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Fixed Point Theoretic Characterization of Generalized Stackelberg Equilibrium Points

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Abstract

Fixed point theoretic characterization of generalized Stackelberg equilibrium points in the case of oligopoly games is given.

1 Introduction

Stackelberg [1] [2] [5] gave the basic exmaple of duopoly in which both players are producers and their gain functions are only dependent on the pair of these two players' productions. On the assumption that the player taking the initiative in producing knows that the follower, namely the opponent, will use the optimal decision rule, Stackelberg proved that the existence of a certain equilibrium point in which the player taking the initiative can yields a larger gain to him and the follower is forced to yield a smaller gain. In this paper, the generalization of Stackelberg equilibrium points from the case of duopoly into the case of oligopoly is given, according to the methods of set valued analysis [3] [4].

2 Superposition of set-valued mappings

Throughout this paper, $\mathbb{N}$ (resp. $\mathbb{R}$) denotes the set of all positive integers (resp. the set of all real numbers). Let X be a compact Hausdorff space, and f, g be two upper semi-continuous, set-valued mappings on X with values in 2^X. Then, the superposition of g and f is defined as

$$(g \circ f)(x) = \bigcup_{y \in f(x)} g(y), \quad x \in X.$$

Then, we have the following:

Lemma 1. $g \circ f$ is upper semi-continuous.

Proof. Let x_0 and z_0 be two elements of X. Let $\{x_\alpha\}$ and $\{z_\alpha\}$ be two nets consisting of elements of X, which converge to x_0 and z_0, respectively. Then, it is sufficient to prove that $z_0 \in (g \circ f)(x_0)$ holds if $z_\alpha \in (g \circ f)(x_\alpha)$ holds for all α. For any α, there exists y_α satisfying

$$z_\alpha \in g(y_\alpha).$$

Since X is compact, there exist y_0 and a subnet $\{y_{\alpha_\beta}\}$ satisfying

$$\lim_\beta y_{\alpha_\beta} = y_0.$$

Therefore, we have

$$\begin{aligned} y_{\alpha_\beta} &\in f(x_{\alpha_\beta}), \\ z_{\alpha_\beta} &\in g(y_{\alpha_\beta}). \end{aligned}$$

Since f and g are upper semi-continuous, we have

$$\begin{aligned} y_0 &\in f(x_0), \\ z_0 &\in g(y_0). \end{aligned}$$

These results conclude the proof.
Let f be a set-valued mapping on X with values in 2^X. Then, for any $S \subset X$, the image of S under the mapping f is defined as

$$f(S) = \bigcup_{x \in S} f(x).$$

Now, we have the following:

Lemma 2. Let X (resp. Y) be a Hausdorff space (resp. a compact Hausdorff space), f be an upper semi-continuous, set-valued mapping on X with values in 2^Y. Then, for any compact subset $S \in 2^X$, $f(S)$ is also compact.

Proof. Let y_0 be an element of Y and $\{y_\alpha\}$ be a net consisting of elements of Y, which converges to y_0. Then, it is sufficient to prove that $y_0 \in f(S)$ holds. For any α, there exists $x_\alpha \in S$ satisfying

$$y_\alpha \in f(x_\alpha).$$

Since $\{x_\alpha\}$ is also a net consisting of elements of S, there exist an accumulating point $x_0 \in X$ and a subnet $\{x_{\alpha_\beta}\}$ which converges to x_0. Since $y_{\alpha_\beta} \in f(x_{\alpha_\beta})$ holds for all β, we obtain

$$y_0 \in f(x_0) \subset f(S).$$

Therefore, this result concludes the proof.
Let X be a metric space with its metric d and f be a bounded closed set-valued mapping on X with values in 2^X. Then, for any $x_0 \in X$, f is said to be continuous at x_0, if f satisfies the following condition:

$$\lim_{n\to\infty} H(f(x_n), f(x_0)) = 0,$$

where $\{x_n\}_{n=1}^\infty$ is a sequence consisting of elements of X, which converges to x_0 and H means Hausdorff's metric. It is clear that f is upper semi-continuous at x_0, if f is continuous at x_0.

3 Generalized Stackelberg equilibrium points

For any positive integer k satisfying $1 \le k \le 3$, let X_k be a metric space with its metric d_k, S_k be a compact subset of X_k and p_k be a continuous function on S_k with values in $\mathbb{R}$. Then, for any $(x_1, x_2, x_3) \in \prod_{i=1}^{3} S_i$, the response function from p_3 to p_1 and p_2 is defined as

$$R_3(x_1, x_2) = \{(x_1, x_2, y_3); y_3 \in S_3, p_3(x_1, x_2, y_3) = \sup_{z_3 \in S_3} p_3(x_1, x_2, z_3)\}.$$

By the same way as above, the response function from p_2 to p_1 is defined as

$$\begin{aligned} R_2(x_1) &= \{(x_1, y_2, y_3); y_2 \in S_2, (x_1, y_2, y_3) \in R_3(x_1, y_2), \\ &\qquad p_2(x_1, y_2, y_3) = \sup_{\substack{z_2 \in S_2 \\ (x_1, z_2, z_3) \in R_3(x_1, z_2)}} p_2(x_1, z_2, z_3)\}. \end{aligned}$$

Finally, the Stackelberg equilibrium set is defined as

$$R_1 = \{(y_1, y_2, y_3); y_1 \in S_1, (y_1, y_2, y_3) \in R_2(y_1), p_1(y_1, y_2, y_3) = \sup_{\substack{z_1 \in S_1 \\ (z_1, z_2, z_3) \in R_2(z_1)}} p_1(z_1, z_2, z_3)\}.$$

Then, we have the following:

Theorem 3. If R_3 is continuous, then the Stackelberg equilibrium set is not empty.

Proof. Since p_3 is continuous on $\prod_{i=1}^{3} S_i$, for any $x_1 \in S_1$ and $x_2 \in S_2$, $R_3(x_1, x_2)$ is nonempty and compact, The assumption that R_3 is continuous on $\prod_{i=1}^{2} S_i$ implies that R_3 is also upper semi-continuous. Therefore, for any $x_1 \in S_1$, $R_2(x_1)$ is nonempty and compact, because f_2 is continuous on $\prod_{i=1}^{3} S_i$ and $R_3(x_1, S_2)$ is nonempty and compact. It is sufficient to prove that R_2 is also upper semi-continuous on S_2. Let x_1^0 be an element of S_1 and $\{x_1^n\}_{n=1}^{\infty}$ be a sequence consisting of elements of $\prod_{i=1}^{3} S_i$, which converges to x_1^0. Let (x_1^0, z_2^0, z_3^0) be an element of $\prod_{i=1}^{3} S_i$ and $\{(x_1^n, y_2^n, y_3^n)\}_{n=1}^{\infty}$ be a sequence consisting of elements of $\prod_{i=1}^{3} S_i$ satisfying

$$\begin{aligned} (x_1^n, y_2^n, y_3^n) &\in R_2(x_1^n), \quad n \in \mathbb{N}, \\ \lim_{n \to \infty} (x_1^n, y_2^n, y_3^n) &= (x_1^0, z_2^0, z_3^0). \end{aligned}$$

We have only to prove that $(x_1^0, z_2^0, z_3^0) \in R_2(x_1^0)$ holds. For any $(x_1^0, z_2^0, w_3^0) \in R_3(x_1^0, z_2^0)$, there exists a sequence $\{(x_1^n, y_2^n, w_3^n)\}_{n=1}^{\infty}$ consisting of elements of $\prod_{i=1}^{3} S_i$ satisfying

$$\begin{aligned} (x_1^n, y_2^n, w_3^n) &\in R_3(x_1^n, y_2^n), \quad n \in \mathbb{N}, \\ \lim_{n \to \infty} (x_1^n, y_2^n, w_3^n) &= (x_1^0, y_2^0, w_3^0), \end{aligned}$$

because the assumption shows the following equality:

$$\begin{aligned} 0 &= \lim_{n \to \infty} H(R_3(x_1^n, y_2^n), R_3(x_1^0, z_2^0)) \\ &\ge \lim_{n \to \infty} H(R_3(x_1^n, y_2^n), \{(x_1^0, z_2^0, w_3^0)\}) \end{aligned}$$

holds. Since the definition of R_3 shows the following inequality:

$$p_2(x_1^n, y_2^n, y_3^n) \geq p_2(x_1^n, y_2^n, w_3^n), \quad n \in \mathbb{N}$$

holds. Therefore, we have

$$\begin{aligned} p_2(x_1^0, z_2^0, z_3^0) &= \lim_{n\to\infty} p_2(x_1^n, y_2^n, y_3^n) \\ &\geq \lim_{n\to\infty} p_2(x_1^n, y_2^n, w_3^n) \\ &= p_2(x_1^0, z_2^0, w_3^0) \end{aligned}$$

Since S_1 is compact, R_2 is upper semi-continuous and p_1 is continuous on $\prod_{i=1}^3 S_i$, R_1 is also nonempty and compact.

References

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