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Nonstandard Representations of Unbounded Self-Adjoint Operators

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1. Introduction

In nonstandard analysis, standardizations of internal (or nonstandard) objects have been studied for constructing standard mathematical objects; e.g. an internal measure space is converted into a measure space in the standard sense, called Loeb space ([1][2][3][4]). The standardization of an internal Hilbert space $\mathcal{H}$ is called the nonstandard hull of $\mathcal{H}$, written as $\hat{\mathcal{H}}$ (Henson and Moore [5]). Then the standardization of an internal operator $A$ on $\mathcal{H}$ with finite norm is naturally defined on $\mathcal{H}$, In this paper, the standardization of $A$ shall be called the standard part of $A$, written as $\hat{A}$. A prominent work of Moore [6] was focused on the case where $\mathcal{H}$ is hyperfinite-dimensional, and studied hyperfinite-dimensional extension of bounded operators on $\hat{\mathcal{H}}$. On the other hand, in the case where the norm of $A$ is not finite, it is not straightforward to give an adequate definition of the standard part of $A$. Albeverio et al. [4] defined $\hat{A}$ only when $\mathcal{H}$ is hyperfinite-dimensional real Hilbert space and $A$ is an internal positive symmetric operator on $\mathcal{H}$.

In this paper, we give a definition of $\hat{A}$ for any internal complex Hilbert space $\mathcal{H}$ and for any internal S-bonded self-adjoint operator $A$ on $\mathcal{H}$, as well as a general consideration on $\hat{A}$ so defined, which suggests the adequacy of the definition.

2. Preliminaries

We work in a $\aleph_1$-saturated nonstandard universe [7]. Note that every nonstandard universe constructed by a bounded ultrapower is $\aleph_1$-saturated.

Let $(V, \| \cdot \|)$ be an internal normed linear space. Define the subspaces $\mu(V, \| \cdot \|)$ and $\mathrm{fin}(V, \| \cdot \|)$ of $V$ by

$$
\mu(V, \| \cdot \|) = \{ \xi \in V \mid \|\xi\| \approx 0 \}, \quad \mathrm{fin}(V, \| \cdot \|) = \{ \xi \in V \mid \|\xi\| < \infty \}.
$$

We often abbreviate them as $\mu(V)$ and $\mathrm{fin}(V)$. Let $\hat{\xi} = \xi + \mu(V)$ and $\hat{V} = \mathrm{fin}(V)/\mu(V)$, the quotient space. We can naturally define the usual norm $\| \cdot \|$ on $\hat{V}$ by $\|\hat{\xi}\| = \circ\|\xi\|$. A countably infinite sequence $\{\xi_i\}_{i \in \mathbb{N}}$, where $\xi_i \in \mathrm{fin}(V, \| \cdot \|)$, approximately converges to $\xi \in V$ in the norm $\| \cdot \|$ if

$$
\forall \varepsilon \in \mathbb{R}^+ \exists n \in \mathbb{N} \forall k \in \mathbb{N} \quad [k > n \Rightarrow \|\xi - \xi_i\| < \varepsilon].
$$
A sequence $\{\xi_i\}_{i \in \mathbb{N}}$ approximately converges to $\xi \in V$ if and only if $\{\xi_i\}_{i \in \mathbb{N}}$ converges to $\xi \in \hat{V}$. A sequence $\{\xi_i\}_{i \in \mathbb{N}}$, where $\xi_i \in \text{fin}(V, \| \cdot \|)$, is $S$-||⋅||^-Cauchy if
\[ \forall \varepsilon \in \mathbb{R^+} \exists n \in \mathbb{N} \forall k, l \in \mathbb{N} \quad [k, l > n \Rightarrow \|\xi_k - \xi_l\| < \varepsilon]. \tag{3} \]
A sequence $\{\xi_i\}_{i \in \mathbb{N}}$ is $S$-||⋅||^-Cauchy if and only if the sequence $\{\xi_i\}_{i \in \mathbb{N}}$ is Cauchy.

A subset $X \subset \text{fin}(V, \| \cdot \|)$ is $S$-||⋅||^-complete if for any $S$-||⋅||^-Cauchy sequence $\{\xi_i\}_{i \in \mathbb{N}}$, there exists $\xi \in X$ such that $\{\xi_i\}$ approximately converges to $\xi$ in the norm $\| \cdot \|$. The subset $X$ is $S$-||⋅||^-complete if and only if $\hat{X}$ is complete in $\hat{V}$, where $\hat{X} = \{\xi | \xi \in X\}$.

The following results, called the hull completeness theorem, is a fundamental property of an internal normed space $(V, \| \cdot \|)$. See Hurd and Loeb [3] for detail.

**Theorem 2.1.** The subspace $\text{fin}(V)$ is $S$-complete in $\| \cdot \|$.

**Corollary 2.2.** (The Hull Completeness Theorem) $\hat{V}$ is a Banach space.

Let $\mathcal{H}$ be an internal Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ an internal bounded linear operator such that the bound $\|T\|$ is finite. The bounded operator $\hat{T} : \mathcal{H} \to \mathcal{H}$, called the standard part of $T$, is defined by the relation $\hat{T}x = \hat{T}x$ for any $x \in \text{fin}(\mathcal{H})$.

For further information on nonstandard real analysis, we refer to Stroyan and Luxemburg [3] and Hurd and Loeb [2].

### 3. Several definitions of standard parts

We give several equivalent definitions of the standard part of an internal bounded self-adjoint operator which is not $S$-bounded.

The following lemma, which is a basic property for self-adjointness, is used to give the first definition of standard parts (see [8]).

**Lemma 3.1.** Let $A$ be a symmetric operator on a Hilbert space $\mathcal{H}$. Then, $A$ is self-adjoint if and only if $\text{Rng}(A \pm i) = \mathcal{H}$.

Let $\mathcal{H}$ be an internal Hilbert space, and $A$ an internal bounded self-adjoint operator on $\mathcal{H}$. Let $\hat{\mathcal{K}} = \text{Ker}([(A + i)^{-1}]^\perp)$. Using the unitarity of $(A + i)(A - i)^{-1}$, we can easily check that $\text{Ker}([(A - i)^{-1}]^\perp) = \hat{\mathcal{K}}$.

**Proposition 3.2.** There exists the unique (possibly unbounded) self-adjoint operator $S$ on $\hat{\mathcal{K}}$ satisfying
\[ (S + i)^{-1} = [(A + i)^{-1}]^\perp |\hat{\mathcal{K}}. \tag{4} \]

**Proof.** We see $\|(A + i)^{-1}\| < \infty$, and $[(A + i)^{-1}]^\perp$ is an bounded normal operator on $\hat{\mathcal{H}}$. The operator $T := [(A + i)^{-1}]^\perp |\hat{\mathcal{K}}$ is a bijection from $\hat{\mathcal{K}}$ to $[(A + i)^{-1}]^\perp \hat{\mathcal{K}}$. Hence the inverse $T^{-1}$ from $[(A + i)^{-1}]^\perp \hat{\mathcal{K}}$ to $\hat{\mathcal{K}}$ is defined. Clearly the operator $S = T^{-1} - i$ satisfies the equation (4).

We will show that $S$ is symmetric. Let $x_1, x_2 \in \text{Dom}(S) (= [(A + i)^{-1}]^\perp \hat{\mathcal{K}})$. Then, we can show that there exist $\xi_i \in x_i$ such that $A\xi_i \in Sx_i$ ($i = 1, 2$) as follows. There
are \( y_i \in \hat{\mathcal{K}} \) and \( \eta_i \in \mathcal{H} \) such that \((S+i)^{-1}y_i = [(A+i)^{-1}]^\sim y_i = x_i \) and \( \eta_i \in y_i \). Let
\[
\xi_i = (A+i)^{-1}\eta_i.
\]
Then \( \xi_i \in x_i \) and \((A+i)\xi_i = \eta_i \in y_i = (S+i)x_i \). Hence \( A\xi_i \in Sx_i \). Thus, \( \langle x_1, Sx_2 \rangle = \circ\langle \xi_1, A\xi_2 \rangle = \circ\langle A\xi_1, \xi_2 \rangle = \langle Sx_1, x_2 \rangle \). Therefore, \( S \) is symmetric.

To prove the self-adjointness, it is sufficient to show \( \text{Rng}(S+i) = \text{Rng}(S-i) = \hat{\mathcal{K}} \) by Lemma 3.1. Clearly \( \text{Rng}(S+i) = \text{Rng}(T^{-1}) = \hat{\mathcal{K}} \). Let \( x \in \text{Dom}(S), \xi \in x \) and \( A\xi \in Sx \). Then we have
\[
(A-i)\mid (S+i) \hat{\mathcal{K}} = \hat{\mathcal{K}}.
\]
Thus, by the equation (4) with \( \text{Ker}([A-i])^{-1} = \hat{\mathcal{K}} \), we have
\[
(S-i)^{-1} = [(A_i-i)^{-1}]^\sim|\hat{\mathcal{K}}.
\]
Therefore, we can show \( \text{Rng}(S-i) = \hat{\mathcal{K}} \) in the similar way to the proof \( \text{Rng}(S+i) = \hat{\mathcal{K}} \).

The uniqueness of \( S \) is clear. \( QED \)

**Definition 3.3.** Under the condition of Proposition 3.2, define the self-adjoint operator \( \text{st}_1(A) \) on \( \hat{\mathcal{K}} \) by \( \text{st}_1(A) + i)^{-1} = [(A+i)^{-1}]^\sim|\hat{\mathcal{K}} \).

The operator \( \text{st}_1(A) \) is called the **standard part** of \( A \). We see that \( \text{st}_1(A) = \hat{A} \) when \( A \) is S-bounded.

**Definition 3.4.** Let \( A \) be an internal bounded operator on \( \mathcal{H} \), an internal Hilbert space. Define \( \text{fin}(A) \subseteq \mathcal{H} \) by
\[
\text{fin}(A) = \{ \xi \in \text{fin}\mathcal{H}|A\xi \in \text{fin}\mathcal{H}\}.
\]

**Definition 3.5.** Let \( A \) be an internal bounded self-adjoint operator on \( \mathcal{H} \). Let \( \hat{\mathcal{K}} \) be the closure of the subspace \( [\text{fin}(A)]^\sim = \{ \hat{\xi}|\xi \in \text{fin}(A)\} \) of \( \mathcal{H} \). Define the self-adjoint operator \( \text{st}_2(A) \) on \( \hat{\mathcal{K}} \) by
\[
e^{it\text{st}_2(A)} = e^{it\hat{A}}|\hat{\mathcal{K}}. \quad t \in \mathbb{R}.
\]

We see that \( \{ e^{it\hat{A}}|\hat{\mathcal{K}}\}_{t \in \mathbb{R}} \) is one-parameter unitary group, since \( \hat{\mathcal{K}} \) is invariant under \( e^{it\hat{A}} \) for all \( t \in \mathbb{R} \). We also see that it is strongly continuous as follows. Let \( \xi \in \text{fin}(A) \). Then, we have \( \|(\ast d/dt)e^{it\hat{A}}\xi\| = \|ie^{it\hat{A}}A\xi\| < \infty \), where \( \ast d/dt \) is the internal differentiation. This implies that \( e^{it\hat{A}}\xi \) is continuous with respect to \( t \in \mathbb{R} \). Thus, \( e^{it\hat{A}} \) is strongly continuous on \( \text{fin}(A) \). Hence by Stone’s theorem, \( \text{st}_2(A) \) is uniquely defined.

If \( A \) is S-bounded, \( \text{st}_2(A) \) coincides with \( \hat{A} \) defined in Section 2. This is seen from the following:

**Proposition 3.6.** Let \( A \) be an internal S-bounded self-adjoint operator. Then,
\[
e^{it\hat{A}} = e^{it\hat{A}},
\]
for all \( t \in \mathbb{R} \).
Proof. For any infinitesimal $\epsilon \in \star \mathbb{R}_0^+$,

$$e^{-1}(e^{itA} - I) \approx iA,$$

holds, because

$$||e^{-1}(e^{itA} - I) - iA|| = ||e^{-1}(i\epsilon\Lambda)\nu||/\nu! \leq \epsilon^{-1}\sum_{\nu=2}^{\infty}(\epsilon||\Lambda||)^{\nu}/\nu!.$$ 

Thus, by the permanence principle,

$$\forall \delta \in \mathbb{R}_+, \exists \epsilon \in \mathbb{R}_+, |t|<\epsilon \Rightarrow ||t^{-1}(e^{itA} - I) - iA|| < \delta.$$ 

Hence, we have

$$\lim_{\epsilon \rightarrow 0} ||e^{-1}(e^{i\epsilon A} - \hat{I}) - i\hat{A}|| = 0.$$ 

Thus we have $(d/dt)e^{itA}|_{t=0} = i\hat{A}$, where $d/dt$ is the usual differentiation. Because $(e^{itA})_{t \in \mathbb{R}}$ is one-parameter unitary group, it follows that $e^{itA} = e^{i\hat{A}}$. QED

Let $E(\cdot)$ be an internal projection-valued measure on $\star \mathbb{R}$, i.e., for each internal Borel set $\Omega \subseteq \star \mathbb{R}$, $E(\Omega)$ is an orthogonal projection on $\mathcal{H}$ such that

1. $E(\phi) = 0$, $E(\star \mathbb{R}) = I$

2. If $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$, then $E(\Omega) = \text{s-lim}_{N \rightarrow \infty} \sum_{n=1}^{N} E(\Omega_n)$

3. $E(\Omega_1)E(\Omega_2) = E(\Omega_1 \cap \Omega_2)$

For $r \in \star \mathbb{R}$, let $\mathcal{H}_r = \text{Rng}(E((-r, r)))$, the range of $E((-r, r))$. Let $D(E) = \bigcup_{r \in \mathbb{R}^+} \mathcal{H}_r \cap \text{finH}$. $D(E)$ is called the standardization domain of $E(\cdot)$. Clearly, $\overline{D(E)}^{\perp\perp} = (\bigcup_{r \in \mathbb{R}^+} \mathcal{H}_r)^{\perp\perp}$.

For $a \in \mathbb{R}$, define the orthogonal projection $\hat{E}_{st}(-\infty, a]$ by

$$\hat{E}_{st}(-\infty, a] = \text{sup}\{\hat{E}(-K, a + \epsilon)\overline{D(E)}^{\perp\perp} | K, \epsilon \in \mathbb{R}^+\}$$

Then we see

$$\text{s-lim}_{a \rightarrow -\infty} \hat{E}_{st}(-\infty, a] = 0$$

$$\text{s-lim}_{\epsilon \downarrow 0} \hat{E}_{st}(-\infty, a + \epsilon] = \hat{E}_{st}(-\infty, a]$$

$$a < b \Rightarrow \hat{E}_{st}(-\infty, a] \leq \hat{E}_{st}(-\infty, b].$$

Hence, $\hat{E}_{st}(-\infty, \cdot]$ defines a projection-valued measure on $\mathbb{R}$.

**Definition 3.7.** For any internal bounded self-adjoint operator $A$, define the self-adjoint operator $\text{st}_3(A)$ on $\overline{D(E)}^{\perp\perp}$ by

$$\text{st}_3(A) = \int \lambda d\hat{E}_{st}(\lambda).$$
Proposition 3.8. Let $A$ be an internal bounded self-adjoint operator, and $E(\cdot)$ the internal projection-valued measure associated with the spectral decomposition of $A$. Then

$$\hat{D}(E)^{\perp\perp} = \overline{\text{fin}(A)^{\perp\perp}}$$

(19)

Proof. $\hat{D}(E)^{\perp\perp} \subseteq \overline{\text{fin}(A)^{\perp\perp}}$ is clear. To prove $\hat{D}(E)^{\perp\perp} \supseteq \overline{\text{fin}(A)^{\perp\perp}}$, it is sufficient to show that for any $\hat{x} \in \overline{\text{fin}(A)^{\perp\perp}}$ there is a sequence $\hat{x}_n \in \hat{D}(E)$ ($n \in \mathbb{N}$) such that $\hat{x}_n \rightarrow \hat{x}$. Let $x_n = E(-n, n)x$ ($n \in \star \mathbb{N}$). Notice that $\|A(x-x_n)\| \geq n\|x-x_n\|$. Suppose $\|x-x_n\| > \epsilon$ for all $n \in \mathbb{N}$. By the permanence principle, there is $N \in \star \mathbb{N}_{\infty}$ such that $\|x-x_N\| > \epsilon$. Hence, $\|A(x-x_n)\| \geq N\|x-x_N\| > N\epsilon \sim \infty$. This contradicts $\|A(x-x_N)\| \leq \|Ax\| < \infty$. $QED$

Theorem 3.9. Let $A$ be an internal bounded self-adjoint operator. Then,

$$\text{st}_2(A) = \int \lambda d\hat{E}_{\text{st}}(\lambda),$$

(20)

and hence $\text{st}_2(A) = \text{st}_3(A)$.

Proof. It is sufficient to show

$$\langle \hat{x}, \exp(it\text{st}_2(A))\hat{x} \rangle = \int e^{it\lambda} \langle \hat{x}, d\hat{E}_{\text{st}}(\lambda)\hat{x} \rangle$$

(21)

for all $\hat{x} \in \overline{\text{fin}(A)^{\perp\perp}}$. Define the internal Borel measure $\mu$ by $\mu(d\lambda) = \langle x, E(d\lambda)x \rangle$. Let $L\mu$ denote the Loeb measure of $\mu$, and $L'\mu$ the Borel measure on $\mathbb{R}$ defined by $L'\mu(\Omega) = L\mu(\text{st}^{-1}[\Omega])$. We can check that $L'\mu$ is well-defined (i.e., $\text{st}^{-1}[\Omega]$ is $L\mu$-measurable for any Borel set $\Omega \subseteq \mathbb{R}$). We also see that $L\mu$ is supported by $\text{fin} \mathbb{R}$, since $L\mu(\star \mathbb{R} \setminus \text{fin} \mathbb{R}) \leq \circ \langle x, E(\star \mathbb{R} \setminus (-n, n))x \rangle = \circ \| (1-E(-n, n))x \|^2 \leq (1/n^2) \circ \|Ax\|^2$ for all $n \in \mathbb{N}$. Therefore

$$\langle \hat{x}, \exp(it\text{st}_2(A))\hat{x} \rangle = \langle \hat{x}, e^{itA}\hat{x} \rangle = \circ \langle x, e^{itA}x \rangle$$

$$= \circ \int_{\mathbb{R}} e^{it\lambda} d\mu(\lambda)$$

$$= \int_{\mathbb{R}} e^{it\lambda} dL'\mu(\lambda).$$

On the other hand, for $a, b \in \mathbb{R}$ with $a < b$,

$$L'\mu(a, b) = L\mu( \bigcup_{\epsilon \in \mathbb{R}^+} (a + \epsilon, b - \epsilon))$$

$$= \lim_{\epsilon \downarrow 0} \circ \langle x, E(a + \epsilon, b - \epsilon)x \rangle$$

$$= \lim_{\epsilon \downarrow 0} \langle \hat{x}, \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle$$

$$= \langle \hat{x}, s\lim_{\epsilon \downarrow 0} \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle$$

$$= \langle \hat{x}, \hat{E}_{\text{st}}(a, b)\hat{x} \rangle.$$  

Hence, $L'\mu(\Omega) = \langle \hat{x}, \hat{E}_{\text{st}}(\Omega)\hat{x} \rangle$ for any Borel set $\Omega \subseteq \mathbb{R}$. $QED$
Let $C \in \mathbb{R}$ be a positive constant, and $h$ be an internal Borel function from $^*\mathbb{R}$ to $^*\mathbb{C}$ satisfying the following properties:

\begin{align*}
  h(x) \approx h(y) \quad \text{iff} \quad x \approx y \quad \text{for all } x, y \text{ with } |x|, |y| < \infty, \\
  |h(x)| < C \quad \text{for all } x \in ^*\mathbb{R}.
\end{align*}

Define the function $\hat{h} : \mathbb{R} \rightarrow \mathbb{C}$ by

\[ \hat{h}(x) = ^{\circ}h(x), \]

for $x \in \mathbb{R}$. We see that $\hat{h}$ is injective and continuous. Let $A$ be an internal bounded self-adjoint operator. Notice that $h(A)$ is an S-bounded internal normal operator.

**Theorem 3.10.** There exists the unique self-adjoint operator $B$ on $\text{fin}(A)^{-\perp\perp}$ such that

\[ \hat{h}(B) = h(A)|\text{fin}(A)^{-\perp\perp}. \quad (22) \]

Moreover, $B$ equals to $\text{st}_3(A)$.

**Proof.** By the argument similar to the proof of Theorem 3.9, we can show

\[
\langle \hat{x}, \overline{h(A)}\hat{x} \rangle = \int_{^*\mathbb{R}} \hat{h}(\lambda)dL'd\mu(\lambda) \\
= \int_{^*\mathbb{R}} \hat{h}(\lambda)\langle \hat{x}, d\hat{E}_{\text{st}}(\lambda)\hat{x} \rangle
\]

for any $\hat{x} \in \text{fin}(A)^{-\perp\perp}$. Thus,

\[ \overline{h(A)}|\text{fin}(A)^{-\perp\perp} = \int_{^*\mathbb{R}} \hat{h}(\lambda)d\hat{E}_{\text{st}}(\lambda). \]

Because $\hat{h}$ is injective, the unique self-adjoint operator $B$ satisfying (22) is $\text{st}_3(A) = \int_{^*\mathbb{R}} \lambda d\hat{E}_{\text{st}}(\lambda)$. QED

**Corollary 3.11.** Definition 3.3, 3.5 and 3.7 are equivalent, that is, $\text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A)$.

**Proof.** Let $h(x) = 1/(x + i)$. QED

In section 2, $\hat{A}$ is defined only when $A$ is an internal S-bounded self-adjoint operator. Now we can extend the definition so as to include the case where $A$ is an internal bounded self-adjoint operator which is not S-bounded; $\hat{A} := \text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A).

**Definition 3.12.** Let $A$ be an internal linear operator on an internal Hilbert space $\mathcal{H}$. Let $D$ be an (external) subspace of fin$\mathcal{H}$. $A$ is standardizable on $D$ if $D \subset \text{fin}(A)$ and if for any $x, y \in D$, $x \approx y$ implies $Ax \approx Ay$. In this case, define the operator $\hat{A}_D$ with domain $D = \{ \hat{x} | x \in D \}$, called the standard part of $A$ on $D$, by

\[ \hat{A}_D \hat{x} = \overline{Ax}, \text{ } x \in D. \quad (23) \]
Clearly, $A$ is standardizable on $D$ if and only if $D \subset \text{fin}(A)$, and if $A\xi \approx 0$ for all $\xi \in D$ with $\xi \approx 0$.

**Lemma 3.13.** An internal bounded operator $A$ is standardizable on $\text{fin}(A^*A)$.

**Proof.** First, we prove $\text{fin}(A^*A) \subset \text{fin}(A)$ as follows. Suppose that $\xi \in \text{fin}(A)$. Let $E(\cdot)$ be the internal spectral-valued measure of the self-adjoint operator $A^*A$. Then, $\|A\xi\|^2 = \langle \xi, A^*A\xi \rangle = \langle \xi, E(0, 1)A^*A\xi \rangle + \langle \xi, (I - E(0, 1))(A^*A)^2\xi \rangle \leq \langle \xi, E(0, 1)A^*A\xi \rangle + \|A^*A\xi\|^2 < \infty$. Thus, $\xi \in \text{fin}(A)$. Second, suppose $x \approx 0$ and $\|A^*Ax\| < \infty$. Then, $\|Ax\|^2 = \langle x, A^*Ax \rangle \leq \|x\|\|A^*Ax\| \approx 0$. $\text{QED}$

**Corollary 3.14.** If $D \subseteq \text{fin}\mathcal{H}$ is invariant under $A$ and $A^*$, $A$ is standardizable on $D$.

The operator $B$ in the above proof is called a hyperfinite extension of $A$ [6].

We use the following lemma in the proof of Theorem 3.16.

**Lemma 3.15.** Let $A$ be a symmetric operator with domain $D \subset \mathcal{H}$, a Hilbert space. Let $D_1 \subset D$ be a dense linear subset of $\mathcal{H}$ and suppose that $A|D_1$ is essentially self-adjoint. Then, $A$ is essentially self-adjoint and $\overline{A} = \overline{A|D_1}$.

**Theorem 3.16.** Let $A$ be an internal self-adjoint operator on $\mathcal{H}$, and $E(\cdot)$ the projector-valued spectral measure of $A$. Then,

$$
\hat{A} = \overline{A_{D(E)}} = \overline{A_{\text{fin}(A^2)}}
$$

(24)

**Proof.** We can show that $\hat{A}_{D(E)}$ is essentially self-adjoint e.g. by Nelson’s analytic vector theorem. Hence, it has one and only one self-adjoint extension, its closure. Thus, it is sufficient to show that $\hat{A}$ is an extension of $\hat{A}_{D(E)}$. If $E(-r, r)\xi = \xi$ ($r \in \mathbb{R}^+$, $\xi \in \mathcal{H}$), then $E_{st}(-s, s)\xi = \xi$ ($s \in \mathbb{R}^+$, $s > r$). Thus, $\hat{A}_D\xi = \hat{A}\xi = \left(\int_{-s}^{s} \lambda dE_\lambda(\lambda)\right)^{-\hat{\xi}} = \int_{-s}^{s} \lambda dE_{st}(\lambda)\xi = \int \lambda dE_{st}(\lambda)\xi = \text{st}_3(A) = \hat{A}\xi$. Therefore $\hat{A} = \hat{A}_{D(E)}$. $\hat{A}_{D(E)} = \hat{A}_{\text{fin}(A^2)}$ follows from $D(E) \subseteq \text{fin}(A^2)$ and Lemma 3.15. $\text{QED}$

4. The domain of $\hat{A}$

**Definition 4.1.** For an internal bounded self-adjoint operator $A$ on $\mathcal{H}$, define $D(A)$ by

$$
D(A) = \{\xi \in \text{fin}\mathcal{H} \mid \text{for all } t \in \mathbb{R}^+, \; e^{-t|A|}\xi \approx A\xi \in \text{fin}\mathcal{H}\}.
$$

Clearly, $D(A)$ is a subspace of $\mathcal{H}$.

**Proposition 4.2.** An internal bounded self-adjoint operator $A$ is standardizable on $D(A)$.
Proof. Let $\xi \in D(A)$ and $||\xi|| \approx 0$. We can easily check $||e^{-t|A|}A|| < \infty$ for all $t > 0$, $t \neq 0$. Hence, $^o||A\xi|| \leq ^o||e^{-t|A|}A\xi|| + ^o||(1-e^{-t|A|})A\xi||$. By the S-boundedness of $e^{-t|A|}A$, the first term equals 0, and by the definition of $D(A)$, the second term equals 0. Thus we have $^o||A\xi|| = 0$. $QED$

The following lemmas are easily shown.

**Lemma 4.3.** Let $f : *N \rightarrow *R^+$ be internal and increasing. If $f(M) < \infty$ for some $M \sim \infty$, then

$$\lim_{n \rightarrow \infty} ^o f(n) < \infty.$$ 

**Lemma 4.4.** Under the same condition to Lemma 4.3, there is $K \sim \infty$ such that for all $L \sim \infty$,

$$f(K) \approx f(L) \text{ if } L \leq K.$$ 

**Proposition 4.5.** Let $\xi \in \text{fin}(H)$. For sufficiently large $t \approx 0$,

$$e^{-t|A|}\xi \in D(A).$$

(25)

**Proof.** Applying Lemma 4.4 to $f(n) = ||e^{-|A|/n}A\xi||$, we find that for sufficiently small $K \sim \infty$ and $L \sim \infty$, $e^{-|A|/K}A\xi \approx e^{-|A|/L}A\xi$. Thus, for sufficiently large $s \approx 0$ and $t \approx 0$, $e^{-|A|}A\xi \approx e^{-|A|}A\xi$. Hence for all $x \approx 0$, $x > 0$,

$$e^{-|x|}Ae^{-t|A|}\xi = e^{-(x+t)|A|}A\xi \approx Ae^{-t|A|}\xi.$$ 

Therefore, $e^{-t|A|}\xi \in D(A)$. $QED$

**Theorem 4.6.** Let $E(\cdot)$ be the spectral resolution of $A$ and $E_K = E(-K, K)$ for $K \in *R^+$. For any $\xi \in \text{fin}(A)$,

$$\xi \in D(A) \text{ iff } A\xi \approx E_KA\xi \text{ for all } K \sim \infty.$$ 

(26)

**Remark.** The right-hand condition is equivalent to

$$\lim_{k \rightarrow \infty \text{ for } K \in *R} ^o ||(I-E_K)A\xi|| = 0.$$ 

(27)

**Proof.** Suppose that $\xi \in \text{fin}(A)$ and $A(I - E_K)\xi \approx 0$ for all $K \sim \infty$. For any $t \approx 0$, there exists a $K \sim \infty$ such that $tK \approx 0$. Thus,

$$||e^{-t|A|}A\xi - A\xi||^2 \approx ||e^{-t|A|}E_KA\xi - E_KA\xi||^2$$

$$= || \int_{-K}^{K} e^{-t|\lambda|}e^{-\lambda t}dE(\lambda)\xi||^2$$

$$= \int_{-K}^{K} |(e^{-t|\lambda|} - 1)\lambda|^2 ||dE(\lambda)\xi||^2$$

$$\leq \sup_{|\lambda| < K} |e^{-t|\lambda|} - 1|^2 \int_{-K}^{K} \lambda^2 ||dE(\lambda)\xi||^2$$

$$= \sup_{|\lambda| < K} |e^{-t|\lambda|} - 1||^2 ||E_KA\xi||^2$$

$$\approx 0.$$
Hence $\xi \in D(A)$.

Conversely, suppose $\xi \in D(A)$ ($\subset \text{fin}(A)$). Applying Lemma 4.4 to $f(n) = \|E_n A \xi\|$, we see that for sufficiently small $K \sim \infty$ and $L \sim \infty$ ($L \leq K$),

$$\|E_L A \xi\| \approx \|E_K A \xi\|.$$ 

Thus, $(E_K - E_L) A \xi \approx 0$, since $\|E_L A \xi - E_K A \xi\|^2 = \|E_K A \xi\|^2 - \|E_L A \xi\|^2 \approx 0$. Let $t \in \mathbb{R}^+_0$ satisfy $tK \sim \infty$ so that

$$\|E_K A \xi - e^{-t|A|} A \xi\| \approx 0.$$ 

Let $L \sim \infty$ satisfy $tL \approx 0$, so that the above

$$\|E_K A \xi - E_L E_K A \xi\| \approx 0.$$ 

Then, for sufficiently small $K \sim \infty$ and for any $t \approx 0$ such that $tK \sim \infty$,

$$E_K A \xi \approx e^{-t|A|} A \xi \approx A \xi.$$ 

Since $\|A \xi - E_K A \xi\| \geq \|A \xi - E_{K'} A \xi\| > 0$ if $K < K'$, we have $E_K A \xi \approx A \xi$ holds for any $K' \sim \infty$. QED

**Proposition 4.7.** Let $\xi \in \text{fin}(A)$. Then, $E_K \xi \in D(A)$ for sufficiently small $K \sim \infty$.

**Proof.** Applying Lemma 4.4 to $f(n) = \|E_n A \xi\|$, we find that for sufficiently small $K, L \sim \infty$, $E_K A \xi \approx E_L A \xi$. Thus, if $L \sim \infty$, $L \leq K$, then $\|(1 - E_L) E_K A \xi\| = \|(E_K - E_L) A \xi\| \approx 0$. If $L > K$, clearly $(1 - E_L) E_K A \xi = 0$. Hence for all $L \sim \infty$, $E_K A \xi \approx E_L E_K A \xi$. Thus $E_K \xi \in D(A)$ by Theorem 4.6. QED

**Corollary 4.8.** $[\text{fin}(A)]^\subset = [D(A)]^\subset$, i.e., if $\xi \in \text{fin}(A)$, then there is $\eta \in D(A)$ such that $\eta \approx \xi$.

**Example** We have seen that the following relations hold:

$\text{fin}(A^2) \subset D(A) \subset \text{fin}(A) \subset \text{fin}\mathcal{H}$,

$[\text{fin}(A^2)]^\subset \subset [D(A)]^\subset = [\text{fin}(A)]^\subset \subset \mathcal{H}$,
\[\text{fin}(A^2)^{\perp\perp} = [D(A)]^{\perp\perp} = [\text{fin}(A)]^{\perp\perp} \subset \mathcal{H}.\]

An example of \(A\) such that \(\text{fin}(A) \setminus D(A) \neq \emptyset\) is given as follows. Let \(\nu\) be an infinite hypernatural number, and \(\mathcal{H} = *\mathbb{C}^\nu\), \(\nu\)-dimensional internal Hilbert space. Define the internal self-adjoint operator \(A\) on \(\mathcal{H}\) by \(A(x_1, x_2, \ldots, x_\nu) = (x_1, 2x_2, \ldots, \nu x_\nu)\). Let \(\xi = (0, 0, \ldots, 0, \nu^{-1})\). Then we see \(\xi \in \text{fin}(A) \setminus D(A)\) from Theorem 4.6.

We also find \(D(A) \setminus \text{fin}(A^2) \neq \emptyset\); let \(\eta = (1^{-2}, 2^{-2}, \ldots, \nu^{-2})\), then we easily see \(\eta \in D(A) \setminus \text{fin}(A^2)\). Moreover we find \(\tilde{\eta} \in [D(A)]^\perp \setminus [\text{fin}(A^2)]^{-}\). In fact, if \(\eta' \approx \eta\), then
\[
\circ \| A^2 \eta' \| \geq \lim_{n \to \infty} \circ \| A^2 E_n \eta' \| = \lim_{n \to \infty} \sqrt{n} = \infty.
\]
Thus, we have \(\tilde{\eta} \notin [\text{fin}(A^2)]^{-}\) by Theorem 4.6.

**Theorem 4.9.** Let \(\xi \in \text{fin}(A)\), then
\[
\xi \in D(A) \iff \lim_{t \downarrow 0, \iota \#^{0}} (e^{-t|A|}-1) = -|\overline{A}|\xi.
\]

**Proof.** Suppose that the right-hand side does not hold. In other words, suppose that
\[
\exists \varepsilon \in \mathbb{R}^+ \forall n \in \mathbb{N} \exists t \in *\mathbb{R}, \ 0 < t < \frac{1}{n} \land \left\| \left( \frac{e^{-t|A|}-1}{t} + |A| \right) \xi \right\| > \varepsilon.
\]
By permanence,
\[
\exists \varepsilon \in \mathbb{R}^+ \exists N \in *\mathbb{N} \exists t \in *\mathbb{R}, \ 0 < t < \frac{1}{n} \land \left\| \left( \frac{e^{-t|A|}-1}{t} + |A| \right) \xi \right\| > \varepsilon.
\]
That is, there is positive infinitesimal \(t\) such that \(t^{-1}(e^{-t|A|}-1)\xi \neq -|A|\xi\).

Thus, for some \(\eta \in \text{fin}(\mathcal{H})\),
\[
\Re \langle \eta, e^{-t|A|} - 1 \rangle \neq \Re \langle \eta, -|A| \rangle.
\]
Let \(f(t) = \Re \langle \eta, e^{-t|A|} \rangle\). By the mean value theorem, for some \(s \in *\mathbb{R}\) with \(0 < s < t\),
\[
f'(s) = \frac{f(t) - f(0)}{t} = \Re \langle \eta, e^{-t|A|} - 1 \rangle \neq \Re \langle \eta, -|A| \rangle.
\]
Therefore, by the definition of \(D(A)\), we have \(\xi \in \text{fin}(A) \setminus D(A)\).

Conversely, suppose \(\xi \in \text{fin}(A) \setminus D(A)\). Then, there is positive infinitesimal \(t_0\) satisfying \(e^{-t_0|A|} |A| \xi \neq |A| \xi\). Let \(\eta = (|A| - e^{-t_0|A|}) \xi \in \text{fin}(\mathcal{H})\). Then this is equivalent to
\[
\langle \eta, e^{-t_0|A|} |A| \xi \rangle \neq \langle \eta, |A| \xi \rangle.
\]
Let \(f(x) = \langle \eta, e^{-x|A|} \rangle\). We see that \(f'\) is increasing and \(-\infty < f' < 0\), and hence \(f\) is decreasing and \(0 < f < \infty\). The relation (31) is equivalent to
\[
f'(t_0) \neq f'(0),
\]
(32)
We have \( f(x) \geq f'(t_0)(x - t_0) + f(t_0) \). Thus we have

\[
0 > \frac{f(x) - f(0)}{x} \geq \frac{f'(t_0)(x - t_0) + f(t_0) - f(0)}{x}.
\]

(33)

Let \( F(x) = (f'(t_0)(x - t_0) + f(t_0) - f(0))/x \), then for \( c \in \mathbb{R}^+ \),

\[
F(ct_0) = f'(t_0) \left( 1 - \frac{1}{c} \right) + \frac{1}{c} \frac{f(t_0) - f(0)}{t_0}.
\]

(34)

By the mean value theorem and \(-\infty < f'(x) < 0\), we have \(|(f(x) - f(0))/x| < \infty\). Hence \( F(ct_0) \approx f'(t_0) \) for all \( c \sim \infty \). Thus, by (32) and (33),

\[
0 > \frac{f(ct_0) - f(0)}{ct_0} \geq F(ct_0) \sim f'(t_0),
\]

(35)

for all \( c \sim \infty \). Thus there is \( \varepsilon \in \mathbb{R}^+ \) such that for sufficiently large \( x \approx 0 \), \( \frac{f(x) - f(0)}{x} - f'(0) > \varepsilon \). By the permanence principle, for sufficiently small \( x \in \mathbb{R}^+ \), \( \frac{f(x) - f(0)}{x} - f'(0) > \varepsilon \). We can check the relations

\[
\left\langle \eta, \left( e^{-x|A|} - 1 \right)/x \right\rangle \xi = \frac{f(x) - f(0)}{x}, \quad \left\langle \eta, |A|\xi \right\rangle = -f'(0), \quad \frac{e^{-x|A|} - 1}{x} > -|A|,
\]

for \( x > 0 \). Therefore, using the increasingness of \((e^{-x||A|-1}|)/x, x, we have

\[
\lim_{x \downarrow 0, x \neq 0} \circ \left\langle \eta, e^{-x|A|} - 1/x \right\rangle \xi \neq \left\langle \eta, -|A|\xi \right\rangle.
\]

QED

**Theorem 4.10.** Let \( A \) be an internal bounded self-adjoint operator. Then, \( \hat{A} = \hat{A}_{D(A)} \).

**Proof.** By Theorem 3.16 and Lemma 3.15, it suffices to show that \( \hat{A}_{D(A)} \) is a closed extension of \( \hat{A}_{\text{fin}}(A_2) \). If \( \xi \in \text{fin}(A^2) \), for any \( K \sim \infty \), \( ||(1 - E_K)A\xi|| \leq \frac{1}{K}||(1 - E_K)A^2\xi|| \leq \frac{1}{K} ||A^2\xi|| \approx 0 \). Hence \( \xi \in D(A) \), and hence \( \hat{A}_{D(A)} \) is an extension of \( \hat{A}_{\text{fin}}(A^2) \).

To prove that \( \hat{A}_{D(A)} \) is closed, it suffices to show that \( \hat{D}(A) = [D(A)]^\perp \) is complete in the norm \( \| \cdot \|_A \) defined by \( \|\xi\|_A = ||\xi|| + ||\hat{A}\xi|| \). Define the internal norm \( \| \cdot \|_A \) on \( \mathcal{H} \) by \( ||\xi||_A = ||\xi|| + ||A\xi|| \). We can check \( \|\xi\|_A = \circ ||\xi||_A \) for \( \xi \in D(A) \).

By Theorem 2.1, \( \text{fin}(A) \) is \( \mathcal{S} \)-\( \| \cdot \|_A \)-complete. Hence, if the sequence \( \{\xi_i\}_{i \in \mathbb{N}} \subset D(A) (\subset \text{fin}(A)) \) is \( \mathcal{S} \)-\( \| \cdot \|_A \)-Cauchy, then there is \( \xi \in \text{fin}(A) \) such that \( \{\xi_i\} \) approximately converges to \( \xi \) in the norm \( \| \cdot \|_A \). This \( \xi \) is shown to be in \( D(A) \) as follows. Regarding Theorem 4.6, and \( \xi_i \in D(A) (i < \infty) \), this relation leads to \( \circ ||(I - E_K)A\xi_i|| = \lim_{i \to \infty} \circ ||(I - E_K)A\xi_i|| = 0 \), for any \( K \sim \infty \). Therefore, from Theorem 4.6, we have \( \xi \in D(A) \) and hence any Cauchy sequence in \( \hat{D}(A) \) converges in \( \hat{D}(A) \) in the norm \( \| \cdot \|_A \). QED
**Theorem 4.11.** The domain $D(A)$ is maximal. That is, if $D(A) \subset S \subset \text{fin}(\mathcal{H})$ and $A$ is standardizable on $S$, then $S = D(A)$.

*Proof.* Suppose that $D(A) \subset S \subset \text{fin}(\mathcal{H})$ and that $A$ is standardizable on $S$. Let $\eta \in S$. By Corollary 4.8 and $\eta \in \text{fin}(A)$, there is $\xi \in D(A)$ such that $\xi \approx \eta$. By the definition of $D(A)$ and the standardizability on $S$, for all positive infinitesimal $t$, $e^{-t|A|A}\eta \approx e^{-t|A|A}\xi \approx A\xi \approx A\eta$, since $\|e^{-t|A|A}\| \leq 1$. Thus, $\eta \in D(A)$. QED

**Proposition 4.12.** Let $A$ be an internal positive operator on $\mathcal{H}$. Then, for any $\eta \in \text{fin}(A^{\frac{1}{2}})$,

$$\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \inf_{\alpha \approx \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle.$$  \hfill (36)

*Proof.* Suppose $\eta \approx \xi$. If $\alpha < \infty$, $\langle \eta, E_{\alpha}A\eta \rangle \approx \langle \xi, E_{\alpha}A\xi \rangle \leq \langle \xi, A\xi \rangle$, that is,

$$\forall \varepsilon \in \mathbb{R}^{+}, \forall \alpha < \infty, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon,$$

Thus, by the permanence principle,

$$\forall \varepsilon \in \mathbb{R}^{+}, \exists K \sim \infty, \forall \alpha \leq K, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon.$$ 

By saturation,

$$\exists K \sim \infty, \forall \varepsilon \in \mathbb{R}^{+}, \forall \alpha \leq K, \quad \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \varepsilon.$$ 

Hence we have

$$\exists K \sim \infty, \quad \circ \langle \eta, E_{K}A\eta \rangle \leq \circ \langle \xi, A\xi \rangle.$$ 

It follows that $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \geq \inf_{\alpha \approx \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle$.

On the other hand, we see that for all $\alpha \sim \infty$, $\|\eta - E_{\alpha}\eta\|^{2} \leq \alpha^{-1}\|A^{\frac{1}{2}}(\eta - E_{\alpha}\eta)\|^{2} \leq \alpha^{-1}\|A^{\frac{1}{2}}\eta\|^{2} \approx 0$. Hence,

$$\forall \alpha \sim \infty, \quad \inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \leq \circ \langle E_{\alpha}\eta, AE_{\alpha}\eta \rangle = \circ \langle \eta, E_{\alpha}A\eta \rangle.$$ 

Thus it follows that $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle \leq \inf_{\alpha \approx \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle$. QED

**Proposition 4.13.** Let $A$ be an internal positive operator and $\eta \in \text{fin}(A)$. Then,

$$\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \langle \hat{\eta}, \hat{A}\hat{\eta} \rangle.$$  \hfill (37)

*Proof.* From Proposition 4.12, we see $\inf_{\xi \approx \eta} \circ \langle \xi, A\xi \rangle = \inf_{\alpha \approx \infty} \circ \langle \eta, E_{\alpha}A\eta \rangle$. By Theorem 4.10 and Proposition 4.7, for sufficiently small $\alpha \sim \infty$, $\circ \langle \eta, E_{\alpha}A\eta \rangle = \circ \langle E_{\alpha}\eta, AE_{\alpha}\eta \rangle = \langle \hat{E}_{\alpha}\eta, \hat{A}\hat{E}_{\alpha}\eta \rangle = \langle \hat{\eta}, \hat{A}\hat{\eta} \rangle$. QED

**Definition 4.14.** Let $A$ be a internal bounded positive operator, and $D \subset \text{fin}(A^{\frac{1}{2}})$. The sesquilinear form $\langle \cdot, A\cdot \rangle$ is standardizable on $D$ if $\langle \xi_{1}, A\eta_{1} \rangle \approx \langle \xi_{2}, A\eta_{2} \rangle$ for all $\xi_{1}, \xi_{2}, \eta_{1}, \eta_{2} \in D$ with $\xi_{1} \approx \xi_{2}$ and $\eta_{1} \approx \eta_{2}$.
Proposition 4.15. Let $D$ be a subspace of $\text{fin}(\mathcal{H})$ and $A \geq 0$. Then, $\langle \cdot, A \cdot \rangle$ is standardizable on $D$ if and only if $A^{\frac{1}{2}}$ is standardizable on $D$.

Proof. Suppose that $A^{\frac{1}{2}}$ is standardizable on $D$. Then $A^{\frac{1}{2}}\xi \approx A^{\frac{1}{2}}\eta$ for any $\xi, \eta \in D$ with $\xi \approx \eta$. Thus, $\langle \xi, A\xi \rangle = \|A^{\frac{1}{2}}\xi\|^2 \approx \|A^{\frac{1}{2}}\eta\|^2 = \langle \eta, A\eta \rangle$. Conversely, suppose that $\langle \cdot, A \cdot \rangle$ is standardizable on $D$. Then for any $\xi, \eta \in D$ with $\xi \approx \eta$, $\|A^{\frac{1}{2}}\xi - A^{\frac{1}{2}}\eta\|^2 = \|A^{\frac{1}{2}}(\xi - \eta)\|^2 = \langle \xi - \eta, A(\xi - \eta) \rangle \approx 0$. QED

Corollary 4.16. The set $D(A^{\frac{1}{2}})$ is a maximal domain of $\langle \cdot, A \cdot \rangle$, and $\langle \xi, A\eta \rangle = \langle A^{\frac{1}{2}}\xi, A^{\frac{1}{2}}\eta \rangle$ for any $\xi, \eta \in D(A^{\frac{1}{2}})$.

References