Nonstandard Representations of Unbounded Self-Adjoint Operators

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1. Introduction

In nonstandard analysis, standardizations of internal (or nonstandard) objects have been studied for constructing standard mathematical objects; e.g. an internal measure space is converted into a measure space in the standard sense, called Loeb space ([1][2][3][4]). The standardization of an internal Hilbert space $\mathcal{H}$ is called the nonstandard hull of $\mathcal{H}$, written as $\hat{\mathcal{H}}$ (Henson and Moore [5]). Then the standardization of an internal operator $A$ on $\mathcal{H}$ with finite norm is naturally defined on $\mathcal{H}$. In this paper, the standardization of $A$ shall be called the standard part of $A$, written as $\hat{A}$. A prominent work of Moore [6] was focused on the case where $\mathcal{H}$ is hyperfinite-dimensional, and studied hyperfinite-dimensional extension of bounded operators on $\hat{\mathcal{H}}$. On the other hand, in the case where the norm of $A$ is not finite, it is not straightforward to give an adequate definition of the standard part of $A$. Albeverio et al. [4] defined $\hat{A}$ only when $\mathcal{H}$ is hyperfinite-dimensional real Hilbert space and $\hat{A}$ is an internal positive symmetric operator on $\mathcal{H}$.

In this paper, we give a definition of $\hat{A}$ for any internal complex Hilbert space $\mathcal{H}$ and for any internal S-bonded self-adjoint operator $A$ on $\mathcal{H}$, as well as a general consideration on $\hat{A}$ so defined, which suggests the adequacy of the definition.

2. Preliminaries

We work in a $\aleph_1$-saturated nonstandard universe [7]. Note that every nonstandard universe constructed by a bounded ultrapower is $\aleph_1$-saturated.

Let $(V, || \cdot ||)$ be an internal normed linear space. Define the subspaces $\mu(V, || \cdot ||)$ and $\text{fin}(V, || \cdot ||)$ of $V$ by

$$
\mu(V, || \cdot ||) = \{ \xi \in V | ||\xi|| \approx 0 \}, \quad \text{fin}(V, || \cdot ||) = \{ \xi \in V | ||\xi|| < \infty \}.
$$

(1)

We often abbreviate them as $\mu(V)$ and $\text{fin}(V)$. Let $\hat{\xi} = \xi + \mu(V)$ and $\hat{V} = \text{fin}(V)/\mu(V)$, the quotient space. We can naturally define the usual norm $|| \cdot ||$ on $\hat{V}$ by $||\xi|| = ^*||\xi||$. A countably infinite sequence $\{\xi_i\}_{i \in \mathbb{N}}$, where $\xi_i \in \text{fin}(V, || \cdot ||)$, approximately converges to $\xi \in V$ in the norm $|| \cdot ||$ if

$$
\forall \varepsilon \in \mathbb{R}^+ \exists n \in \mathbb{N} \forall k \in \mathbb{N} \quad [k > n \Rightarrow ||\xi - \xi_i|| < \varepsilon].
$$

(2)
A sequence $\{\xi_i\}_{i \in \mathbb{N}}$ approximately converges to $\xi \in V$ if and only if $\{\hat{\xi}_i\}_{i \in \mathbb{N}}$ converges to $\hat{\xi} \in \hat{V}$. A sequence $\{\xi_i\}_{i \in \mathbb{N}}$, where $\xi_i \in \text{fin}(V, \|\cdot\|)$, is $S\|\cdot\|$-Cauchy if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n \in \mathbb{N} \forall k, l \in \mathbb{N} \quad [k, l > n \Rightarrow \|\xi_k - \xi_l\| < \varepsilon].$$

(3)

A sequence $\{\xi_i\}_{i \in \mathbb{N}}$ is $S\|\cdot\|$-Cauchy if and only if the sequence $\{\hat{\xi}_i\}_{i \in \mathbb{N}}$ is Cauchy.

A subset $X \subset \text{fin}(V, \|\cdot\|)$ is $S\|\cdot\|$-complete if for any $S\|\cdot\|$-Cauchy sequence $\{\xi_i\}_{i \in \mathbb{N}}$, there exists $\xi \in X$ such that $\{\xi_i\}$ approximately converges to $\xi$ in the norm $\|\cdot\|$. The subset $X$ is $S\|\cdot\|$-complete if and only if $\hat{X}$ is complete in $\hat{V}$, where $\hat{X} = \{\hat{\xi} | \xi \in X\}$.

The following results, called the hull completeness theorem, is a fundamental property of an internal normed space $(V, \|\cdot\|)$. See Hurd and Loeb [3] for detail.

**Theorem 2.1.** The subspace $\text{fin}(V)$ is $S$-complete in $\|\cdot\|$. 

**Corollary 2.2.** (The Hull Completeness Theorem) $\hat{V}$ is a Banach space.

Let $\mathcal{H}$ be an internal Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ an internal bounded linear operator such that the bound $\|T\|$ is finite. The bounded operator $\hat{T} : \mathcal{H} \to \hat{\mathcal{H}}$, called the standard part of $T$, is defined by the relation $\hat{T} \hat{x} = \overline{Tx}$ for any $x \in \text{fin}(\mathcal{H})$.

For further information on nonstandard real analysis, we refer to Stroyan and Luxemburg [3] and Hurd and Loeb [2].

3. **Several definitions of standard parts**

We give several equivalent definitions of the standard part of an internal bounded self-adjoint operator which is not $S$-bounded.

The following lemma, which is a basic property for self-adjointness, is used to give the first definition of standard parts (see [8]).

**Lemma 3.1.** Let $A$ be a symmetric operator on a Hilbert space $\mathcal{H}$. Then, $A$ is self-adjoint if and only if $\text{Rng}(A \pm i) = \mathcal{H}$.

Let $\mathcal{H}$ be an internal Hilbert space, and $A$ an internal bounded self-adjoint operator on $\mathcal{H}$. Let $\hat{\mathcal{K}} = \text{Ker}([(A+i)^{-1}]^\perp)$. Using the unitarity of $(A+i)(A-i)^{-1}$, we can easily check that $\text{Ker}([(A-i)^{-1}]^\perp) = \hat{\mathcal{K}}$.

**Proposition 3.2.** There exists the unique (possibly unbounded) self-adjoint operator $S$ on $\hat{\mathcal{K}}$ satisfying

$$(S + i)^{-1} = [(A+i)^{-1}]^\perp |\hat{\mathcal{K}}. \quad (4)$$

**Proof.** We see $\|(A+i)^{-1}\| < \infty$, and $[(A+i)^{-1}]^\perp$ is an bounded normal operator on $\hat{\mathcal{H}}$. The operator $T := [(A+i)^{-1}]^\perp |\hat{\mathcal{K}}$ is a bijection from $\hat{\mathcal{K}}$ to $[(A+i)^{-1}]^\perp |\hat{\mathcal{K}}$. Hence the inverse $T^{-1}$ is $[(A+i)^{-1}]^\perp |\hat{\mathcal{K}}$ to $\mathcal{K}$ is defined. Clearly the operator $S = T^{-1} - i$ satisfies the equation (4).

We will show that $S$ is symmetric. Let $x_1, x_2 \in \text{Dom}(S)$ ($= [(A+i)^{-1}]^\perp |\hat{\mathcal{K}}$). Then, we can show that there exist $\xi_i \in x_i$ such that $A\xi_i \in Sx_i$ ($i = 1, 2$) as follows. There
are \( y_i \in \hat{\mathcal{K}} \) and \( \eta_i \in \mathcal{H} \) such that \((S + i)^{-1}y_i = [(A + i)^{-1}]^\perp y_i = x_i \) and \( \eta_i \in y_i \). Let \( \xi_i = (A + i)^{-1}\eta_i \). Then \( \xi_i \in x_i \) and \((A + i)\xi_i = \eta_i \in y_i = (S + i)x_i \). Hence \( A\xi_i \in Sx_i \). Thus, \( \langle x_1, Sx_2 \rangle = \circ \langle \xi_1, A\xi_2 \rangle = \circ \langle A\xi_1, \xi_2 \rangle = (Sx_1, x_2) \). Therefore, \( S \) is symmetric.

To prove the self-adjointness, it is sufficient to show \( \operatorname{Rng}(S + i) = \operatorname{Rng}(S - i) = \hat{\mathcal{K}} \) by Lemma 3.1. Clearly \( \operatorname{Rng}(S + i) = \operatorname{Rng}(T^{-1}) = \hat{\mathcal{K}} \). Let \( x \in \operatorname{Dom}(S) \), \( \xi \in x \) and \( A\xi \in Sx \). Then we have
\[
\begin{align*}
\frac{(A - i)}{(A + i)} \wedge (S + i)x &= \left\langle \frac{A - i}{A + i} (A + i)\xi \right\rangle ^\perp = (S - i)x. \\
\end{align*}
\]
Thus, by the equation (4) with \( \operatorname{Ker}([(A - i)^{-1}]^\perp) = \hat{\mathcal{K}} \), we have
\[
(S - i)^{-1} = [(A - i)^{-1}]^\perp|\hat{\mathcal{K}}.
\]
Therefore, we can show \( \operatorname{Rng}(S - i) = \hat{\mathcal{K}} \) in the similar way to the proof of \( \operatorname{Rng}(S + i) = \hat{\mathcal{K}} \). The uniqueness of \( S \) is clear. \( QED \)

\textbf{Definition 3.3.} Under the condition of Proposition 3.2, define the self-adjoint operator \( \operatorname{st}_1(A) \) on \( \hat{\mathcal{K}} \) by \( \operatorname{st}_1(A) + i)^{-1} = [(A + i)^{-1}]^\perp|\hat{\mathcal{K}}. \)

The operator \( \operatorname{st}_1(A) \) is called the \textit{standard part} of \( A \). We see that \( \operatorname{st}_1(A) = \hat{A} \) when \( A \) is \( S \)-bounded.

\textbf{Definition 3.4.} Let \( A \) be an internal bounded operator on \( \mathcal{H} \), an internal Hilbert space. Define \( \operatorname{fin}(A) \subseteq \mathcal{H} \) by
\[
\operatorname{fin}(A) = \{\xi \in \operatorname{fin}\mathcal{H} | A\xi \in \operatorname{fin}\mathcal{H}\}.
\]

\textbf{Definition 3.5.} Let \( A \) be an internal bounded self-adjoint operator on \( \mathcal{H} \). Let \( \hat{\mathcal{K}} \) be the closure of the subspace \( \operatorname{fin}(A)^\perp = \{\hat{\xi}, \xi \in \operatorname{fin}(A)\} \) of \( \mathcal{H} \). Define the self-adjoint operator \( \operatorname{st}_2(A) \) on \( \hat{\mathcal{K}} \) by
\[
e^{it\operatorname{st}_2(A)} = e^{it\hat{A}}|\hat{\mathcal{K}}. \quad t \in \mathbb{R}.
\]

We see that \( \{e^{it\hat{A}}|\hat{\mathcal{K}}\}_{t \in \mathbb{R}} \) is one-parameter unitary group, since \( \hat{\mathcal{K}} \) is invariant under \( e^{it\hat{A}} \) for all \( t \in \mathbb{R} \). We also see that it is strongly continuous as follows. Let \( \xi \in \operatorname{fin}(A) \). Then, we have \( \|*(d/dt)e^{it\hat{A}}\xi \| = \|ie^{it\hat{A}}A\xi \| < \infty \), where \( *d/dt \) is the internal differentiation. This implies that \( e^{it\hat{A}}\xi \) is continuous with respect to \( t \in \mathbb{R} \). Thus, \( e^{it\hat{A}} \) is strongly continuous on \( \operatorname{fin}(A)^\perp \). Hence by Stone’s theorem, \( \operatorname{st}_2(A) \) is uniquely defined.

If \( A \) is \( S \)-bounded, \( \operatorname{st}_2(A) \) coincides with \( \hat{A} \) defined in Section 2. This is seen from the following:

\textbf{Proposition 3.6.} Let \( A \) be an internal \( S \)-bounded self-adjoint operator. Then,
\[
e^{it\hat{A}} = e^{it\hat{A}},
\]
for all \( t \in \mathbb{R} \).
Proof. For any infinitesimal $\epsilon \in \star \mathbb{R}_0^+$,
\begin{equation}
\epsilon^{-1}(e^{i\epsilon A} - I) \approx iA,
\end{equation}
holds, because
\begin{align*}
\|\epsilon^{-1}(e^{i\epsilon A} - I) - iA\| &= \|\epsilon^{-1} \sum_{\nu=2}^{\infty} (i\epsilon^\nu A^\nu) / \nu! \| \\
&\leq \epsilon^{-1} \sum_{\nu=2}^{\infty} (\nu! \epsilon^\nu) / \nu! = \epsilon^{-1} (\epsilon^\nu \|A\| - \epsilon) - \|A\| \approx 0.
\end{align*}
Thus, by the permanence principle,
\begin{equation}
\forall \delta \in \mathbb{R}_+, \exists \epsilon \in \mathbb{R}_+, |t| < \epsilon \Rightarrow \|t^{-1}(e^{itA} - I) - iA\| < \delta.
\end{equation}
Hence, we have
\begin{equation}
\lim_{\epsilon \rightarrow 0} \|\epsilon^{-1}(\overline{e^{i\epsilon A}} - \hat{I}) - i\hat{A}\| = 0.
\end{equation}
Thus we have $(d/dt)e^{it\hat{A}}|_{t=0} = i\hat{A}$, where $d/dt$ is the usual differentiation. Because $(e^{it\hat{A}})_{t \in \mathbb{R}}$ is one-parameter unitary group, it follows that $e^{it\hat{A}} = e^{itA}$. QED

Let $E(\cdot)$ be an internal projection-valued measure on $\star \mathbb{R}$, i.e., for each internal Borel set $\Omega \subseteq \star \mathbb{R}$, $E(\Omega)$ is an orthogonal projection on $\mathcal{H}$ such that
\begin{enumerate}
\item $E(\phi) = 0$, $E(\star \mathbb{R}) = I$
\item If $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \phi$ if $n \neq m$, then $E(\Omega) = \text{s-lim}_{N \rightarrow \infty} \sum_{n=1}^{N} E(\Omega_n)$
\item $E(\Omega_1)E(\Omega_2) = E(\Omega_1 \cap \Omega_2)$.
\end{enumerate}
For $r \in \star \mathbb{R}$, let $\mathcal{H}_r = \text{Rng}(E(-r, r))$, the range of $E((-r, r))$. Let $D(E) = \bigcup_{r \in \mathbb{R}^+} \mathcal{H}_r \cap \text{fin} \mathcal{H}$. $D(E)$ is called the standardization domain of $E(\cdot)$. Clearly, $\overline{D(E)}^{\perp\perp} = (\bigcup_{r \in \mathbb{R}^+} \mathcal{H}_r)^{\perp\perp}$.

For $a \in \mathbb{R}$, define the orthogonal projection $\hat{E}_{st}(-\infty, a]$ by
\begin{align*}
\hat{E}_{st}(-\infty, a] &= \sup \{ \hat{E}(-K, a + \epsilon) \|\hat{D}(E)^{\perp\perp} |K, \epsilon \in \mathbb{R}^+ \} \\
&= \text{s-lim}_{n \rightarrow \infty} \hat{E}(-n, a \frac{1}{n}) \|\hat{D}(E)^{\perp\perp}.
\end{align*}
Then we see
\begin{align*}
\text{s-lim}_{a \rightarrow -\infty} \hat{E}_{st}(-\infty, a] &= 0 \\
\text{s-lim}_{\epsilon \downarrow 0} \hat{E}_{st}(-\infty, a + \epsilon] &= \hat{E}_{st}(-\infty, a] \\
a < b \Rightarrow \hat{E}_{st}(-\infty, a] \leq \hat{E}_{st}(-\infty, b].
\end{align*}
Hence, $\hat{E}_{st}(-\infty, \cdot]$ defines a projection-valued measure on $\mathbb{R}$.

Definition 3.7. For any internal bounded self-adjoint operator $A$, define the self-adjoint operator $\text{st}_3(A)$ on $\overline{D(E)}^{\perp\perp}$ by
\begin{equation}
\text{st}_3(A) = \int \lambda d\hat{E}_{st}(\lambda).
\end{equation}
Proposition 3.8. Let $A$ be an internal bounded self-adjoint operator, and $E(\cdot)$ the internal projection-valued measure associated with the spectral decomposition of $A$. Then

\[ \hat{D}(E)^{\perp\perp} = \overline{\text{fin}(A)^{\perp\perp}}. \] (19)

Proof. $\hat{D}(E)^{\perp\perp} \subseteq \overline{\text{fin}(A)^{\perp\perp}}$ is clear. To prove $\hat{D}(E)^{\perp\perp} \supseteq \overline{\text{fin}(A)^{\perp\perp}}$, it is sufficient to show that for any $\hat{x} \in \overline{\text{fin}(A)^{\perp\perp}}$ there is a sequence $\hat{x}_n \in \hat{D}(E)$ ($n \in \mathbb{N}$) such that $\hat{x}_n \to \hat{x}$. Let $x_n = E(-n, n)x \ (n \in \star\mathbb{N})$. Notice that $\|A(x-x_n)\| \geq n\|x-x_n\|$. Suppose $\|x-x_n\| > \epsilon$ for all $n \in \mathbb{N}$. By the permanence principle, there is $N \in \star\mathbb{N}_{\infty}$ such that $\|x-x_N\| > \epsilon$. Hence, $\|A(x-x_n)\| \geq N\|x-x_N\| > N\epsilon \sim \infty$. This contradicts $\|A(x-x_N)\| \leq \|Ax\| < \infty$. \(QED\)

Theorem 3.9. Let $A$ be an internal bounded self-adjoint operator. Then,

\[ \text{st}_{2}(A) = \int \lambda d\hat{E}_{\text{st}}(\lambda), \] (20)

and hence $\text{st}_{2}(A) = \text{st}_{3}(A)$.

Proof. It is sufficient to show

\[ \langle \hat{x}, \exp(it\text{st}_{2}(A))\hat{x} \rangle = \int e^{it\lambda} \langle \hat{x}, d\hat{E}_{\text{st}}(\lambda)\hat{x} \rangle \] (21)

for all $\hat{x} \in \overline{\text{fin}(A)^{\perp\perp}}$. Define the internal Borel measure $\mu$ by $\mu(d\lambda) = \langle x, E(d\lambda)x \rangle$. Let $L\mu$ denote the Loeb measure of $\mu$, and $L'\mu$ the Borel measure on $\mathbb{R}$ defined by $L'\mu(\Omega) = L\mu(\text{st}^{-1}[\Omega])$. We can check that $L'\mu$ is well-defined (i.e., $\text{st}^{-1}[\Omega]$ is $L\mu$-measurable for any Borel set $\Omega \subseteq \mathbb{R}$). We also see that $L\mu$ is supported by $\text{fin} \star\mathbb{R}$, since $L\mu(\star\mathbb{R} \setminus \text{fin} \star\mathbb{R}) \leq \circ\langle x, E(\star\mathbb{R} \setminus (-n, n))x \rangle = \circ\|(1-E(-n, n))x\|^2 \leq (1/n^2)^\circ\|Ax\|^2$ for all $n \in \mathbb{N}$. Therefore

\[ \langle \hat{x}, \exp(it\text{st}_{2}(A))\hat{x} \rangle = \langle \hat{x}, e^{itA}\hat{x} \rangle = \circ\langle x, e^{itA}x \rangle \]

\[ = \circ \int_{\mathbb{R}} e^{it\lambda} d\mu(\lambda) \]

\[ = \int_{\mathbb{R}} e^{it\lambda} dL'\mu(\lambda). \]

On the other hand, for $a, b \in \mathbb{R}$ with $a < b$,

\[ L'\mu(a, b) = L\mu( \bigcup_{\epsilon \in \mathbb{R}^+} (a + \epsilon, b - \epsilon)) \]

\[ = \lim_{\epsilon \downarrow 0} \circ\langle x, E(a + \epsilon, b - \epsilon)x \rangle \]

\[ = \lim_{\epsilon \downarrow 0} \langle \hat{x}, \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle \]

\[ = \langle \hat{x}, \text{s-lim}_{\epsilon \downarrow 0} \hat{E}(a + \epsilon, b - \epsilon)\hat{x} \rangle \]

\[ = \langle \hat{x}, \hat{E}_{\text{st}}(a, b)\hat{x} \rangle. \]

Hence, $L'\mu(\Omega) = \langle \hat{x}, \hat{E}_{\text{st}}(\Omega)\hat{x} \rangle$ for any Borel set $\Omega \subseteq \mathbb{R}$. \(QED\)
Let $C \in \mathbb{R}$ be a positive constant, and $h$ be an internal Borel function from $\star \mathbb{R}$ to $\star \mathbb{C}$ satisfying the following properties:

\[ h(x) \approx h(y) \quad \text{iff} \quad x \approx y \quad \text{for all } x, y \text{ with } |x|, |y| < \infty, \]

\[ |h(x)| < C \quad \text{for all } x \in \star \mathbb{R}. \]

Define the function $\hat{h} : \mathbb{R} \to \mathbb{C}$ by

\[ \hat{h}(x) = \circ h(x), \]

for $x \in \mathbb{R}$. We see that $\hat{h}$ is injective and continuous. Let $A$ be an internal bounded self-adjoint operator. Notice that $h(A)$ is an S-bounded internal normal operator.

**Theorem 3.10.** There exists the unique self-adjoint operator $B$ on $\text{fin}(A) \cap \perp$ such that

\[ \hat{h}(B) = h(\overline{A})|\text{fin}(A) \cap \perp. \] (22)

Moreover, $B$ equals to $\text{st}_3(A)$.

**Proof.** By the argument similar to the proof of Theorem 3.9, we can show

\[ \langle \hat{x}, h(\overline{A})\hat{x} \rangle = \int_{\mathbb{R}} \hat{h}(\lambda) dL' \mu(\lambda) \]

\[ = \int_{\mathbb{R}} \hat{h}(\lambda) \langle \hat{x}, d\hat{E}_{\text{st}}(\lambda)\hat{x} \rangle \]

for any $\hat{x} \in \text{fin}(A) \cap \perp$. Thus,

\[ h(\overline{A})|\text{fin}(A) \cap \perp = \int_{\mathbb{R}} \hat{h}(\lambda) d\hat{E}_{\text{st}}(\lambda). \]

Because $\hat{h}$ is injective, the unique self-adjoint operator $B$ satisfying (22) is $\text{st}_3(A) = \int_{\mathbb{R}} \lambda d\hat{E}_{\text{st}}(\lambda)$. QED

**Corollary 3.11.** Definition 3.3, 3.5 and 3.7 are equivalent, that is, $\text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A)$.

**Proof.** Let $h(x) = 1/(x + i)$. QED

In section 2, $\hat{A}$ is defined only when $A$ is an internal S-bounded self-adjoint operator. Now we can extend the definition so as to include the case where $A$ is an internal bounded self-adjoint operator which is not S-bounded; $\hat{A} := \text{st}_1(A) = \text{st}_2(A) = \text{st}_3(A)$.

**Definition 3.12.** Let $A$ be an internal linear operator on an internal Hilbert space $\mathcal{H}$. Let $D$ be an (external) subspace of $\text{fin}\mathcal{H}$. $A$ is standardizable on $D$ if $D \subset \text{fin}(A)$ and if for any $x, y \in D$, $x \approx y$ implies $Ax \approx Ay$. In this case, define the operator $\hat{A}_D$ with domain $D = \{ \hat{x} | x \in D \}$, called the standard part of $A$ on $D$, by

\[ \hat{A}_D \hat{x} = \overline{Ax}, \quad x \in D. \] (23)
Clearly, $A$ is standardizable on $D$ if and only if $D \subseteq \text{fin}(A)$, and if $A\xi \approx 0$ for all $\xi \in D$ with $\xi \approx 0$.

**Lemma 3.13.** An internal bounded operator $A$ is standardizable on $\text{fin}(A^*A)$.

**Proof.** First, we prove $\text{fin}(A^*A) \subseteq \text{fin}(A)$ as follows. Suppose that $\xi \in \text{fin}(A)$. Let $E(\cdot)$ be the internal spectral-projection-valued measure of the self-adjoint operator $A^*A$. Then, $\|A\xi\|^2 = \langle \xi, A^*A\xi \rangle = \langle \xi, E[0, 1]A^*A\xi \rangle + \langle \xi, (I - E[0, 1])(A^*A)^2\xi \rangle \leq \langle \xi, E[0, 1]A^*A\xi \rangle + \|A^*A\xi\|^2 < \infty$. Thus, $\xi \in \text{fin}(A)$. Second, suppose $x \approx 0$ and $\|A^*Ax\| < \infty$. Then, $\|Ax\|^2 = \langle x, A^*Ax \rangle \leq \|x\|\|A^*Ax\| \approx 0$. QED

**Corollary 3.14.** If $D \subseteq \text{fin}\mathcal{H}$ is invariant under $A$ and $A^*$, $A$ is standardizable on $D$.

The operator $B$ in the above proof is called a hyperfinite extension of $A$ [6]. We use the following lemma in the proof of Theorem 3.16.

**Lemma 3.15.** Let $A$ be a symmetric operator with domain $D \subseteq \mathcal{H}$, a Hilbert space. Let $D_1 \subseteq D$ be a dense linear subset of $\mathcal{H}$ and suppose that $A|D_1$ is essentially self-adjoint. Then, $A$ is essentially self-adjoint and $\overline{A} = A|D_1$.

**Theorem 3.16.** Let $A$ be an internal self-adjoint operator on $\mathcal{H}$, and $E(\cdot)$ the projector-valued spectral measure of $A$. Then,

$$\hat{A} = \overline{A_{D(E)}} = \overline{A_{\text{fin}(A^2)}}$$

(24)

**Proof.** We can show that $\hat{A}_{D(E)}$ is essentially self-adjoint e.g. by Nelson's analytic vector theorem. Hence, it has one and only one self-adjoint extension, its closure. Thus, it is sufficient to show that $\hat{A}$ is an extension of $\hat{A}_{D(E)}$. If $E(-r, r)\xi = \xi$ ($r \in \mathbb{R}^+$, $\xi \in \mathcal{H}$), then $E_{st}(-r, r)\xi = \hat{\xi}$ ($s \in \mathbb{R}^+$, $s > r$). Thus,

$$\hat{A}D\xi = \hat{A}\xi = \int_s^\infty \lambda dE_\lambda(\xi) = \int_s^\infty \lambda d\hat{E}_{st}(\lambda)\xi = \hat{A}\xi = \hat{A}\xi,$$

Therefore, $\hat{A} = \hat{A}_{D(E)}$. $\hat{A}_{D(E)}$ follows from $D(E) \subseteq \text{fin}(A^2)$ and Lemma 3.15. QED

4. The domain of $\hat{A}$

**Definition 4.1.** For an internal bounded self-adjoint operator $A$ on $\mathcal{H}$, define $D(A)$ by

$$D(A) = \{\xi \in \text{fin}\mathcal{H} \mid \text{for all } t \in \mathbb{R}_0^+, \; e^{-t|A|}\xi \approx A\xi \in \text{fin}\mathcal{H}\}.$$  

Clearly, $D(A)$ is a subspace of $\mathcal{H}$.

**Proposition 4.2.** An internal bounded self-adjoint operator $A$ is standardizable on $D(A)$. 

Proof. Let \( \xi \in D(A) \) and \( \|\xi\| \approx 0 \). We can easily check \( \|e^{-t|A|}A\| < \infty \) for all \( t > 0, t \neq 0 \). Hence, \( \|\xi\| \leq \|e^{-t|A|}A\xi\| + \|\xi\| \|1-e^{-t|A|}A\| \). By the S-boundedness of \( e^{-t|A|}A \), the first term equals 0, and by the definition of \( D(A) \), the second term equals 0. Thus we have \( \|\xi\| = 0 \). \( QED \)

The following lemmas are easily shown.

**Lemma 4.3.** Let \( f : \star \mathbb{N} \rightarrow \star \mathbb{R}^+ \) be internal and increasing. If \( f(M) < \infty \) for some \( M \sim \infty \), then
\[
\lim_{n \rightarrow \infty} ^{\circ} f(n) < \infty.
\]

**Lemma 4.4.** Under the same condition to Lemma 4.3, there is \( K \sim \infty \) such that for all \( L \sim \infty \),
\[
f(K) \approx f(L) \quad \text{if} \quad L \leq K.
\]

**Proposition 4.5.** Let \( \xi \in \text{fin}(\mathcal{H}) \). For sufficiently large \( t \approx 0 \),
\[
e^{-t|A|} \xi \in D(A).
\] (25)

**Proof.** Applying Lemma 4.4 to \( f(n) = \|e^{-|A|/n}A\xi\| \), we find that for sufficiently small \( K \sim \infty \) and \( L \sim \infty \), \( e^{-|A|/K}A\xi \approx e^{-|A|/L}A\xi \). Thus, for sufficiently large \( s \approx 0 \) and \( t \approx 0 \), \( e^{-t|A|}A\xi \approx e^{-t|A|}A\xi \). Hence for all \( x \approx 0, x > 0 \),
\[
e^{-t|A|}Ae^{-t|A|} \xi = e^{-(s+t)|A|}A\xi \approx Ae^{-t|A|} \xi.
\]
Therefore, \( e^{-t|A|} \xi \in D(A) \). \( QED \)

**Theorem 4.6.** Let \( E(\cdot) \) be the spectral resolution of \( A \) and \( E_K = E(-K, K) \) for \( K \in \star \mathbb{R}^+ \). For any \( \xi \in \text{fin}(\mathcal{A}) \),
\[
\xi \in D(A) \quad \text{iff} \quad A\xi \approx E_KA\xi \quad \text{for all} \quad K \sim \infty.
\] (26)

**Remark.** The right-hand condition is equivalent to
\[
\lim_{K \rightarrow \infty} \quad ^{\circ} \| (I-E_K)A\xi \| = 0.
\] (27)

**Proof.** Suppose that \( \xi \in \text{fin}(\mathcal{A}) \) and \( A(I-E_K)\xi \approx 0 \) for all \( K \sim \infty \). For any \( t \approx 0 \), there exists a \( K \sim \infty \) such that \( tK \approx 0 \). Thus,
\[
\|e^{-t|A|}A\xi - A\xi\|^2 \approx \|e^{-t|A|}E_KA\xi - E_KA\xi\|^2
\]
\[
= \left\| \int_{-K}^{K} e^{-t|\lambda|} \lambda - \lambda dE(\lambda)\xi \right\|^2
\]
\[
= \left\| \int_{-K}^{K} |(e^{-t|\lambda|} - 1)\lambda|^2 dE(\lambda)\xi \right\|^2
\]
\[
\leq \sup_{|\lambda| < K} |e^{-t|\lambda|} - 1|^2 \int_{-K}^{K} \lambda^2 dE(\lambda)\xi \right\|^2
\]
\[
= \sup_{|\lambda| < K} |e^{-t|\lambda|} - 1|^2 \|E_KA\xi\|^2
\]
\[
\approx 0.
\]
Hence \( \xi \in D(A) \).

Conversely, suppose \( \xi \in D(A) \) \((\subset \text{fin}(A))\). Applying Lemma 4.4 to \( f(n) = \|E_nA\xi\| \), we see that for sufficiently small \( K \sim \infty \) and \( L \sim \infty \) \((L \leq K)\),

\[
\|E_LA\xi\| \approx \|E_KA\xi\|.
\]

Thus, \((E_K - E_L)A\xi \approx 0\), since \(\|E_LA\xi - E_KA\xi\|^2 = \|E_KA\xi\|^2 - \|E_LA\xi\|^2 \approx 0\). Let \( t \in \mathbb{R}_0^+ \) satisfy \( tK \sim \infty \) so that

\[
\|E_KA\xi - e^{-t|A|}A\xi\| \approx \|E_KA\xi\|.
\]

Let \( L \sim \infty \) satisfy \( tL \approx 0 \), so that the above

\[
\|E_KA\xi - E_LE_KA\xi\| \approx (1-E_K)E_LE_KA\xi = 0.
\]

Thus, for sufficiently small \( K \sim \infty \) and for any \( t \approx 0 \) such that \( tK \sim \infty \),

\[
E_KA\xi \approx e^{-t|A|}A\xi \approx A\xi.
\]

Since \(\|A\xi - E_KA\xi\| \geq \|A\xi - E_{K'}A\xi\| > 0\) if \( K < K' \), we have \(E_{K'}A\xi \approx A\xi\) holds for any \( K' \sim \infty \). \( \text{QED} \)

**Proposition 4.7.** Let \( \xi \in \text{fin}(A) \). Then, \( E_K\xi \in D(A) \) for sufficiently small \( K \sim \infty \).

**Proof.** Applying Lemma 4.4 to \( f(n) = \|E_nA\xi\| \), we find that for sufficiently small \( K, L \sim \infty \), \( E_KA\xi \approx E_LA\xi \). Thus, if \( L \sim \infty \), \( L \leq K \), then \(\|(1 - E_L)E_KA\xi\| = \|(E_K - E_L)A\xi\| \approx 0\). If \( L > K \), clearly \( (1 - E_L)E_KA\xi = 0 \). Hence for all \( L \sim \infty \), \( E_KA\xi \approx E_LE_KA\xi \). Thus \( E_K\xi \in D(A) \) by Theorem 4.6. \( \text{QED} \)

**Corollary 4.8.** \([\text{fin}(A)]^\wedge = [D(A)]^\wedge\), i.e., if \( \xi \in \text{fin}(A) \), then there is \( \eta \in D(A) \) such that \( \eta \approx \xi \).

**Example** We have seen that the following relations hold:

\[
\text{fin}(A^2) \subset D(A) \subset \text{fin}(A) \subset \text{fin}\mathcal{H},
\]

\[
[\text{fin}(A^2)]^\wedge \subset [D(A)]^\wedge = [\text{fin}(A)]^\wedge \subset \hat{\mathcal{H}},
\]
\[ [\text{fin}(A^2)]^\perp = [D(A)]^\perp = [\text{fin}(A)]^\perp \subset \mathcal{H}. \]

An example of \( A \) such that \( \text{fin}(A) \setminus D(A) \neq \emptyset \) is given as follows. Let \( \nu \) be an infinite hypernatural number, and \( \mathcal{H} = \star \mathbb{C}^\nu \), \( \nu \)-dimensional internal Hilbert space. Define the internal self-adjoint operator \( A \) on \( \mathcal{H} \) by \( A(x_1, x_2, \ldots, x_\nu) = (x_1, 2x_2, \ldots, \nu x_\nu) \). Let \( \xi = (0, 0, \ldots, 0, \nu^{-1}) \). Then we see \( \xi \in \text{fin}(A) \setminus D(A) \) from Theorem 4.6.

We also find \( D(A) \setminus \text{fin}(A^2) \neq \emptyset \); let \( \eta = (1^{-2}, 2^{-2}, \ldots, \nu^{-2}) \), then we easily see \( \eta \in D(A) \setminus \text{fin}(A^2) \). Moreover we find \( \hat{\eta} \in [D(A)]^\perp \setminus [\text{fin}(A^2)]^\perp \). In fact, if \( \eta' \approx \eta \), then
\[
\circ ||A^2\eta'|| \geq \lim_{\nu \to \infty} \circ ||A^2E_n\eta|| = \lim_{\nu \to \infty} \sqrt{n} = \infty.
\]
Thus, we have \( \hat{\eta} \not\in [\text{fin}(A^2)]^\perp \) by Theorem 4.6.

**Theorem 4.9.** Let \( \xi \in \text{fin}(A) \), then

\[
\xi \in D(A) \iff \lim_{t \downarrow 0, \iota \#^0} \left( \frac{e^{-t|A|} - 1}{t} \xi \right) = - |A| \xi. \tag{28}
\]

**Proof.** Suppose that the right-hand side does not hold. In other words, suppose that

\[
\exists \epsilon \in \mathbb{R}^+ \forall n \in \mathbb{N} \exists t \in \star \mathbb{R}, 0 < t < \frac{1}{n} \wedge \left\| \left( \frac{e^{-t|A|} - 1}{t} + |A| \mathbb{I} \xi \right) \right\| > \epsilon. \tag{29}
\]

By permanence,

\[
\exists \epsilon \in \mathbb{R}^+ \exists N \in \star \mathbb{N}_\infty \exists t \in \star \mathbb{R}, 0 < t < \frac{1}{n} \wedge \left\| \left( \frac{e^{-t|A|} - 1}{t} + |A| \right) \xi \right\| > \epsilon. \tag{30}
\]

That is, there is positive infinitesimal \( t \) such that \( t^{-1}(e^{-t|A|} - 1)\xi \not\approx -|A|\xi \).

Thus, for some \( \eta \in \text{fin}(\mathcal{H}) \),

\[
\Re \langle \eta, e^{-t|A|} - 1 \xi \rangle \not\approx \Re \langle \eta, -|A|\xi \rangle.
\]

Let \( f(t) = \Re \langle \eta, e^{-t|A|} \xi \rangle \). By the mean value theorem, for some \( s \in \star \mathbb{R} \) with \( 0 < s < t \),

\[
f'(s) = \frac{f(t) - f(0)}{t} = \Re \langle \eta, e^{-|A|s} - 1 \xi \rangle \not\approx \Re \langle \eta, -|A|\xi \rangle.
\]

Therefore, by the definition of \( D(A) \), we have \( \xi \in \text{fin}(A) \setminus D(A) \).

Conversely, suppose \( \xi \in \text{fin}(A) \setminus D(A) \). Then, there is positive infinitesimal \( t_0 \) satisfying

\[
e^{-t_0|A|}|A|\xi \not\approx |A|\xi. \]

Let \( \eta = (|A| - e^{t_0|A|})\xi \) (\( \in \text{fin}(\mathcal{H}) \)). Then this is equivalent to

\[
\langle \eta, e^{-t_0|A|} |A|\xi \rangle \not\approx \langle \eta, |A|\xi \rangle. \tag{31}
\]

Let \( f(x) = \langle \eta, e^{-x|A|} \xi \rangle \) (\( x \in \star \mathbb{R}^+ \)). We see that \( f' \) is increasing and \( -\infty < f' < 0 \), and hence \( f \) is decreasing and \( 0 < f < \infty \). The relation (31) is equivalent to

\[
f'(t_0) \not\approx f'(0), \tag{32}
\]
We have \( f(x) \geq f'(t_0)(x - t_0) + f(t_0) \). Thus we have

\[
0 > \frac{f(x) - f(0)}{x} \geq \frac{f'(t_0)(x - t_0) + f(t_0) - f(0)}{x}.
\] (33)

Let \( F(x) = [f'(t_0)(x - t_0) + f(t_0) - f(0)]/x \), then for \( c \in \mathbb{R}^+ \),

\[
F(cx) = f'(t_0) \left( 1 - \frac{1}{c} \right) + \frac{1}{c} \frac{f(t_0) - f(0)}{t_0}.
\] (34)

By the mean value theorem and \(-\infty < f'(x) < 0\), we have \( |(f(x) - f(0))/x| < \infty \). Hence \( F(cx) \approx f'(t_0) \) for all \( c \sim \infty \). Thus, by (32) and (33),

\[
0 > \frac{f(cx) - f(0)}{cx} \geq F(cx) \gg f'(0),
\] (35)

for all \( c \sim \infty \). Thus there is \( \epsilon \in \mathbb{R}^+ \) such that for sufficiently large \( x \approx 0 \), \( \frac{f(x) - f(0)}{x} > \epsilon \). By the permanence principle, for sufficiently small \( x \in \mathbb{R}^+ \), \( \frac{f(x) - f(0)}{x} > \epsilon \). We can check the relations

\[
\langle \eta, \frac{e^{-x|A|} - 1}{x} \xi \rangle = \frac{f(x) - f(0)}{x}, \quad \langle \eta, |A| \xi \rangle = -f'(0), \quad \frac{e^{-x|A|} - 1}{x} > -|A|,
\]

for \( x > 0 \). Therefore, using the increasingness of \((e^{-x||A||^{-1}})/x, x, \) we have

\[
\lim_{x \downarrow 0, x \neq 0} \circ \langle \eta, \frac{e^{-x|A|} - 1}{x} \xi \rangle \neq \langle \eta, |A| \xi \rangle.
\]

\[QED\]

**Theorem 4.10.** Let \( A \) be an internal bounded self-adjoint operator. Then, \( \hat{A} = \hat{A}_{D(A)} \).

**Proof.** By Theorem 3.16 and Lemma 3.15, it suffices to show that \( \hat{A}_{D(A)} \) is a closed extension of \( \hat{A}_{\text{fin}(A^2)} \). If \( \xi \in \text{fin}(A^2) \), for any \( K \sim \infty \), \( \| (I-E_K)A \xi \| \leq \frac{1}{K} \| (I-E_K)A^2 \xi \| \leq \frac{1}{K} \| A^2 \xi \| \approx 0 \). Hence \( \xi \in D(A) \), and hence \( \hat{A}_{D(A)} \) is an extension of \( \hat{A}_{\text{fin}(A^2)} \).

To prove that \( \hat{A}_{D(A)} \) is closed, it suffices to show that \( \hat{D}(A) = [D(A)]^\circ \) is complete in the norm \( \| \cdot \|_A \) defined by \( \| \xi \|_A = \| \xi \| + \| \hat{A} \xi \| \). Define the internal norm \( \| \cdot \|_{A} \) on \( \mathcal{H} \) by \( \| \xi \|_{A} = \| \xi \| + \| A \xi \| \). We can check \( \| \xi \|_{A} = \circ \| \xi \|_A \) for \( \xi \in D(A) \).

By Theorem 2.1, \( \text{fin}(A) \) is \( S \)-\( \| \cdot \|_A \)-complete. Hence, if the sequence \( \{ \xi_i \}_{i \in \mathbb{N}} \subset D(A) \subset \text{fin}(A) \) is \( S \)-\( \| \cdot \|_A \)-Cauchy, then there is \( \xi \in \text{fin}(A) \) such that \( \{ \xi_i \} \) approximately converges to \( \xi \) in the norm \( \| \cdot \|_A \). This \( \xi \) is shown to be in \( D(A) \) as follows. Regarding Theorem 4.6, and \( \xi_i \in D(A) \), this relation leads to \( \circ \| (I-E_K)A \xi_i \| = \lim_{i \rightarrow \infty} \circ \| (I-E_K)A \xi_i \| = 0 \), for any \( K \sim \infty \). Therefore, from Theorem 4.6, we have \( \xi \in D(A) \) and hence any Cauchy sequence in \( \hat{D}(A) \) converges in \( \hat{D}(A) \) in the norm \( \| \cdot \|_A \). \( QED \)
Theorem 4.11. The domain $D(A)$ is maximal. That is, if $D(A) \subset S \subset \text{fin}({\mathcal H})$ and $A$ is standardizable on $S$, then $S = D(A)$.

Proof. Suppose that $D(A) \subset S \subset \text{fin}({\mathcal H})$ and that $A$ is standardizable on $S$. Let $\eta \in S$. By Corollary 4.8 and $\eta \in \text{fin}(A)$, there is $\xi \in D(A)$ such that $\xi \approx \eta$. By the definition of $D(A)$ and the standardizability on $S$, for all positive infinitesimal $t$, $e^{-t|A|A}\eta \approx e^{-t|A|A}\xi \approx A\xi \approx A\eta$, since $\|e^{-t|A|A}\| \leq 1$. Thus, $\eta \in D(A)$. QED

Proposition 4.12. Let $A$ be an internal positive operator on $\mathcal H$. Then, for any $\eta \in \text{fin}(A^{\frac{1}{2}})$,

$$\inf_{\xi \approx \eta} \langle \langle \xi, A\xi \rangle \rangle = \inf_{\alpha \rightarrow \infty} \langle \langle \eta, E_{\alpha}A\eta \rangle \rangle.$$

(36)

Proof. Suppose $\eta \approx \xi$. If $\alpha < \infty$, $\langle \eta, E_{\alpha}A\eta \rangle \approx \langle \xi, E_{\alpha}A\xi \rangle \leq \langle \xi, A\xi \rangle$, that is,

$$\forall \epsilon \in \mathbb{R}^+, \forall \alpha < \infty, \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \epsilon,$$

Thus, by the permanence principle,

$$\forall \epsilon \in \mathbb{R}^+, \exists \alpha \sim \infty, \forall \alpha \leq K, \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \epsilon.$$

By saturation,

$$\exists \alpha \sim \infty, \forall \epsilon \in \mathbb{R}^+, \forall \alpha \leq K, \langle \eta, E_{\alpha}A\eta \rangle \leq \langle \xi, A\xi \rangle + \epsilon.$$

Hence we have

$$\exists \alpha \sim \infty, \langle \langle \eta, E_{K}A\eta \rangle \rangle \leq \langle \langle \xi, A\xi \rangle \rangle.$$

It follows that $\inf_{\xi \approx \eta} \langle \langle \xi, A\xi \rangle \rangle \geq \inf_{\alpha \rightarrow \infty} \langle \langle \eta, E_{\alpha}A\eta \rangle \rangle$.

On the other hand, we see that for all $\alpha \sim \infty$, $\|\eta - E_{\alpha}\eta\|^{2} \leq \alpha^{-1}\|A^{\frac{1}{2}}(\eta - E_{\alpha}\eta)\|^{2} \leq \alpha^{-1}\|A^{\frac{1}{2}}\eta\|^{2} \leq 1$. Hence,

$$\forall \alpha \sim \infty, \inf_{\xi \approx \eta} \langle \langle \xi, A\xi \rangle \rangle \leq \langle \langle E_{\alpha}\eta, A\eta \rangle \rangle \approx \langle \langle \eta, E_{\alpha}A\eta \rangle \rangle.$$

Thus it follows that $\inf_{\xi \approx \eta} \langle \langle \xi, A\xi \rangle \rangle \leq \inf_{\alpha \sim \infty} \langle \langle \eta, E_{\alpha}A\eta \rangle \rangle$. QED

Proposition 4.13. Let $A$ be an internal positive operator and $\eta \in \text{fin}(A)$. Then,

$$\inf_{\xi \approx \eta} \langle \langle \xi, A\xi \rangle \rangle = \langle \langle \eta, \hat{A}\eta \rangle \rangle.$$

(37)

Proof. From Proposition 4.12, we see $\inf_{\xi \approx \eta} \langle \langle \xi, A\xi \rangle \rangle = \inf_{\alpha \sim \infty} \langle \langle \eta, E_{\alpha}A\eta \rangle \rangle$. By Theorem 4.10 and Proposition 4.7, for sufficiently small $\alpha \sim \infty$, $\langle \langle \eta, E_{\alpha}A\eta \rangle \rangle = \langle \langle E_{\alpha}\eta, A\eta \rangle \rangle = \langle \langle \eta, \hat{A}\eta \rangle \rangle$. QED

Definition 4.14. Let $A$ be a internal bounded positive operator, and $D \subset \text{fin}(A^{\frac{1}{2}})$. The sesquilinear form $\langle \cdot, A\cdot \rangle$ is standardizable on $D$ if $\langle \xi_{1}, A\eta_{1} \rangle \approx \langle \xi_{2}, A\eta_{2} \rangle$ for all $\xi_{1}, \xi_{2}, \eta_{1}, \eta_{2} \in D$ with $\xi_{1} \approx \xi_{2}$ and $\eta_{1} \approx \eta_{2}$. 
Proposition 4.15. Let $D$ be a subspace of $\text{fin}(\mathcal{H})$ and $A \geq 0$. Then, $\langle \cdot, A \cdot \rangle$ is standardizable on $D$ if and only if $A^{\frac{1}{2}}$ is standardizable on $D$.

Proof. Suppose that $A^{\frac{1}{2}}$ is standardizable on $D$. Then $A^{\frac{1}{2}}\xi \approx A^{\frac{1}{2}}\eta$ for any $\xi, \eta \in D$ with $\xi \approx \eta$. Thus, $\langle \xi, A\xi \rangle = \|A^{\frac{1}{2}}\xi\|^2 \approx \|A^{\frac{1}{2}}\eta\|^2 = \langle \eta, A\eta \rangle$. Conversely, suppose that $\langle \cdot, A \cdot \rangle$ is standardizable on $D$. Then for any $\xi, \eta \in D$ with $\xi \approx \eta$, $\|A^{\frac{1}{2}}\xi - A^{\frac{1}{2}}\eta\|^2 = \|A^{\frac{1}{2}}(\xi - \eta)\|^2 = \langle \xi - \eta, A(\xi - \eta) \rangle \approx 0$. QED

Corollary 4.16. The set $D(A^{\frac{1}{2}})$ is a maximal domain of $\langle \cdot, A \cdot \rangle$, and $\langle \cdot, A \xi \rangle = \langle A^{\frac{1}{2}}\xi, A^{\frac{1}{2}}\eta \rangle$ for any $\xi, \eta \in D(A^{\frac{1}{2}})$.

References


