<table>
<thead>
<tr>
<th>Title</th>
<th>Fixed Point Theoretic Characterization of Generalized Stackelberg Equilibrium Points (Nonlinear Analysis and Convex Analysis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Akashi, Shigeo</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2001(1187), 228-231</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64670">http://hdl.handle.net/2433/64670</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher Kyoto University</td>
</tr>
</tbody>
</table>
Fixed Point Theoretic Characterization of Generalized Stackelberg Equilibrium Points

Shigeo AKASHI
Department of Mathematics, Faculty of Science, Niigata University
8050, 2-nochou, Ikarashi, Niigata-shi, 950-2181 JAPAN

Abstract
Fixed point theoretic characterization of generalized Stackelberg equilibrium points in the case of oligopoly games is given.

1 Introduction
Stackelberg [1] [2] [5] gave the basic example of duopoly in which both players are producers and their gain functions are only dependent on the pair of these two players' productions. On the assumption that the player taking the initiative in producing knows that the follower, namely the opponent, will use the optimal decision rule, Stackelberg proved that the existence of a certain equilibrium point in which the player taking the initiative can yield a larger gain to him and the follower is forced to yield a smaller gain. In this paper, the generalization of Stackelberg equilibrium points from the case of duopoly into the case of oligopoly is given, according to the methods of set valued analysis [3] [4].

2 Superposition of set-valued mappings
Throughout this paper, \( \mathbb{N} \) (resp. \( \mathbb{R} \)) denotes the set of all positive integers (resp. the set of all real numbers). Let \( X \) be a compact Hausdorff space, and \( f, g \) be two upper semi-continuous, set-valued mappings on \( X \) with values in \( 2^X \). Then, the superposition of \( g \) and \( f \) is defined as

\[
(g \circ f)(x) = \bigcup_{y \in f(x)} g(y), \quad x \in X.
\]

Then, we have the following:

**Lemma 1.** \( g \circ f \) is upper semi-continuous.

**Proof.** Let \( x_0 \) and \( z_0 \) be two elements of \( X \). Let \( \{x_\alpha\} \) and \( \{z_\alpha\} \) be two nets consisting of elements of \( X \), which converge to \( x_0 \) and \( z_0 \), respectively. Then, it is sufficient to prove
that $z_0 \in (g \circ f)(x_0)$ holds if $z_\alpha \in (g \circ f)(x_\alpha)$ holds for all $\alpha$. For any $\alpha$, there exists $y_\alpha$

satisfying

$$z_\alpha \in g(y_\alpha).$$

Since $X$ is compact, there exist $y_0$ and a subnet $\{y_{\alpha_\beta}\}$ satisfying

$$\lim_{\beta} y_{\alpha_\beta} = y_0.$$

Therefore, we have

$$y_{\alpha_\beta} \in f(x_{\alpha_\beta}),$$

$$z_{\alpha_\beta} \in g(y_{\alpha_\beta}).$$

Since $f$ and $g$ are upper semi-continuous, we have

$$y_0 \in f(x_0),$$

$$z_0 \in g(y_0).$$

These results conclude the proof.

Let $f$ be a set-valued mapping on $X$ with values in $2^X$. Then, for any $S \subset X$, the image of $S$ under the mapping $f$ is defined as

$$f(S) = \bigcup_{x \in S} f(x).$$

Now, we have the following:

**Lemma 2.** Let $X$ (resp. $Y$) be a Hausdorff space (resp. a compact Hausdorff space), $f$ be an upper semi-continuous, set-valued mapping on $X$ with values in $2^Y$. Then, for any compact subset $S \subset 2^X$, $f(S)$ is also compact.

**Proof.** Let $y_0$ be an element of $Y$ and $\{y_\alpha\}$ be a net consisting of elements of $Y$, which converges to $y_0$. Then, it is sufficient to prove that $y_0 \in f(S)$ holds. For any $\alpha$, there exists $x_\alpha \in S$ satisfying

$$y_\alpha \in f(x_\alpha).$$

Since $\{x_\alpha\}$ is also a net consisting of elements of $S$, there exist an accumulating point $x_0 \in X$ and a subnet $\{x_{\alpha_\beta}\}$ which converges to $x_0$. Since $y_{\alpha_\beta} \in f(x_{\alpha_\beta})$ holds for all $\beta$, we obtain

$$y_0 \in f(x_0) \subset f(S).$$

Therefore, this result concludes the proof.

Let $X$ be a metric space with its metric $d$ and $f$ be a bounded closed set-valued mapping on $X$ with values in $2^X$. Then, for any $x_0 \in X$, $f$ is said to be continuous at $x_0$, if $f$ satisfies the following condition:

$$\lim_{n \to \infty} H(f(x_n), f(x_0)) = 0,$$

where $\{x_n\}_{n=1}^\infty$ is a sequence consisting of elements of $X$, which converges to $x_0$ and $H$ means Hausdorff's metric. It is clear that $f$ is upper semi-continuous at $x_0$, if $f$ is continuous at $x_0$. 
3 Generalized Stackelberg equilibrium points

For any positive integer \( k \) satisfying \( 1 \leq k \leq 3 \), let \( X_k \) be a metric space with its metric \( d_k \), \( S_k \) be a compact subset of \( X_k \) and \( p_k \) be a continuous function on \( S_k \) with values in \( \mathbb{R} \). Then, for any \( (x_1, x_2, x_3) \in \prod_{i=1}^{3} S_i \), the response function from \( p_3 \) to \( p_1 \) and \( p_2 \) is defined as

\[
R_3(x_1, x_2) = \{(x_1, x_2, y_3); y_3 \in S_3, p_3(x_1, x_2, y_3) = \sup_{z_3 \in S_3} p_3(x_1, x_2, z_3)\}.
\]

By the same way as above, the response function from \( p_2 \) to \( p_1 \) is defined as

\[
R_2(x_1) = \{(x_1, y_2, y_3); y_2 \in S_2, (x_1, y_2, y_3) \in R_3(x_1, y_2), p_2(x_1, y_2, y_3) = \sup_{z_2 \in S_2} p_2(x_1, y_2, z_2)\}.
\]

Finally, the Stackelberg equilibrium set is defined as

\[
R_1 = \{(y_1, y_2, y_3); y_1 \in S_1, (y_1, y_2, y_3) \in R_2(y_1), p_1(y_1, y_2, y_3) = \sup_{z_1 \in S_1} p_1(z_1, y_2, z_3)\}.
\]

Then, we have the following:

**Theorem 3.** If \( R_3 \) is continuous, then the Stackelberg equilibrium set is not empty.

**Proof.** Since \( p_3 \) is continuous on \( \prod_{i=1}^{3} S_i \), for any \( x_1 \in S_1 \) and \( x_2 \in S_2 \), \( R_3(x_1, x_2) \) is nonempty and compact, The assumption that \( R_3 \) is continuous on \( \prod_{i=1}^{2} S_i \) implies that \( R_3 \) is also upper semi-continuous. Therefore, for any \( x_1 \in S_1 \), \( R_2(x_1) \) is nonempty and compact, because \( f_2 \) is continuous on \( \prod_{i=1}^{3} S_i \) and \( R_3(x_1, S_2) \) is nonempty and compact. It is sufficient to prove that \( R_2 \) is also upper semi-continuous on \( S_2 \). Let \( x_1^0 \) be an element of \( S_1 \) and \( \{x_1^n\}_{n=1}^{\infty} \) be a sequence consisting of elements of \( \prod_{i=1}^{3} S_i \), which converges to \( x_1^0 \). Let \( (x_1^0, z_2^0, z_3^0) \) be an element of \( \prod_{i=1}^{3} S_i \) and \( \{(x_1^n, y_2^n, y_3^n)\}_{n=1}^{\infty} \) be a sequence consisting of elements of \( \prod_{i=1}^{3} S_i \) satisfying

\[
(x_1^n, y_2^n, y_3^n) \in R_2(x_1^n), \quad n \in \mathbb{N},
\]

\[
\lim_{n \to \infty} (x_1^n, y_2^n, y_3^n) = (x_1^0, z_2^0, z_3^0).
\]

We have only to prove that \( (x_1^0, z_2^0, z_3^0) \in R_2(x_1^0) \) holds. For any \( (x_1^0, z_2^0, w_3^0) \in R_3(x_1^0, z_2^0) \), there exists a sequence \( \{(x_1^n, y_2^n, w_3^n)\}_{n=1}^{\infty} \) consisting of elements of \( \prod_{i=1}^{3} S_i \) satisfying

\[
(x_1^n, y_2^n, w_3^n) \in R_3(x_1^n, y_2^n), \quad n \in \mathbb{N},
\]

\[
\lim_{n \to \infty} (x_1^n, y_2^n, w_3^n) = (x_1^0, y_2^0, w_3^0),
\]

because the assumption shows the following equality:

\[
0 = \lim_{n \to \infty} H(R_3(x_1^n, y_2^n), R_3(x_1^0, z_2^0)) \geq \lim_{n \to \infty} H(R_3(x_1^n, y_2^n), \{(x_1^0, z_2^0, w_3^0)\})
\]
holds. Since the definition of $R_3$ shows the following inequality:

$$p_2(x_1^n, y_2^n, y_3^n) \geq p_2(x_1^n, y_2^n, w_3^n), \quad n \in \mathbb{N}$$

holds. Therefore, we have

$$p_2(x_1^0, z_2^0, z_3^0) = \lim_{n \to \infty} p_2(x_1^n, y_2^n, y_3^n) \geq \lim_{n \to \infty} p_2(x_1^n, y_2^n, w_3^n) = p_2(x_1^0, z_2^0, w_3^0)$$

Since $S_1$ is compact, $R_2$ is upper semi-continuous and $p_1$ is continuous on $\prod_{i=1}^{3} S_i$, $R_1$ is also nonempty and compact.

References


