb{R}$ with $a < b$,

$$L'\mu(a, b) = L\mu(\bigcup_{\epsilon \in \mathbb{R}^+} (a + \epsilon, b - \epsilon))$$

$$= \lim_{\epsilon \downarrow 0} \circ \langle x, E(a + \epsilon, b - \epsilon)x \rangle$$

$$= \lim_{\epsilon \downarrow 0} \langle \hat{x}, \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle$$

$$= \langle \hat{x}, s\text{-lim}_{\epsilon \downarrow 0} \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle$$

$$= \langle \hat{x}, \hat{E}_{\text{st}}(a, b)\hat{x} \rangle.$$

Hence, $L'\mu(\Omega) = \langle \hat{x}, \hat{E}_{\text{st}}(\Omega)\hat{x} \rangle$ for any Borel set $\Omega \subseteq \mathbb{R}$. \(QED\)
Let $C \in \mathbb{R}$ be a positive constant, and $h$ be an internal Borel function from $\ast \mathbb{R}$ to $\ast \mathbb{C}$ satisfying the following properties:

\[ h(x) \approx h(y) \quad \text{iff} \quad x \approx y \quad \text{for all } x, y \text{ with } |x|, |y| < \infty, \]

\[ |h(x)| < C \quad \text{for all } x \in \ast \mathbb{R}. \]

Define the function $\hat{h} : \mathbb{R} \to \mathbb{C}$ by

\[ \hat{h}(x) = \circ h(x), \]

for $x \in \mathbb{R}$. We see that $\hat{h}$ is injective and continuous. Let $A$ be an internal bounded self-adjoint operator. Notice that $h(A)$ is an $S$-bounded internal normal operator.

**Theorem 3.10.** There exists the unique self-adjoint operator $B$ on $\text{fin}(A)^\perp$ such that

\[ \hat{h}(B) = h(A)\mid \text{fin}(A)^\perp. \]  

(22)

Moreover, $B$ equals to $\text{st}_3(A)$.

**Proof.** By the argument similar to the proof of Theorem 3.9, we can show

\[ \langle \hat{x}, h(A)\hat{x} \rangle = \int_{\mathbb{R}} \hat{h}(\lambda) dL' \mu(\lambda) \]

\[ = \int_{\mathbb{R}} \hat{h}(\lambda) \langle \hat{x}, d\hat{E}_{st}(\lambda) \hat{x} \rangle \]

for any \( \hat{x} \in \text{fin}(A)^{\perp \perp} \). Thus,

\[ h(A)\mid \text{fin}(A)^{\perp \perp} = \int_{\mathbb{R}} \hat{h}(\lambda) d\hat{E}_{st}(\lambda). \]

Because $\hat{h}$ is injective, the unique self-adjoint operator $B$ satisfying (22) is $\text{st}_3(A) = \int_{\mathbb{R}} \lambda d\hat{E}_{st}(\lambda)$. QED

**Corollary 3.11.** Definition 3.3, 3.5 and 3.7 are equivalent, that is, $\text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A)$.

**Proof.** Let $h(x) = 1/(x + i)$. QED

In section 2, $\hat{A}$ is defined only when $A$ is an internal S-bounded self-adjoint operator. Now we can extend the definition so as to include the case where $A$ is an internal bounded self-adjoint operator which is not S-bounded; $\hat{A} := \text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A)$.

**Definition 3.12.** Let $A$ be an internal linear operator on an internal Hilbert space $\mathcal{H}$. Let $D$ be an (external) subspace of $\text{fin}\mathcal{H}$. $A$ is standardizable on $D$ if $D \subset \text{fin}(A)$ and if for any $x, y \in D$, $x \approx y$ implies $Ax \approx Ay$. In this case, define the operator $A_D$ with domain $D = \{ \hat{x} | x \in D \}$, called the standard part of $A$ on $D$, by

\[ \hat{A}_D \hat{x} = \overline{Ax}, \quad x \in D. \]  

(23)
Clearly, $A$ is standardizable on $D$ if and only if $D \subset \operatorname{fin}(A)$, and if $A\xi \approx 0$ for all $\xi \in D$ with $\xi \approx 0$.

**Lemma 3.13.** An internal bounded operator $A$ is standardizable on $\operatorname{fin}(A^*A)$.

**Proof.** First, we prove $\operatorname{fin}(A^*A) \subset \operatorname{fin}(A)$ as follows. Suppose that $\xi \in \operatorname{fin}(A)$. Let $E(\cdot)$ be the internal spectral-valued measure of the self-adjoint operator $A^*A$. Then, $||A\xi||^2 = \langle \xi, A^*A\xi \rangle = \langle \xi, E[0, 1]A^*A\xi \rangle + \langle \xi, (I - E[0, 1])A^*A\xi \rangle + \langle \xi, (I - E[0, 1])(A^*A)^2\xi \rangle \leq \langle \xi, E[0, 1]A^*A\xi \rangle + ||A^*A\xi||^2 < \infty$. Thus, $\xi \in \operatorname{fin}(A)$. Suppose $x \approx 0$ and $||A^*Ax|| < \infty$. Then, $||Ax||^2 = \langle x, A^*Ax \rangle \leq ||x||||A^*Ax|| \approx 0$. $QED$

**Corollary 3.14.** If $D \subset \operatorname{fin}\mathcal{H}$ is invariant under $A$ and $A^*A$, $A$ is standardizable on $D$.

The operator $B$ in the above proof is called a hyperfinite extension of $A$ [6].

We use the following lemma in the proof of Theorem 3.16.

**Lemma 3.15.** Let $A$ be a symmetric operator with domain $D \subset \mathcal{H}$, a Hilbert space. Let $D_1 \subset D$ be a dense linear subset of $\mathcal{H}$ and suppose that $A|D_1$ is essentially self-adjoint. Then, $A$ is essentially self-adjoint and $\overline{A} = A|D_1$.

**Theorem 3.16.** Let $A$ be an internal self-adjoint operator on $\mathcal{H}$, and $E(\cdot)$ the projector-valued spectral measure of $A$. Then,

$$\hat{A} = \overline{A_{D(E)}} = \overline{A_{\operatorname{fin}(A^2)}}$$

**Proof.** We can show that $\hat{A}_{D(E)}$ is essentially self-adjoint e.g. by Nelson's analytic vector theorem. Hence, it has one and only one self-adjoint extension, its closure. Thus, it is sufficient to show that $\hat{A}$ is an extension of $\hat{A}_{D(E)}$. If $E(-r,r)\xi = \xi$ ($r \in \mathbb{R}^+, \xi \in \mathcal{H}$), then $\hat{E}_s(-s,s)\xi = \hat{\xi}$ ($s \in \mathbb{R}^+, s > r$). Thus, $\hat{A}_{D}(\xi) = \hat{A}_s \xi = \hat{\xi} = \int_{-s}^{s} \lambda d\hat{E}_s(\lambda)\hat{\xi} = \int \lambda d\hat{E}_s(\lambda)\hat{\xi} = \text{st}_3(A) = \hat{A}\hat{\xi}$. Therefore $\hat{A} = \hat{A}_{D(E)}$. $\hat{A}_{D(E)} = \hat{A}_{\operatorname{fin}(A^2)}$ follows from $D(E) \subset \operatorname{fin}(A^2)$ and Lemma 3.15. $QED$

4. The domain of $\hat{A}$

**Definition 4.1.** For an internal bounded self-adjoint operator $A$ on $\mathcal{H}$, define $D(A)$ by

$$D(A) = \{ \xi \in \operatorname{fin}\mathcal{H} \mid \text{for all } t \in \mathbb{R}_0^+, e^{-t|A|}\xi \approx A\xi \in \operatorname{fin}\mathcal{H} \}.$$ 

Clearly, $D(A)$ is a subspace of $\mathcal{H}$.

**Proposition 4.2.** An internal bounded self-adjoint operator $A$ is standardizable on $D(A)$. 


Proof. Let $\xi \in D(A)$ and $\|\xi\| \approx 0$. We can easily check $\|e^{-t|A|}A\| < \infty$ for all $t > 0$, $t \neq 0$. Hence, $^0\|A\xi\| \leq ^0\|e^{-t|A|}A\xi\| + ^0\|(1 - e^{-t|A|})A\|$. By the S-boundedness of $e^{-t|A|}A$, the first term equals 0, and by the definition of $D(A)$, the second term equals 0. Thus we have $^0\|A\xi\| = 0$. QED

The following lemmas are easily shown.

Lemma 4.3. Let $f : \star N \to \star \mathbb{R}^+$ be internal and increasing. If $f(M) < \infty$ for some $M \sim \infty$, then
\[ \lim_{n \to \infty} ^0 f(n) < \infty. \]

Lemma 4.4. Under the same condition to Lemma 4.3, there is $K \sim \infty$ such that for all $L \sim \infty$,
\[ f(K) \approx f(L) \text{ if } L \leq K. \]

Proposition 4.5. Let $\xi \in \text{fin}(\mathcal{H})$. For sufficiently large $t \approx 0$,
\[ e^{-t|A|}\xi \in D(A). \] (25)

Proof. Applying Lemma 4.4 to $f(n) = \|e^{-|A|/n}A\xi\|$, we find that for sufficiently small $K \sim \infty$ and $L \sim \infty$, $e^{-|A|/K}A\xi \approx e^{-|A|/L}A\xi$. Thus, for sufficiently large $s \approx 0$ and $t \approx 0$, $e^{-s|A|}A\xi \approx e^{-t|A|}A\xi$. Hence, for all $x \approx 0$, $x > 0$,
\[ e^{-s|A|}Ae^{-t|A|}\xi = e^{-(x+t)|A|}A\xi \approx Ae^{-t|A|}\xi. \]

Therefore, $e^{-t|A|}\xi \in D(A)$. QED

Theorem 4.6. Let $E(\cdot)$ be the spectral resolution of $A$ and $E_K = E(-K, K)$ for $K \in \star \mathbb{R}^+$. For any $\xi \in \text{fin}(A)$,
\[ \xi \in D(A) \text{ iff } A\xi \approx E_K A\xi \text{ for all } K \sim \infty. \] (26)

Remark. The right-hand condition is equivalent to
\[ \lim_{K \to \infty \text{ for } K \neq 0} ^0\|(I - E_K)A\xi\| = 0. \] (27)

Proof. Suppose that $\xi \in \text{fin}(A)$ and $A(I - E_K)\xi \approx 0$ for all $K \sim \infty$. For any $t \approx 0$, there exists a $K \sim \infty$ such that $tK \approx 0$. Thus,
\[ \|e^{-t|A|}A\xi - A\xi\|^2 \approx \|e^{-t|A|}E_K A\xi - E_K A\xi\|^2 \]
\[ = \| \int_{-K}^{K} e^{-t|\lambda|} (\lambda - \lambda dE(\lambda))\xi\|^2 \]
\[ = \int_{-K}^{K} |(e^{-t|\lambda|} - 1)|\lambda|^2 dE(\lambda)\xi||^2 \]
\[ \leq \sup_{|\lambda| < K} |e^{-t|\lambda|} - 1|^2 \int_{-K}^{K} \lambda^2 dE(\lambda)\xi||^2 \]
\[ = \sup_{|\lambda| < K} |e^{-t|\lambda|} - 1|^2 \|E_K A\xi\|^2 \]
\[ \approx 0. \]
Hence $\xi \in D(A)$.

Conversely, suppose $\xi \in D(A) (\subset \text{fin}(A))$. Applying Lemma 4.4 to $f(n) = \|E_nA\xi\|$, we see that for sufficiently small $K \sim \infty$ and $L \sim \infty (L \leq K)$,

$$\|E_LA\xi\| \approx \|E_KA\xi\|.$$  

Thus, $(E_K - E_L)A\xi \approx 0$, since $\|E_LA\xi - E_KA\xi\|^2 = \|E_KA\xi\|^2 - \|E_LA\xi\|^2 \approx 0$. Let $t \in \mathbb{R}_0^+$ satisfy $tK \sim \infty$ so that

$$\|E_KA\xi - e^{-t|A|}A\xi\|$$  

$$= \|\int_{-K}^K \lambda(1 - e^{-t|A|})dE(\lambda)\xi - \int_{(-\infty,-K) \cup (K, \infty)} e^{-t|A|}\lambda dE(\lambda)\xi\|$$  

$$\leq \|\int_{-K}^K \lambda(1 - e^{-t|A|})dE(\lambda)\xi\| + e^{-tK}\|A\xi\|$$  

$$\approx \|\int_{-K}^K \lambda(1 - e^{-t|A|})dE(\lambda)\xi\|.$$  

Let $L \sim \infty$ satisfy $tL \approx 0$, so that the above

$$\leq \|\int_{-L}^{L} \lambda(1 - e^{-tL})dE(\lambda)\xi\| + \|(E_K - E_L)A\xi\|$$  

$$\approx 0.$$  

Thus, for sufficiently small $K \sim \infty$ and for any $t \approx 0$ such that $tK \sim \infty$,

$$E_KA\xi \approx e^{-t|A|}A\xi \approx A\xi.$$  

Since $\|A\xi - E_KA\xi\| \geq \|A\xi - E_{K'}A\xi\| > 0$ if $K < K'$, we have $E_{K'}A\xi \approx A\xi$ holds for any $K' \sim \infty$. $QED$

**Proposition 4.7.** Let $\xi \in \text{fin}(A)$. Then, $E_K\xi \in D(A)$ for sufficiently small $K \sim \infty$.

**Proof.** Applying Lemma 4.4 to $f(n) = \|E_nA\xi\|$, we find that for sufficiently small $K, L \sim \infty$, $E_KA\xi \approx E_LA\xi$. Thus, if $L \sim \infty$, $L \leq K$, then $\|(1 - E_L)E_KA\xi\| = \|(E_K - E_L)A\xi\| \approx 0$. If $L > K$, clearly $(1 - E_L)E_KA\xi = 0$. Hence for all $L \sim \infty$, $E_KA\xi \approx E_LE_KA\xi$. Thus $E_K\xi \in D(A)$ by Theorem 4.6. $QED$

**Corollary 4.8.** $[\text{fin}(A)]^\sim = [D(A)]^\sim$, i.e., if $\xi \in \text{fin}(A)$, then there is $\eta \in D(A)$ such that $\eta \approx \xi$.

**Example** We have seen that the following relations hold:

$$\text{fin}(A^2) \subset D(A) \subset \text{fin}(A) \subset \text{fin}\mathcal{H},$$

$$[\text{fin}(A^2)]^\sim \subset [D(A)]^\sim = [\text{fin}(A)]^\sim \subset \hat{\mathcal{H}}.$$
\[ \text{An example of } A \text{ such that } \text{fin}(A) \setminus D(A) \neq \emptyset \text{ is given as follows. Let } \nu \text{ be an infinite hypernatural number, and } \mathcal{H} = \star \mathbb{C}^{\nu}, \nu \text{-dimensional internal Hilbert space. Define the internal self-adjoint operator } A \text{ on } \mathcal{H} \text{ by } A(x_1, x_2, \ldots, x_\nu) = (x_1, 2x_2, \ldots, \nu x_\nu). \text{ Let } \xi = (0, 0, \ldots, 0, \nu^{-1}). \text{ Then we see } \xi \in \text{fin}(A) \setminus D(A) \text{ from Theorem 4.6.}
\]

We also find \( D(A) \setminus \text{fin}(A^2) \neq \emptyset; \) let \( \eta = (1^{-2}, 2^{-2}, \ldots, \nu^{-2}) \), then we easily see \( \eta \in D(A) \setminus \text{fin}(A^2) \). Moreover, we find \( \hat{\eta} \in [D(A)]^{\wedge} \setminus [\text{fin}(A^2)]^{-} \). In fact, if \( \eta' \approx \eta \), then

\[ \|A^2\eta'\| \geq \lim_{n \to \infty} \|A^2 E_n \eta'\| = \lim_{n \to \infty} \sqrt{n} = \infty. \]

Thus, we have \( \hat{\eta} \not\in [\text{fin}(A^2)]^{-} \) by Theorem 4.6.

**Theorem 4.9.** Let \( \xi \in \text{fin}(A) \), then

\[ \xi \in D(A) \text{ iff } \lim_{t \downarrow 0, t \#^0} \left( \frac{e^{-t|A|} - 1}{t} \xi \right) \approx -|\overline{A}| \xi. \quad (28) \]

**Proof.** Suppose that the right-hand side does not hold. In other words, suppose that

\[ \exists \varepsilon \in \mathbb{R}^{+} \forall n \in \mathbb{N} \exists t \in \star \mathbb{R}, \ 0 < t < \frac{1}{n} \land \left\| \left( \frac{e^{-t|A|} - 1}{t} + |A| \right) \xi \right\| > \varepsilon. \quad (29) \]

By permanence,

\[ \exists \varepsilon \in \mathbb{R}^{+} \exists N \in \star \mathbb{N}_\infty \exists t \in \star \mathbb{R}, \ 0 < t < \frac{1}{n} \land \left\| \left( \frac{e^{-t|A|} - 1}{t} + |A| \right) \xi \right\| > \varepsilon. \quad (30) \]

That is, there is positive infinitesimal \( t \) such that \( t^{-1}(e^{-t|A|} - 1) \xi \not\approx -|A| \xi \).

Thus, for some \( \eta \in \text{fin}(\mathcal{H}) \),

\[ \Re \left( \eta, \frac{e^{-t|A|} - 1}{t} \xi \right) \not\approx \Re(\eta, -|A| \xi). \]

Let \( f(t) = \Re(\eta, e^{-t|A|} \xi) \). By the mean value theorem, for some \( s \in \star \mathbb{R} \) with \( 0 < s < t \),

\[ f'(s) = \frac{f(t) - f(0)}{t} = \Re \left( \eta, \frac{e^{-t|A|} - 1}{t} \xi \right) \not\approx \Re(\eta, -|A| \xi). \]

Therefore, by the definition of \( D(A) \), we have \( \xi \in \text{fin}(A) \setminus D(A) \).

Conversely, suppose \( \xi \in \text{fin}(A) \setminus D(A) \). Then, there is positive infinitesimal \( t_0 \) satisfying \( e^{-t_0|A|} \xi \not\approx A \xi \). Let \( \eta = (|A| - e^{-t_0|A|} |A|) \xi (\in \text{fin}(\mathcal{H})) \). Then this is equivalent to

\[ \langle \eta, e^{-t_0|A|} |A| \xi \rangle \neq \langle \eta, |A| \xi \rangle. \quad (31) \]

Let \( f(x) = \langle \eta, e^{-x|A|} \xi \rangle \) (\( x \in \star \mathbb{R}^+ \)). We see that \( f' \) is increasing and \(-\infty < f' < 0 \), and hence \( f \) is decreasing and \( 0 < f < \infty \). The relation (31) is equivalent to

\[ f'(t_0) \not\approx f'(0), \quad (32) \]
We have $f(x) \geq f'(t_0)(x - t_0) + f(t_0)$. Thus we have

$$0 > \frac{f(x) - f(0)}{x} \geq \frac{f'(t_0)(x - t_0) + f(t_0) - f(0)}{x}.$$  \hfill (33)

Let $F(x) = [f'(t_0)(x - t_0) + f(t_0) - f(0)]/x$, then for $c \in \mathbb{R}^+$,

$$F(ct_0) = f'(t_0) \left( 1 - \frac{1}{c} \right) + \frac{1}{c} \frac{f(t_0) - f(0)}{t_0}.$$ \hfill (34)

By the mean value theorem and $-\infty < f'(x) < 0$, we have $|(f(x) - f(0))/x| < \infty$. Hence $F(ct_0) \approx f'(t_0)$ for all $c \sim \infty$. Thus, by (32) and (33),

$$0 > \frac{f(ct_0) - f(0)}{ct_0} \geq \frac{f'(t_0) > f'(0)},$$ \hfill (35)

for all $c \sim \infty$. Thus there is $\epsilon \in \mathbb{R}^+$ such that for sufficiently large $x \approx 0$, $\frac{f(x) - f(0)}{x} - f'(0) > \epsilon$. By the permanence principle, for sufficiently small $x \in \mathbb{R}^+$, $\frac{f(x) - f(0)}{x} - f'(0) > \epsilon$. We can check the relations

$$\langle \eta, \left( e^{-x|A|} - 1 \right) / x \rangle = \frac{f(x) - f(0)}{x}, \quad \langle \eta, |A|\xi \rangle = -f'(0), \quad \frac{e^{-x|A|} - 1}{x} > -|A|,$$

for $x > 0$. Therefore, using the increasingness of $(e^{-x|||\cdot|||_{A}} - 1)/x$, we have

$$\lim_{x \downarrow 0, x \# 0} \langle \eta, e^{-x|A|} \rangle \neq \langle \eta, |A|\xi \rangle.$$

**QED**

**Theorem 4.10.** Let $A$ be an internal bounded self-adjoint operator. Then, $\hat{A} = \hat{A}_{D(A)}$.

**Proof.** By Theorem 3.16 and Lemma 3.15, it suffices to show that $\hat{A}_{D(A)}$ is a closed extension of $\hat{A}_{\text{fin}(A^2)}$. If $\xi \in \text{fin}(A^2)$, for any $K \sim \infty$, $||(1 - E_K) A \xi|| \leq 1/K ||(1 - E_K) A^2 \xi|| \leq 1/K ||A^2 \xi|| \approx 0$. Hence $\xi \in D(A)$, and hence $\hat{A}_{D(A)}$ is an extension of $\hat{A}_{\text{fin}(A^2)}$.

To prove that $\hat{A}_{D(A)}$ is closed, it suffices to show that $\hat{D}(A) = [D(A)]^c$ is complete in the norm $\| \cdot \|_A$ defined by $\| \xi \|_A = \| \xi \| + \| \hat{A} \xi \|$. Define the internal norm $\| \cdot \|_A$ on $\mathcal{H}$ by $\| \xi \|_A = \| \xi \| + \| A \xi \|$. We can check $\| \xi \|_A = \| \xi \|_A$ for $\xi \in D(A)$.

By Theorem 2.1, $\text{fin}(A)$ is S-$\| \cdot \|_A$-complete. Hence, if the sequence $\{\xi_i\}_{i \in \mathbb{N}} \subset D(A)$ (\subset $\text{fin}(A)$) is S-$\| \cdot \|_A$-Cauchy, then there is $\xi \in \text{fin}(A)$ such that $\{\xi_i\}$ approximately converges to $\xi$ in the norm $\| \cdot \|_A$. This $\xi$ is shown to be in $D(A)$ as follows. Regarding Theorem 4.6, and $\xi_i \in D(A)$ ($i < \infty$), this relation leads to $\| (I - E_K) A \xi_i \| = \lim_{i \rightarrow \infty} \circ \| (I - E_K) A \xi_i \| = 0$, for any $K \sim \infty$. Therefore, from Theorem 4.6, we have $\xi \in D(A)$ and hence any Cauchy sequence in $\hat{D}(A)$ converges in $\hat{D}(A)$ in the norm $\| \cdot \|_A$. **QED**
**Theorem 4.11.** The domain $D(A)$ is maximal. That is, if $D(A) \subset S \subset \text{fin}(\mathcal{H})$ and $A$ is standardizable on $S$, then $S = D(A)$.

**Proof.** Suppose that $D(A) \subset S \subset \text{fin}(\mathcal{H})$ and that $A$ is standardizable on $S$. Let $\eta \in S$. By Corollary 4.8 and $\eta \in \text{fin}(A)$, there is $\xi \in D(A)$ such that $\xi \approx \eta$. By the definition of $D(A)$ and the standardizability on $S$, for all positive infinitesimal $t$, $e^{-t|A|}A\eta \approx e^{-t|A|}A\xi \approx A\xi \approx A\eta$, since $\|e^{-t|A|}\| \leq 1$. Thus, $\eta \in D(A)$. QED

**Proposition 4.12.** Let $A$ be an internal positive operator on $\mathcal{H}$. Then, for any $\eta \in \text{fin}(A^{1/2})$,

$$\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \inf_{\alpha \sim \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle. \quad (36)$$

**Proof.** Suppose $\eta \approx \xi$. If $\alpha < \infty$, $\langle \eta, E_{\alpha}A\eta \rangle \approx \langle \xi, E_{\alpha}A\xi \rangle \leq \langle \xi, A\xi \rangle$, that is,

$$\forall \varepsilon \in \mathbb{R}^+, \forall \alpha < \infty, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon,$$

Thus, by the permanence principle,

$$\forall \varepsilon \in \mathbb{R}^+, \exists \alpha \sim \infty, \forall \alpha \leq \alpha, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon.$$

By saturation,

$$\exists \alpha \sim \infty, \forall \varepsilon \in \mathbb{R}^+, \forall \alpha \leq \alpha, \quad \langle \eta, E\alpha A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon.$$

Hence we have

$$\exists \alpha \sim \infty, \quad \circ \langle \eta, E_{\alpha}A\eta \rangle \leq \circ \langle \xi, A\xi \rangle.$$

It follows that $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \geq \inf_{\alpha \sim \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle$.

On the other hand, we see that for all $\alpha \sim \infty$, $\|\eta - E_{\alpha}\eta\|^2 \leq \alpha^{-1}\|A^{1/2}(\eta - E_{\alpha}\eta)\|^2 \leq \alpha^{-1}\|A^{1/2}\eta\|^2 \approx 0$. Hence,

$$\forall \alpha \sim \infty, \quad \inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \leq \circ \langle E_{\alpha}\eta, AE_{\alpha}\eta \rangle = \circ \langle \eta, E_{\alpha}A\eta \rangle.$$

Thus it follows that $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \leq \inf_{\alpha \sim \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle$. QED

**Proposition 4.13.** Let $A$ be an internal positive operator and $\eta \in \text{fin}(A)$. Then,

$$\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \langle \hat{\eta}, \hat{A}\hat{\eta} \rangle. \quad (37)$$

**Proof.** From Proposition 4.12, we see $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \inf_{\alpha \sim \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle$. By Theorem 4.10 and Proposition 4.7, for sufficiently small $\alpha \sim \infty$, $\circ \langle \eta, E_{\alpha}A\eta \rangle = \circ \langle E_{\alpha}\eta, AE_{\alpha}\eta \rangle = \langle \hat{E}_{\alpha}\eta, \hat{A}\hat{E}_{\alpha}\eta \rangle = \langle \hat{\eta}, \hat{A}\hat{\eta} \rangle$. QED

**Definition 4.14.** Let $A$ be a internal bounded positive operator, and $D \subset \text{fin}(A^{1/2})$. The sesquilinear form $\langle \cdot, A \cdot \rangle$ is standardizable on $D$ if $\langle \xi_1, A\eta_1 \rangle \approx \langle \xi_2, A\eta_2 \rangle$ for all $\xi_1, \xi_2, \eta_1, \eta_2 \in D$ with $\xi_1 \approx \xi_2$ and $\eta_1 \approx \eta_2$. 
Proposition 4.15. Let $D$ be a subspace of $\text{fin}(\mathcal{H})$ and $A \geq 0$. Then, $\langle \cdot, A \cdot \rangle$ is standardizable on $D$ if and only if $A^{\frac{1}{2}}$ is standardizable on $D$.

Proof. Suppose that $A^{\frac{1}{2}}$ is standardizable on $D$. Then $A^{\frac{1}{2}}\xi \approx A^{\frac{1}{2}}\eta$ for any $\xi, \eta \in D$ with $\xi \approx \eta$. Thus, $\langle \xi, A\xi \rangle = \|A^{\frac{1}{2}}\xi\|^2 \approx \|A^{\frac{1}{2}}\eta\|^2 = \langle \eta, A\eta \rangle$. Conversely, suppose that $\langle \cdot, A \cdot \rangle$ is standardizable on $D$. Then for any $\xi, \eta \in D$ with $\xi \approx \eta$, $\|A^{\frac{1}{2}}\xi - A^{\frac{1}{2}}\eta\|^2 = \|A^{\frac{1}{2}}(\xi - \eta)\|^2 \approx \langle \xi - \eta, A(\xi - \eta) \rangle \approx 0$. QED

Corollary 4.16. The set $D(A^{\frac{1}{2}})$ is a maximal domain of $\langle \cdot, A \cdot \rangle$, and $\langle \xi, A\eta \rangle = \langle \widetilde{A^{\frac{1}{2}}\xi}, A^{\frac{1}{2}}\widetilde{\eta} \rangle$ for any $\xi, \eta \in D(A^{\frac{1}{2}})$.

References


