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Kyoto University
Nonstandard Representations of Unbounded Self-Adjoint Operators

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1. Introduction

In nonstandard analysis, standardizations of internal (or nonstandard) objects have been studied for constructing standard mathematical objects; e.g. an internal measure space is converted into a measure space in the standard sense, called Loeb space ([1][2][3][4]). The standardization of an internal Hilbert space $\mathcal{H}$ is called the nonstandard hull of $\mathcal{H}$, written as $\hat{\mathcal{H}}$ (Henson and Moore [5]). Then the standardization of an internal operator $A$ on $\mathcal{H}$ with finite norm is naturally defined on $\mathcal{H}$, In this paper, the standardization of $A$ shall be called the standard part of $A$, written as $\hat{A}$. A prominent work of Moore [6] was focused on the case where $\mathcal{H}$ is hyperfinite-dimensional, and studied hyperfinite-dimensional extension of bounded operators on $\hat{\mathcal{H}}$. On the other hand, in the case where the norm of $A$ is not finite, it is not straightforward to give an adequate definition of the standard part of $A$. Albeverio et al. [4] defined $\hat{A}$ only when $\mathcal{H}$ is hyperfinite-dimensional real Hilbert space and $A$ is an internal positive symmetric operator on $\mathcal{H}$.

In this paper, we give a definition of $\hat{A}$ for any internal complex Hilbert space $\mathcal{H}$ and for any internal S-bonded self-adjoint operator $A$ on $\mathcal{H}$, as well as a general consideration on $\hat{A}$ so defined, which suggests the adequacy of the definition.

2. Preliminaries

We work in a $\aleph_1$-saturated nonstandard universe [7]. Note that every nonstandard universe constructed by a bounded ultrapower is $\aleph_1$-saturated.

Let $(V, \| \cdot \|)$ be an internal normed linear space. Define the subspaces $\mu(V, \| \cdot \|)$ and $\text{fin}(V, \| \cdot \|)$ of $V$ by

$$\mu(V, \| \cdot \|) = \{ \xi \in V \mid \|\xi\| \approx 0 \}, \quad \text{fin}(V, \| \cdot \|) = \{ \xi \in V \mid \|\xi\| < \infty \}. \quad (1)$$

We often abbreviate them as $\mu(V)$ and $\text{fin}(V)$. Let $\hat{\xi} = \xi + \mu(V)$ and $\hat{V} = \text{fin}(V)/\mu(V)$, the quotient space. We can naturally define the usual norm $\| \cdot \|$ on $\hat{V}$ by $\|\xi\| = \circ \|\xi\|$. A countably infinite sequence $\{\xi_i\}_{i \in \mathbb{N}}$, where $\xi_i \in \text{fin}(V, \| \cdot \|)$, approximately converges to $\xi \in V$ in the norm $\| \cdot \|$ if

$$\forall \epsilon \in \mathbb{R}^+ \exists n \in \mathbb{N} \forall k \in \mathbb{N} \quad [k > n \Rightarrow \|\xi - \xi_i\| < \epsilon]. \quad (2)$$
A sequence \( \{\xi_i\}_{i \in \mathbb{N}} \) approximately converges to \( \xi \in V \) if and only if \( \{\hat{\xi}_i\}_{i \in \mathbb{N}} \) converges to \( \hat{\xi} \in \hat{V} \). A sequence \( \{\xi_i\}_{i \in \mathbb{N}} \), where \( \xi_i \in \text{fin}(V, \| \cdot \|) \), is S-\( \| \cdot \| \)-Cauchy if
\[
\forall \epsilon \in \mathbb{R}^+ \, \exists n \in \mathbb{N} \, \forall k, l \in \mathbb{N} \, \left[ k, l > n \Rightarrow \| \xi_k - \xi_l \| < \epsilon \right].
\] (3)
A sequence \( \{\xi_i\}_{i \in \mathbb{N}} \) is S-\( \| \cdot \| \)-Cauchy if and only if the sequence \( \{\hat{\xi}_i\}_{i \in \mathbb{N}} \) is Cauchy.

A subset \( X \subset \text{fin}(V, \| \cdot \|) \) is S-\( \| \cdot \| \)-complete if for any S-\( \| \cdot \| \)-Cauchy sequence \( \{\xi_i\}_{i \in \mathbb{N}} \), there exists \( \xi \in X \) such that \( \{\xi_i\} \) approximately converges to \( \xi \) in the norm \( \| \cdot \| \). The subset \( X \) is S-\( \| \cdot \| \)-complete if and only if \( \hat{X} \) is complete in \( \hat{V} \), where \( \hat{X} = \{\hat{\xi} | \xi \in X\} \).

The following results, called the hull completeness theorem, is a fundamental property of an internal normed space \((V, \| \cdot \|)\). See Hurd and Loeb [3] for detail.

**Theorem 2.1.** The subspace \( \text{fin}(V) \) is S-complete in \( \| \cdot \| \).

**Corollary 2.2.** (The Hull Completeness Theorem) \( \hat{V} \) is a Banach space.

Let \( \mathcal{H} \) be an internal Hilbert space, and \( T : \mathcal{H} \rightarrow \mathcal{H} \) an internal bounded linear operator such that the bound \( \|T\| \) is finite. The bounded operator \( \hat{T} : \mathcal{H} \rightarrow \hat{\mathcal{H}} \), called the standard part of \( T \), is defined by the relation \( \hat{T} \hat{x} = \hat{T} \hat{x} \) for any \( x \in \text{fin}(\mathcal{H}) \).

For further information on nonstandard real analysis, we refer to Stroyan and Luxemburg [3] and Hurd and Loeb [2].

### 3. Several definitions of standard parts

We give several equivalent definitions of the standard part of an internal bounded self-adjoint operator which is not S-bounded.

The following lemma, which is a basic property for self-adjointness, is used to give the first definition of standard parts (see [8]).

**Lemma 3.1.** Let \( A \) be a symmetric operator on a Hilbert space \( \mathcal{H} \). Then, \( A \) is self-adjoint if and only if \( \text{Rng}(A \pm i) = \mathcal{H} \).

Let \( \mathcal{H} \) be an internal Hilbert space, and \( A \) an internal bounded self-adjoint operator on \( \mathcal{H} \). Let \( \hat{\mathcal{K}} = \text{Ker}([(A + i)^{-1}]^{-} \hat{\mathcal{K}} \). Using the unitarity of \( (A + i)(A - i)^{-1} \), we can easily check that \( \text{Ker}([(A - i)^{-1}]^{-} \hat{\mathcal{K}} \) is \( \hat{\mathcal{K}} \).

**Proposition 3.2.** There exists the unique (possibly unbounded) self-adjoint operator \( S \) on \( \hat{\mathcal{K}} \) satisfying
\[
(S + i)^{-1} = [(A + i)^{-1}]^{-} \hat{\mathcal{K}}.
\] (4)

**Proof.** We see \( \|(A + i)^{-1}\| < \infty \), and \( [(A + i)^{-1}]^{-} \) is an bounded normal operator on \( \hat{\mathcal{H}} \). The operator \( T := [(A + i)^{-1}]^{-} \hat{\mathcal{K}} \) is a bijection from \( \hat{\mathcal{K}} \) to \( [(A + i)^{-1}]^{-} \hat{\mathcal{K}} \). Hence the inverse \( T^{-1} \) from \( [(A + i)^{-1}]^{-} \hat{\mathcal{K}} \) to \( \hat{\mathcal{K}} \) is defined. Clearly the operator \( S = T^{-1} - i \) satisfies the equation (4).

We will show that \( S \) is symmetric. Let \( x_1, x_2 \in \text{Dom}(S) \) (\( = [(A + i)^{-1}]^{-} \hat{\mathcal{K}} \)). Then, we can show that there exist \( \xi_i \in x_i \) such that \( A\xi_i \in Sx_i \) (\( i = 1, 2 \)) as follows. There
are $y_i \in \hat{\cal K}$ and $\eta_i \in \cal H$ such that $(S + i)^{-1}y_i = [(A + i)^{-1}]^\perp y_i = x_i$ and $\eta_i \in y_i$. Let 
$\xi_i = (A + i)^{-1}\eta_i$. Then $\xi_i \in x_i$ and $(A + i)\xi_i = \eta_i \in y_i = (S + i)x_i$. Hence $A\xi_i \in Sx_i$. Thus, 
$\langle x_1, Sx_2 \rangle = \langle \xi_1, A\xi_2 \rangle = \langle (A\xi_1, \xi_2) = \langle Sx_1, x_2 \rangle$. Therefore, $S$ is symmetric.

To prove the self-adjointness, it is sufficient to show Rng$(S + i) = \text{Rng}(S - i) = \hat{\cal K}$ by Lemma 3.1. Clearly Rng$(S + i) = \text{Rng}(T^{-1}) = \hat{\cal K}$. Let $x \in \text{Dom}(S)$, $\xi \in x$ and $A\xi \in Sx$. Then we have

$$
\left( \frac{A - i}{A + i} \right)^\perp (S + i)x = \left( \frac{A - i}{A + i} (A + i)\xi \right)^\perp = (S - i)x.
$$

Thus, by the equation (4) with Ker$([A - i]^{-1})^\perp = \hat{\cal K}$, we have

$$
(S - i)^{-1} = [(A - i)\xi \in \hat{\cal K}.
$$

Therefore, we can show Rng$(S - i) = \hat{\cal K}$ in the similar way to the proof of Rng$(S + i) = \hat{\cal K}$. The uniqueness of $S$ is clear. QED

**Definition 3.3.** Under the condition of Proposition 3.2, define the self-adjoint operator $\text{st}_1(A)$ on $\hat{\cal K}$ by $(\text{st}_1(A) + i)^{-1} = [(A + i)^{-1}]^\perp|\hat{\cal K}$.

The operator $\text{st}_1(A)$ is called the standard part of $A$. We see that $\text{st}_1(A) = \hat{A}$ when $A$ is $S$-bounded.

**Definition 3.4.** Let $A$ be an internal bounded operator on $\cal H$, an internal Hilbert space. Define fin$(A) \subseteq \cal H$ by

$$
\text{fin}(A) = \{ \xi \in \text{fin}\cal H | A\xi \in \text{fin}\cal H \}.
$$

**Definition 3.5.** Let $A$ be an internal bounded self-adjoint operator on $\cal H$. Let $\hat{\cal K} = \text{fin}(A)^\perp = \{ \xi \in \text{fin}(A) \}$ be the closure of the subspace $\text{fin}(A)^\perp = \{ \xi \xi \in \text{fin}(A) \}$ of $\cal H$. Define the self-adjoint operator $\text{st}_2(A)$ on $\hat{\cal K}$ by

$$
\hat{e}^{it\text{st}_2(A)} = e^{it\hat{A}}|\hat{\cal K}, \quad t \in \mathbb{R}.
$$

We see that $\{e^{it\hat{A}}|\hat{\cal K}\}_{t \in \mathbb{R}}$ is one-parameter unitary group, since $\hat{\cal K}$ is invariant under $e^{it\hat{A}}$ for all $t \in \mathbb{R}$. We also see that it is strongly continuous as follows. Let $\xi \in \text{fin}(A)$. Then, we have $||(*d/dt)e^{it\hat{A}}\xi|| = ||ie^{it\hat{A}}A\xi|| < \infty$, where $*d/dt$ is the internal differentiation. This implies that $e^{it\hat{A}}\hat{\xi}$ is continuous with respect to $t \in \mathbb{R}$. Thus, $e^{it\hat{A}}$ is strongly continuous on $\text{fin}(A)^\perp$. Hence by Stone's theorem, $\text{st}_2(A)$ is uniquely defined.

If $A$ is $S$-bounded, $\text{st}_2(A)$ coincides with $\hat{A}$ defined in Section 2. This is seen from the following:

**Proposition 3.6.** Let $A$ be an internal $S$-bounded self-adjoint operator. Then,

$$
e^{it\hat{A}} = e^{it\hat{A}}, \quad t \in \mathbb{R}.
$$

for all $t \in \mathbb{R}$.
Proof. For any infinitesimal $\epsilon \in \star \mathbb{R}_0^+$,
\[
\epsilon^{-1}(e^{i\epsilon A} - I) \approx iA,
\] (10)
holds, because
\[
\|\epsilon^{-1}(e^{i\epsilon A} - I) - iA\| = \|\epsilon^{-1}\sum_{\nu=2}^{\infty}(i\epsilon A)^\nu/\nu!\| \leq \epsilon^{-1}\sum_{\nu=2}^{\infty}(\|A\|)^\nu/\nu!
\]
\[
= \epsilon^{-1}(e^{\|A\|} - 1) - \|A\| \approx 0.
\]
Thus, by the permanence principle,
\[
\forall \delta \in \mathbb{R}_+, \exists \epsilon \in \mathbb{R}_+, |t| < \epsilon \Rightarrow \|t^{-1}(e^{itA} - I) - iA\| < \delta
\] (11)
Hence, we have
\[
\lim_{\epsilonarrow 0} \|\epsilon^{-1}(\overline{e^{i\epsilon A}} - \hat{I}) - i\hat{A}\| = 0
\] (12)
Thus we have $(d/dt)e^{i\epsilon A}|_{t=0} = i\hat{A}$, where $d/dt$ is the usual differentiation. Because $(e^{i\epsilon A})_{\epsilon \in \mathbb{R}}$ is one-parameter unitary group, it follows that $e^{i\epsilon A} = e^{it\hat{A}}$. QED

Let $E(\cdot)$ be an internal projection-valued measure on $\star \mathbb{R}$, i.e., for each internal Borel set $\Omega \subseteq \star \mathbb{R}$, $E(\Omega)$ is an orthogonal projection on $\mathcal{H}$ such that
1. $E(\phi) = 0$, $E(\star \mathbb{R}) = I$
2. If $\Omega = \bigcup_{n=1}^{\infty} \Omega_n \star$ with $\Omega_n \cap \Omega_m = \phi$ if $n \neq m$, then $E(\Omega) = s-lim_{N \arrow \infty} \sum_{n=1}^{N} E(\Omega_n)$
3. $E(\Omega_1)E(\Omega_2) = E(\Omega_1 \cap \Omega_2)$.

For $r \in \star \mathbb{R}$, let $\mathcal{H}_r = \text{Rng}(E(\cdot-r))$, the range of $E((-r, r))$. Let $D(E) = \bigcup_{r \in \mathbb{R}^+} \mathcal{H}_{r} \cap \text{fin}\mathcal{H}$. $D(E)$ is called the standardization domain of $E(\cdot)$. Clearly, $D(E)^{\perp\perp} = (\bigcup_{r \in \mathbb{R}^+} \mathcal{H}_{r})^{\perp\perp}$.

For $a \in \mathbb{R}$, define the orthogonal projection $\hat{E}_{st}(-\infty, a]$ by
\[
\hat{E}_{st}(-\infty, a] = \sup\{\hat{E}(-K, a + \epsilon]\hat{D}(E)^{\perp\perp}K, \epsilon \in \mathbb{R}^+\}
\] (13)
\[
= s-lim_{n \arrow \infty} \hat{E}(-n, a + \frac{1}{n}]\hat{D}(E)^{\perp\perp}.
\] (14)
Then we see
\[
s-lim_{a \arrow -\infty} \hat{E}_{st}(-\infty, a] = 0
\] (15)
\[
s-lim_{\epsilon \downarrow 0} \hat{E}_{st}(-\infty, a + \epsilon] = \hat{E}_{st}(-\infty, a]
\] (16)
\[
a < b \Rightarrow \hat{E}_{st}(-\infty, a] \leq \hat{E}_{st}(-\infty, b].
\] (17)
Hence, $\hat{E}_{st}(-\infty, \cdot]$ defines a projection-valued measure on $\mathbb{R}$.

Definition 3.7. For any internal bounded self-adjoint operator $A$, define the self-adjoint operator $\text{st}_3(A)$ on $\overline{D(E)}^{\perp\perp}$ by
\[
\text{st}_3(A) = \int \lambda d\hat{E}_{st}(\lambda).
\] (18)
Proposition 3.8. Let $A$ be an internal bounded self-adjoint operator, and $E(\cdot)$ the internal projection-valued measure associated with the spectral decomposition of $A$. Then

$$\hat{D}(E)^{\perp\perp} = \overline{\text{fin}(A)^{\perp\perp}}.$$  \hspace{1cm} (19)

**Proof.** $\hat{D}(E)^{\perp\perp} \subseteq \overline{\text{fin}(A)^{\perp\perp}}$ is clear. To prove $\hat{D}(E)^{\perp\perp} \supseteq \overline{\text{fin}(A)^{\perp\perp}}$, it is sufficient to show that for any $\hat{x} \in \text{fin}(A)^{\perp}$ there is a sequence $\hat{x}_n \in \hat{D}(E)$ ($n \in \mathbb{N}$) such that $\hat{x}_n \to \hat{x}$. Let $x_n = E(-n, n)x$ ($n \in \mathbb{N}$). Notice that $\|A(x - x_n)\| \geq n\|x - x_n\|$. Suppose $\|x - x_n\| > \epsilon$ for all $n \in \mathbb{N}$. By the permanence principle, there is $N \in \mathbb{N}$ such that $\|A(x - x_n)\| \geq N\|x - x_N\| > N\epsilon \sim \infty$. This contradicts $\|A(x - x_N)\| \leq \|Ax\| < \infty$. \hspace{1cm} \textit{QED}

Theorem 3.9. Let $A$ be an internal bounded self-adjoint operator. Then,

$$\text{st}_2(A) = \int \lambda \text{d}\hat{E}_{\text{st}}(\lambda),$$  \hspace{1cm} (20)

and hence $\text{st}_2(A) = \text{st}_3(A)$.

**Proof.** It is sufficient to show

$$\langle \hat{x}, \exp(it\text{st}_2(A))\hat{x} \rangle = \int e^{it\lambda} \langle \hat{x}, \hat{E}_{\text{st}}(\lambda)\hat{x} \rangle$$  \hspace{1cm} (21)

for all $\hat{x} \in \text{fin}(A)^{\perp\perp}$. Define the internal Borel measure $\mu$ by $\mu(d\lambda) = \langle x, E(d\lambda)x \rangle$. Let $L_{\mu}$ denote the Loeb measure of $\mu$, and $L'\mu$ the Borel measure on $\mathbb{R}$ defined by $L'\mu(\Omega) = L_{\mu}(\text{st}^{-1}[\Omega])$. We can check that $L'\mu$ is well-defined (i.e., $\text{st}^{-1}[\Omega]$ is $L_{\mu}$-measurable for any Borel set $\Omega \subseteq \mathbb{R}$). We also see that $L_{\mu}$ is supported by $\text{fin}^*\mathbb{R}$, since $L_{\mu}([*\mathbb{R} \setminus \text{fin}^*\mathbb{R}]) \leq \circ\langle x, E([-n, n])x \rangle = \circ\|1 - \hat{E}(-n, n)\| \leq (1/n^2)^\circ\|Ax\|^2$ for all $n \in \mathbb{N}$. Therefore

$$\langle \hat{x}, \exp(it\text{st}_2(A))\hat{x} \rangle = \langle \hat{x}, e^{itA}\hat{x} \rangle = \circ\langle x, e^{itA}x \rangle = \circ\int_{\mathbb{R}} e^{it\lambda} d\mu(\lambda) = \int_{\mathbb{R}} e^{it\lambda} dL_{\mu}(\lambda) = \int_{\mathbb{R}} e^{it\lambda} dL'(\lambda).$$

On the other hand, for $a, b \in \mathbb{R}$ with $a < b$,

$$L'\mu(a, b) = L_{\mu}( \bigcup_{\epsilon \in \mathbb{R}^+} (a + \epsilon, b - \epsilon) ) = \lim_{\epsilon \downarrow 0} \circ\langle x, E(a + \epsilon, b - \epsilon)x \rangle = \lim_{\epsilon \downarrow 0} \langle \hat{x}, \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle = \langle \hat{x}, s\text{-lim}_{\epsilon \downarrow 0} \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle = \langle \hat{x}, \hat{E}_{\text{st}}(a, b)\hat{x} \rangle.$$

Hence, $L'\mu(\Omega) = \langle \hat{x}, \hat{E}_{\text{st}}(\Omega)\hat{x} \rangle$ for any Borel set $\Omega \subseteq \mathbb{R}$. \hspace{1cm} \textit{QED}
Let $C \in \mathbb{R}$ be a positive constant, and $h$ be an internal Borel function from $^*\mathbb{R}$ to $^*\mathbb{C}$ satisfying the following properties:

$$h(x) \approx h(y) \text{ iff } x \approx y \text{ for all } x, y \text{ with } |x|, |y| < \infty,$$

$$|h(x)| < C \text{ for all } x \in \mathbb{R}.$$ 

Define the function $\hat{h} : \mathbb{R} \to \mathbb{C}$ by

$$\hat{h}(x) = \circ h(x),$$

for $x \in \mathbb{R}$. We see that $\hat{h}$ is injective and continuous. Let $A$ be an internal bounded self-adjoint operator. Notice that $h(A)$ is an $S$-bounded internal normal operator.

**Theorem 3.10.** There exists the unique self-adjoint operator $B$ on $\text{fin}(A)^{_{\perp \perp}}$ such that

$$\hat{h}(B) = h\overline{(A)}|\text{fin}(A)^{_{\perp \perp}}. \quad (22)$$

Moreover, $B$ equals to $\text{st}_3(A)$.

**Proof.** By the argument similar to the proof of Theorem 3.9, we can show

$$\langle \hat{x}, \overline{h(A)}\hat{x} \rangle = \int_{\mathbb{R}} \hat{h}(\lambda) dL'\mu(\lambda)$$

$$= \int_{\mathbb{R}} \hat{h}(\lambda) \langle \hat{x}, d\hat{E}_\lambda(\lambda)\hat{x} \rangle$$

for any $\hat{x} \in \text{fin}(A)^{_{\perp \perp}}$. Thus,

$$h\overline{(A)}|\text{fin}(A)^{_{\perp \perp}} = \int_{\mathbb{R}} \hat{h}(\lambda) d\hat{E}_\lambda(\lambda).$$

Because $\hat{h}$ is injective, the unique self-adjoint operator $B$ satisfying (22) is $\text{st}_3(A) = \int_{\mathbb{R}} \lambda d\hat{E}_\lambda(\lambda)$. \textit{QED}

**Corollary 3.11.** Definition 3.3, 3.5 and 3.7 are equivalent, that is, $\text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A)$.

**Proof.** Let $h(x) = 1/(x + i)$. \textit{QED}

In section 2, $\hat{A}$ is defined only when $A$ is an internal $S$-bounded self-adjoint operator. Now we can extend the definition so as to include the case where $A$ is an internal bounded self-adjoint operator which is not $S$-bounded; $\hat{A} := \text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A)$.

**Definition 3.12.** Let $A$ be an internal linear operator on an internal Hilbert space $\mathcal{H}$. Let $D$ be an (external) subspace of $\text{fin}\mathcal{H}$. $A$ is standardizable on $D$ if $D \subset \text{fin}(A)$ and if for any $x, y \in D$, $x \approx y$ implies $Ax \approx Ay$. In this case, define the operator $\hat{A}_D$ with domain $\hat{D} = \{\hat{x} | x \in D\}$, called the standard part of $A$ on $D$, by

$$\hat{A}_D\hat{x} = \overline{Ax}, \quad x \in D. \quad (23)$$
Clearly, $A$ is standardizable on $D$ if and only if $D \subset \text{fin}(A)$, and if $A\xi \approx 0$ for all $\xi \in D$ with $\xi \approx 0$.

**Lemma 3.13.** An internal bounded operator $A$ is standardizable on $\text{fin}(A^*A)$.

**Proof.** First, we prove $\text{fin}(A^*A) \subset \text{fin}(A)$ as follows. Suppose that $\xi \in \text{fin}(A)$. Let $E(\cdot)$ be the internal spectral-valued mesure of the self-adjoint operator $A^*A$. Then, $\|A\xi\|^2 = \langle \xi, A^*A\xi \rangle = \langle \xi, E[0, 1]A^*A\xi \rangle + \langle \xi, (I - E[0, 1])A^*A\xi \rangle \leq \langle \xi, E[0, 1]A^*A\xi \rangle + \|A^*A\xi\|^2 < \infty$. Thus, $\xi \in \text{fin}(A)$. Second, suppose $x \approx 0$ and $\|A^*Ax\| < \infty$. Then, $\|Ax\|^2 = \langle x, A^*Ax \rangle < \|x\|\|A^*Ax\| \approx 0$. QED

**Corollary 3.14.** If $D \subseteq \text{fin}\mathcal{H}$ is invariant under $A$ and $A^*$, $A$ is standardizable on $D$.

The operator $B$ in the above proof is called a hyperfinite extension of $A$ [6]. We use the following lemma in the proof of Theorem 3.16.

**Lemma 3.15.** Let $A$ be a symmetric operator with domain $D \subset \mathcal{H}$, a Hilbert space. Let $D_1 \subset D$ be a dense linear subset of $\mathcal{H}$ and suppose that $A|D_1$ is essentially self-adjoint. Then, $A$ is essentially self-adjoint and $A \approx A|D_1$.

**Theorem 3.16.** Let $A$ be an internal self-adjoint operator on $\mathcal{H}$, and $E(\cdot)$ the projector-valued spectral measure of $A$. Then,

$$
\hat{A} = \overline{A_{D(E)}} = \overline{A_{\text{fin}(A^2)}}
$$

**Proof.** We can show that $\hat{A}_{D(E)}$ is essentially self-adjoint e.g. by Nelson’s analytic vector theorem. Hence, it has one and only one self-adjoint extension, its closure. Thus, it is sufficient to show that $\hat{A}$ is an extension of $\hat{A}_{D(E)}$. If $E(-r, r)\xi = \xi$ ($r \in \mathbb{R}^+, \xi \in \mathcal{H}$), then $E_{st}(-s, s)\xi = \hat{\xi}$ ($s \in \mathbb{R}^+, s > r$). Thus, $\hat{A}_{D(E)} = \hat{A}\xi = \{\mathcal{F}^\lambda \hat{E}_{st}(\lambda)\} = \int_\lambda \mathcal{F}^\lambda \hat{E}_{st}(\lambda)\lambda = \mathcal{S}_t\mathcal{S}_t(A) = \hat{A}\xi$. Therefore $\hat{A} = \hat{A}_{D(E)}$. $\hat{A}_{D(E)} = \hat{A}_{\text{fin}(A^2)}$ follows from $D(E) \subseteq \text{fin}(A^2)$ and Lemma 3.15. QED

4. The domain of $\hat{A}$

**Definition 4.1.** For an internal bounded self-adjoint operator $A$ on $\mathcal{H}$, define $D(A)$ by

$$
D(A) = \{\xi \in \text{fin}\mathcal{H} \mid \text{for all } t \in \mathbb{R}_0^+, e^{-t|A|A}\xi \approx A\xi \in \text{fin}\mathcal{H}\}.
$$

Clearly, $D(A)$ is a subspace of $\mathcal{H}$.

**Proposition 4.2.** An internal bounded self-adjoint operator $A$ is standardizable on $D(A)$. 

Proof. Let $\xi \in D(A)$ and $\|\xi\| \approx 0$. We can easily check $\|e^{-t|A|}A\| < \infty$ for all $t > 0$, $t \neq 0$. Hence, $^0\|A\xi\| \leq ^0\|e^{-t|A|}A\xi\| + ^0\|(1 - e^{-t|A|})A\xi\|$. By the S-boundedness of $e^{-t|A|}A$, the first term equals 0, and by the definition of $D(A)$, the second term equals 0. Thus we have $^0\|A\xi\| = 0$. $QED$

The following lemmas are easily shown.

Lemma 4.3. Let $f : \cdot \mapsto \cdot \mathbb{R}^+$ be internal and increasing. If $f(M) < \infty$ for some $M \sim \infty$, then

$$\lim_{n \to \infty} ^0f(n) < \infty.$$ 

Lemma 4.4. Under the same condition to Lemma 4.3, there is $K \sim \infty$ such that for all $L \sim \infty$,

$$f(K) \approx f(L) \text{ if } L \leq K.$$ 

Proposition 4.5. Let $\xi \in \text{fin}(\mathcal{H})$. For sufficiently large $t \approx 0$,

$$e^{-t|A|}\xi \in D(A).$$

(25)

Proof. Applying Lemma 4.4 to $f(n) = \|e^{-|A|/n}A\xi\|$, we find that for sufficiently small $K \sim \infty$ and $L \sim \infty$, $e^{-|A|/K}A\xi \approx e^{-|A|/L}A\xi$. Thus, for sufficiently large $s \approx 0$ and $t \approx 0$, $e^{-s|A|}A\xi \approx e^{-t|A|}A\xi$. Hence for all $x \approx 0$, $x > 0$,

$$e^{-t|A|}Ae^{-t|A|}\xi = e^{-x+t}|A|A\xi \approx Ae^{-t|A|}\xi.$$ 

Therefore, $e^{-t|A|}\xi \in D(A)$. $QED$

Theorem 4.6. Let $E(\cdot)$ be the spectral resolution of $A$ and $E_K = E(-K, K)$ for $K \in \mathbb{R}^+$. For any $\xi \in \text{fin}(A)$,

$$\xi \in D(A) \text{ iff } A\xi \approx E_KA\xi \text{ for all } K \sim \infty.$$ 

(26)

Remark. The right-hand condition is equivalent to

$$\lim_{K \to \infty} \sup_{K \leq K} \|I - E_K\|A\xi\| = 0.$$ 

(27)

Proof. Suppose that $\xi \in \text{fin}(A)$ and $A(I - E_K)\xi \approx 0$ for all $K \sim \infty$. For any $t \approx 0$, there exists a $K \sim \infty$ such that $tK \approx 0$. Thus,

$$\|e^{-t|A|}A\xi - A\xi\|^2 \approx \|e^{-t|A|}E_KA\xi - E_KA\xi\|^2 \approx \|e^{-t|A|}E_KA\xi - E_KA\xi\|^2 \approx \int_{-K}^{K} e^{-t|A|}\lambda - \lambda dE(\lambda)\xi\|^2 \approx \int_{-K}^{K} |(e^{-t|A|} - 1)\lambda| dE(\lambda)\xi\|^2 \approx \sup_{|\lambda| < K} |e^{-t|A|} - 1|^2 \int_{-K}^{K} \lambda^2 dE(\lambda)\xi\|^2 \approx 0.$$
Hence $\xi \in D(A)$.

Conversely, suppose $\xi \in D(A) \subset \text{fin}(A)$. Applying Lemma 4.4 to $f(n) = \|E_n A \xi\|$, we see that for sufficiently small $K \sim \infty$ and $L \sim \infty$ ($L \leq K$),

$$\|E_L A \xi\| \approx \|E_K A \xi\|.$$  

Thus, $(E_K - E_L) A \xi \approx 0$, since $\|E_L A \xi - E_K A \xi\|^2 = \|E_K A \xi\|^2 - \|E_L A \xi\|^2 \approx 0$. Let $t \in \mathbb{R}_0^+$ satisfy $tK \sim \infty$ so that

$$\|E_K A \xi - e^{-t|A|} A \xi\|$$

$$= \| \int_{-K}^{K} \lambda(1 - e^{-t|\lambda|})dE(\lambda)\xi - \int_{-\infty}^{K} e^{-t|\lambda|}dE(\lambda)\xi \|$$

$$\leq \| \int_{-K}^{K} \lambda(1 - e^{-t|\lambda|})dE(\lambda)\xi \| + e^{-tK}\|A\xi\|$$

$$\approx \| \int_{-K}^{K} \lambda(1 - e^{-t|\lambda|})dE(\lambda)\xi \|.$$  

Let $L \sim \infty$ satisfy $tL \approx 0$, so that the above

$$\leq \| \int_{-L}^{L} \lambda(1 - e^{-t|\lambda|})dE(\lambda)\xi \| + \|E_K - E_L\|A\xi\|$$

$$\approx 0.$$  

Thus, for sufficiently small $K \sim \infty$ and for any $t \approx 0$ such that $tK \sim \infty$,

$$E_K A \xi \approx e^{-t|A|} A \xi \approx A \xi.$$  

Since $\|A \xi - E_K A \xi\| \geq \|A \xi - E_K' A \xi\| > 0$ if $K < K'$, we have $E_K A \xi \approx A \xi$ holds for any $K' \sim \infty$. $QED$

**Proposition 4.7.** Let $\xi \in \text{fin}(A)$. Then, $E_K \xi \in D(A)$ for sufficiently small $K \sim \infty$.

**Proof.** Applying Lemma 4.4 to $f(n) = \|E_n A \xi\|$, we find that for sufficiently small $K, L \sim \infty$, $E_K A \xi \approx E_L A \xi$. Thus, if $L \sim \infty$, $L \leq K$, then $\|(1 - E_L)E_K A \xi\| = \|(E_K - E_L)A \xi\| \approx 0$. If $L > K$, clearly $(1 - E_L)E_K A \xi = 0$. Hence for all $L \sim \infty$, $E_K A \xi \approx E_L E_K A \xi$. Thus $E_K \xi \in D(A)$ by Theorem 4.6. $QED$

**Corollary 4.8.** $[\text{fin}(A)]^\sim = [D(A)]^\sim$, i.e., if $\xi \in \text{fin}(A)$, then there is $\eta \in D(A)$ such that $\eta \approx \xi$.

**Example** We have seen that the following relations hold:

$$\text{fin}(A^2) \subset D(A) \subset \text{fin}(A) \subset \text{fin} \mathcal{H},$$

$$[\text{fin}(A^2)]^\sim \subset [D(A)]^\sim = [\text{fin}(A)]^\sim \subset \hat{\mathcal{H}},$$
An example of $A$ such that $\text{fin}(A) \setminus D(A) \neq \emptyset$ is given as follows. Let $\nu$ be an infinite hypernatural number, and $\mathcal{H} = {}^*\mathbb{C}^\nu$, $\nu$-dimensional internal Hilbert space. Define the internal self-adjoint operator $A$ on $\mathcal{H}$ by $A(x_1, x_2, ..., x_\nu) = (x_1, 2x_2, ..., \nu x_\nu)$. Let $\xi = (0, 0, ..., 0, \nu^{-1})$. Then we see $\xi \in \text{fin}(A) \setminus D(A)$ from Theorem 4.6.

We also find $D(A) \setminus \text{fin}(A^2) \neq \emptyset$; let $\eta = (1^{-2}, 2^{-2}, ..., \nu^{-2})$, then we easily see $\eta \in D(A) \setminus \text{fin}(A^2)$. Moreover we find $\tilde{\eta} \in [D(A)]^\wedge \setminus [\text{fin}(A^2)]^{-}$. In fact, if $\eta' \approx \eta$, then $\circ||A^2\eta'|| \geq \lim_{n \to \infty} n \circ||A^2E_n\eta'|| = \lim_{n \to \infty} \sqrt{n} = \infty$. Thus, we have $\tilde{\eta} \not\in [\text{fin}(A^2)]^{-}$ by Theorem 4.6.

**Theorem 4.9.** Let $\xi \in \text{fin}(A)$, then

$$\xi \in D(A) \iff \lim_{t \to 0, \#^{0}} \left( \frac{e^{-t|A|}-1}{t} \xi \right)^{-} = -|A|\xi.$$  \hfill (28)

**Proof.** Suppose that the right-hand side does not hold. In other words, suppose that

$$\exists \varepsilon \in \mathbb{R}^{+} \forall n \in \mathbb{N} \exists t \in {}^*\mathbb{R}, \ 0 < t < \frac{1}{n} \land \left\| \left( \frac{e^{-t|A|}-1}{t} + |A|I \xi \right) \right\| > \varepsilon. \hfill (29)$$

By permanence,

$$\exists \varepsilon \in \mathbb{R}^{+} \exists N \in {}^*\mathbb{N}_{\infty} \exists t \in {}^*\mathbb{R}, \ 0 < t < \frac{1}{n} \land \left\| \left( \frac{e^{-t|A|}-1}{t} + |A| \right) \xi \right\| > \varepsilon. \hfill (30)$$

That is, there is positive infinitesimal $t$ such that $t^{-1}(e^{-t|A|} - 1)\xi \not\approx -|A|\xi$.

Thus, for some $\eta \in \text{fin}(\mathcal{H})$,

$$\Re\left( \eta, \frac{e^{-t|A|}-1}{t} \xi \right) \not\approx \Re(\eta, -|A|\xi).$$

Let $f(t) = \Re(\eta, e^{-t|A|} \xi)$. By the mean value theorem, for some $s \in {}^*\mathbb{R}$ with $0 < s < t$,

$$f'(s) = \frac{f(t) - f(0)}{t} = \Re\left( \eta, \frac{e^{-t|A|}-1}{t} \xi \right) \not\approx \Re(\eta, -|A|\xi).$$

Therefore, by the definition of $D(A)$, we have $\xi \in \text{fin}(A) \setminus D(A)$.

Conversely, suppose $\xi \in \text{fin}(A) \setminus D(A)$. Then, there is positive infinitesimal $t_0$ satisfying $e^{-t_0|A|}|A|\xi \not\approx |A|\xi$. Let $\eta = (|A| - e^{t_0|A|}|A|)\xi \in \text{fin}(\mathcal{H})$. Then this is equivalent to

$$\langle \eta, e^{-t_0|A|}|A|\xi \rangle \not\approx \langle \eta, |A|\xi \rangle. \hfill (31)$$

Let $f(x) = \langle \eta, e^{-|A|x} \xi \rangle$ ($x \in {}^*\mathbb{R}^{+}$). We see that $f'$ is increasing and $-\infty < f' < 0$, and hence $f$ is decreasing and $0 < f < \infty$. The relation (31) is equivalent to

$$f'(t_0) \not\approx f'(0), \hfill (32)$$
We have $f(x) \geq f'(t_0)(x - t_0) + f(t_0)$. Thus we have:

$$0 > \frac{f(x) - f(0)}{x} \geq \frac{f'(t_0)(x - t_0) + f(t_0) - f(0)}{x}.$$  \hspace{1cm} (33)

Let $F(x) = [f'(t_0)(x - t_0) + f(t_0) - f(0)]/x$, then for $c \in ^* \mathbb{R}^+$,

$$F(ct_0) = f'(t_0) \left( 1 - \frac{1}{c} \right) + \frac{1}{c} \frac{f(t_0) - f(0)}{t_0}.$$  \hspace{1cm} (34)

By the mean value theorem and $-\infty < f'(x) < 0$, we have $|f(x) - f(0)/x| < \infty$. Hence $F(ct_0) \approx f'(t_0)$ for all $c \sim \infty$. Thus, by (32) and (33),

$$0 > \frac{f(ct_0) - f(0)}{ct_0} \geq F(ct_0) \geq f'(t_0),$$  \hspace{1cm} (35)

for all $c \sim \infty$. Thus there is $\epsilon \in \mathbb{R}^+$ such that for sufficiently large $x \approx 0$, $\frac{f(x) - f(0)}{x} - f'(0) > \epsilon$. By the permanence principle, for sufficiently small $x \in \mathbb{R}^+$, $\frac{f(x) - f(0)}{x} - f'(0) > \epsilon$. We can check the relations

$$\langle \eta, \left( \frac{e^{-x|A|} - 1}{x} \right) \xi \rangle = \frac{f(x) - f(0)}{x}, \quad \langle \eta, |A|\xi \rangle = -f'(0), \quad \frac{e^{-x|A|} - 1}{x} > -|A|,$$

for $x > 0$. Therefore, using the increasingness of $(e^{-x|A|^{-1}})/x, x$, we have

$$\lim_{x \downarrow 0, x \neq 0} \omega \langle \eta, \frac{e^{-x|A|} - 1}{x} \xi \rangle \neq \langle \eta, -|A|\xi \rangle.$$

QED

**Theorem 4.10.** Let $A$ be an internal bounded self-adjoint operator. Then, $\hat{A} = \hat{A}_{D(A)}$.

**Proof.** By Theorem 3.16 and Lemma 3.15, it suffices to show that $\hat{A}_{D(A)}$ is a closed extension of $\hat{A}_{\text{fin}(A^2)}$. If $\xi \in \text{fin}(A^2)$, for any $K \sim \infty$, $\| (I - E_K)A \xi \| \leq \frac{1}{K} \| (I - E_K)A^2 \xi \| \leq \frac{1}{K} \| A^2 \xi \| \approx 0$. Hence $\xi \in D(A)$, and hence $\hat{A}_{D(A)}$ is an extension of $\hat{A}_{\text{fin}(A^2)}$.

To prove that $\hat{A}_{D(A)}$ is closed, it suffices to show that $\hat{D}(A) = [\hat{D}(A)]^c$ is complete in the norm $\| \cdot \|_A$ defined by $\| \xi \|_A = \| \xi \| + \| \hat{A} \xi \|$. Define the internal norm $\| \cdot \|_A$ on $\mathcal{H}$ by $\| \xi \|_A = \| \xi \| + \| A \xi \|$. We can check $\| \xi \|_A = \omega \| \xi \|_A$ for $\xi \in D(A)$.

By Theorem 2.1, $\text{fin}(A)$ is $S$-$\| \cdot \|_A$-complete. Hence, if the sequence $\{ \xi_i \}_{i \in \mathbb{N}} \subset D(A)$ ($\subset \text{fin}(A)$) is $S$-$\| \cdot \|_A$-Cauchy, then there is $\xi \in \text{fin}(A)$ such that $\xi_i$ approximately converges to $\xi$ in the norm $\| \cdot \|_A$. This $\xi$ is shown to be in $D(A)$ as follows. Regarding Theorem 4.6, and $\xi_i \in D(A)$ ($i < \infty$), this relation leads to $\omega \| (I - E_K)A \xi_i \| = \lim_{i \to \infty} \omega \| (I - E_K)A \xi_i \| = 0$, for any $K \sim \infty$. Therefore, from Theorem 4.6, we have $\xi \in D(A)$ and hence any Cauchy sequence in $\hat{D}(A)$ converges in $\hat{D}(A)$ in the norm $\| \cdot \|_A$. QED
Theorem 4.11. The domain \( D(A) \) is maximal. That is, if \( D(A) \subset S \subset \text{fin}(\mathcal{H}) \) and \( A \) is standardizable on \( S \), then \( S = D(A) \).

Proof. Suppose that \( D(A) \subset S \subset \text{fin}(\mathcal{H}) \) and that \( A \) is standardizable on \( S \). Let \( \eta \in S \). By Corollary 4.8 and \( \eta \in \text{fin}(A) \), there is \( \xi \in D(A) \) such that \( \xi \approx \eta \). By the definition of \( D(A) \) and the standardizability on \( S \), for all positive infinitesimal \( t \), \( e^{-t|A|A} \eta \approx e^{-t|A|} A \xi \approx A \xi \approx \eta \), since \( ||e^{-t|A||}|| \leq 1 \). Thus, \( \eta \in D(A) \). QED

Proposition 4.12. Let \( A \) be an internal positive operator on \( \mathcal{H} \). Then, for any \( \eta \in \text{fin}(A^{\frac{1}{2}}) \),

\[
\inf_{\xi\approx\eta} \langle \xi, A\xi \rangle = \inf_{\alpha\sim\infty} \langle \eta, E_{\alpha}A\eta \rangle. \tag{36}
\]

Proof. Suppose \( \eta \approx \xi \). If \( \alpha < \infty \), \( \langle \eta, E_{\alpha}A\eta \rangle \approx \langle \xi, E_{\alpha}A\xi \rangle \leq \langle \xi, A\xi \rangle \), that is,

\[
\forall \varepsilon \in \mathbb{R}^{+}, \forall \alpha < \infty, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon,
\]

Thus, by the permanence principle,

\[
\forall \varepsilon \in \mathbb{R}^{+}, \exists K \sim \infty, \forall \alpha \leq K, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon.
\]

By saturation,

\[
\exists K \sim \infty, \forall \varepsilon \in \mathbb{R}^{+}, \forall \alpha \leq K, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon.
\]

Hence we have

\[
\exists K \sim \infty, \quad \langle \eta, E_{K}A\eta \rangle \leq \langle \xi, A\xi \rangle.
\]

It follows that \( \inf_{\xi\approx\eta} \langle \xi, A\xi \rangle \geq \inf_{\alpha\sim\infty} \langle \eta, E_{\alpha}A\eta \rangle \).

On the other hand, we see that for all \( \alpha \sim \infty \), \( ||\eta - E_{\alpha}\eta||^{2} \leq \alpha^{-1}||A^{\frac{1}{2}}(\eta - E_{\alpha}\eta)||^{2} \leq \alpha^{-1}||A^{\frac{1}{2}}\eta||^{2} \approx 0 \). Hence,

\[
\forall \alpha \sim \infty, \quad \inf_{\xi\approx\eta} \langle \xi, A\xi \rangle \leq \langle E_{\alpha}\eta, AE_{\alpha}\eta \rangle = \langle \eta, E_{\alpha}A\eta \rangle.
\]

Thus it follows that \( \inf_{\xi\approx\eta} \langle \xi, A\xi \rangle \leq \inf_{\alpha\sim\infty} \langle \eta, E_{\alpha}A\eta \rangle \). QED

Proposition 4.13. Let \( A \) be an internal positive operator and \( \eta \in \text{fin}(A) \). Then,

\[
\inf_{\xi\approx\eta} \langle \xi, A\xi \rangle = \langle \hat{\eta}, \hat{A}\hat{\eta} \rangle. \tag{37}
\]

Proof. From Proposition 4.12, we see \( \inf_{\xi\approx\eta} \langle \xi, A\xi \rangle = \inf_{\alpha\sim\infty} \langle \eta, E_{\alpha}A\eta \rangle \). By Theorem 4.10 and Proposition 4.7, for sufficiently small \( \alpha \sim \infty \), \( \langle \eta, E_{\alpha}A\eta \rangle = \langle E_{\alpha}\eta, AE_{\alpha}\eta \rangle = \langle \hat{E}_{\alpha}\eta, \hat{A}\hat{E}_{\alpha}\eta \rangle = \langle \hat{\eta}, \hat{A}\hat{\eta} \rangle \). QED

Definition 4.14. Let \( A \) be a internal bounded positive operator, and \( D \subset \text{fin}(A^{\frac{1}{2}}) \). The sesquilinear form \( \langle \cdot, A\cdot \rangle \) is standardizable on \( D \) if \( \langle \xi_{1}, A\eta_{1} \rangle \approx \langle \xi_{2}, A\eta_{2} \rangle \) for all \( \xi_{1}, \xi_{2}, \eta_{1}, \eta_{2} \in D \) with \( \xi_{1} \approx \xi_{2} \) and \( \eta_{1} \approx \eta_{2} \).
Proposition 4.15. Let $D$ be a subspace of $\text{fin}(\mathcal{H})$ and $A \geq 0$. Then, $\langle \cdot, A \cdot \rangle$ is standardizable on $D$ if and only if $A^{\frac{1}{2}}$ is standardizable on $D$.

Proof. Suppose that $A^{\frac{1}{2}}$ is standardizable on $D$. Then $A^{\frac{1}{2}}\xi \approx A^{\frac{1}{2}}\eta$ for any $\xi, \eta \in D$ with $\xi \approx \eta$. Thus, $\langle \xi, A\xi \rangle = \|A^{\frac{1}{2}}\xi\|^2 \approx \|A^{\frac{1}{2}}\eta\|^2 = \langle \eta, A\eta \rangle$. Conversely, suppose that $\langle \cdot, A \cdot \rangle$ is standardizable on $D$. Then for any $\xi, \eta \in D$ with $\xi \approx \eta$, $\|A^{\frac{1}{2}}\xi - A^{\frac{1}{2}}\eta\|^2 = \|A^{\frac{1}{2}}(\xi - \eta)\|^2 = \langle \xi - \eta, A(\xi - \eta) \rangle \approx 0$. QED

Corollary 4.16. The set $D(A^{\frac{1}{2}})$ is a maximal domain of $\langle \cdot, A \cdot \rangle$, and $^0\langle \xi, A\eta \rangle = \langle A^{\frac{1}{2}}\xi, A^{\frac{1}{2}}\eta \rangle$ for any $\xi, \eta \in D(A^{\frac{1}{2}})$.

References