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MODULI SPACE OF THE POLYNOMIALS WITH DEGREE N

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1 Introduction

The subject of this paper is a new coordinate system, so-called multiplier cooridinates, introduced into the moduli space, $\mathbb{M}_n$, of the polynomial maps $\mathrm{Poly}_n(\mathbb{C})$ from the Riemann sphere, $\widehat{\mathbb{C}}$, to itself, with degree n.

In study of its geometry and topology from a viewpoint of complex dynamical systems, we make use of this system in order to express singular part, and dynamical loci as algebraic curves or surfaces. And to exhibit the moduli space with a higher degree under this system deserves particular attention: for example, a problem of characterization of exceptinal part, $\mathcal{E}_n(=\mathbb{C}^{n-1}\setminus\mathbb{M}_n)$. This problem is our main subject.

The initiator of the use of multiplier cooridinates is J. Milnor ([Mil93]), to the case of the quadratic rational maps.

First, we investigate the moduli space $\mathbb{M}_n$ consisting of all holomorphic (affine) conjugacy classes of $\mathrm{Poly}_n(\mathbb{C})$. A polynomial map p of degree n is monic and centered if it has the form $p(z)=z^n+c_{n-2}z^{n-2}+\cdots+c_1z+c_0$. Every polynomial map from $\widehat{\mathbb{C}}$ to itself is conjugate under an affine change of variable to a monic centered one, and this is uniquely determined up to conjugacy under the action of the group $G(n-1)$ of $(n-1)$-st roots of unity. Hence the affine space $\mathcal{P}_1(n)$ of all monic centered polynomials of degree n with coordinate $(c_0,c_1,\cdots,c_{n-2})$ is regarded as an $(n-1)$-sheeted covering space of $\mathbb{M}_n$ Thus we can use $\mathcal{P}_1(n)$ as a coordinate space for the moduli space $\mathbb{M}_n$, though it remains the ambiguity up to the group $G(n-1)$. This coordinate space has the advantages of being easy to be treated.

However, it would be also worthwhile to introduce another coordinate system having any merit different from $\mathcal{P}_1(n)$'s. In fact, Milnor successfully introduced coordinates in the moduli space of the space of all quadratic rational maps using the elementary symmetric functions of the multipliers at the fixed points of a map ([Mil93]). In the case of $\mathrm{Poly}_n(\mathbb{C})$, we try to explore an analogy to this in section 2.

2 Polynomials of degree n

2.1 Moduli space of polynomial maps

Let $\widehat{\mathbb{C}}$ be the Riemann sphere, and $\mathrm{Poly}_n(\mathbb{C})$ be the space of all polynomial maps of degree n from $\widehat{\mathbb{C}}$ to itself: $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \quad (a_n \neq 0)$. The group $\mathfrak{A}(\mathbb{C})$ of all affine transformations acts on $\mathrm{Poly}_n(\mathbb{C})$ by conjugation:

$$g \circ p \circ g^{-1} \in \mathrm{Poly}_n(\mathbb{C}) \quad \text{for} \quad g \in \mathfrak{A}(\mathbb{C}),\ p \in \mathrm{Poly}_n(\mathbb{C}).$$

Two maps $p_1, p_2 \in \mathrm{Poly}_n(\mathbb{C})$ are **holomorphically conjugate** if and only if there exists $g \in \mathfrak{A}(\mathbb{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. Under the conjugacy of the action of $\mathfrak{A}(\mathbb{C})$, it can be assumed that any map in $\mathrm{Poly}_n(\mathbb{C})$ is "monic" and "centered", i.e.,

$$p(z) = z^n + c_{n-2}z^{n-2} + c_{n-3}z^{n-3} \cdots + c_0.$$

This p is determined up to the action of the group $G(n-1)$ of $(n-1)$-st roots of unity, where each $\eta \in G(n-1)$ acts on $p \in \mathrm{Poly}_n(\mathbb{C})$ by the transformation $p(z) \mapsto p(\eta z)/\eta$. For example, in the case of $n = 4$ the following three monic and centered polynomials belong to the same conjugacy class:

$$\begin{aligned} &z^4 + az^2 + bz + c \\ &z^4 + a\omega z^2 + bz + c\omega^2 \\ &z^4 + a\omega^2 z^2 + bz + c\omega \end{aligned}$$

where ω is a third root of unity.

The quotient space of $\mathrm{Poly}_n(\mathbb{C})$ under this action will be denoted by $\mathbb{M}_n$, and called the **moduli space** of holomorphic conjugacy classes $\langle p \rangle$ of polynomial maps p of degree n. Let $\mathcal{P}_1(n)$ be the affine space of all monic centered polynomials of degree n with coordinate $(c_0, c_1, \cdots, c_{n-2})$. Then we have an $(n-1)$-to-one canonical projection Φ from $\mathcal{P}_1(n)$ onto $\mathbb{M}_n$. Thus we can use $\mathcal{P}_1(n)$ as coordinate space for $\mathbb{M}_n$ though there remains the ambiguity up to the group $G(n-1)$.

2.2 Multiplier coordinates

Now we intend to explore another coordinate space for $\mathbb{M}_n$. For each $p(z) \in \mathrm{Poly}_n(\mathbb{C})$, let $z_1, \cdots, z_n, z_{n+1}(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i ; $\mu_i = p'(z_i)$ $(1 \le i \le n)$, and $\mu_{n+1} = 0$. Consider the elementary symmetric functions of the n multipliers,

$$\begin{aligned}
\sigma_{n,1} &= \mu_1 + \cdots + \mu_n, \\
\sigma_{n,2} &= \mu_1\mu_2 + \cdots + \mu_{n-1}\mu_n = \textstyle\sum_{i=1}^{n-1} \mu_i \sum_{j>i}^{n} \mu_j, \\
&\cdots \\
\sigma_{n,n} &= \mu_1\mu_2 \cdots \mu_n, \\
\sigma_{n,n+1} &= 0.
\end{aligned}$$

Note that these are well defined on the moduli space $\mathbb{M}_n$, since μ_i's are invariant by affine conjugacy.

2.2.1 The holomorphic index fixed point formula

For an isolated fixed point $f(x_0) = x_0,\quad x_0 \ne \infty$ we define the holomorphic index of f at x_0 to be the residue

$$\iota(f,\ x_0) = \frac{1}{2\pi i}\oint \frac{1}{z - f(z)}dz$$

For the point at infinity, we define the residue of f at ∞ to be equal to the residue of $\phi \circ f \circ \phi$ at origin, where $\phi(z) = \frac{1}{z}$. The Fatou index theorem (see [?]) is as follows: For any rational map $f : \mathbf{C} \to \mathbf{C}$ with $f(z)$ not identically equal to z, we have the relation $\sum_{f(z)=z} \iota(f,\ z) = 1$. This theorem can be applied to these μ_i's ; $\sum_{i=1}^{n} \frac{1}{1-\mu_i} + \frac{1}{1-0} = 1$, provided $\mu_i \ne 1 (1 < i < n)$. Arranging this equation for the form of elementary symmetric functions, we have

$$\gamma_0 + \gamma_1\sigma_{n,1} + \gamma_2\sigma_{n,2} + \cdots + \gamma_{n-1}\sigma_{n,n-1} = 0$$

where

$$\gamma_k = (-1)^k n \binom{n-1}{k} \Big/ \binom{n}{k} = (-1)^k(n-k).$$

Note that $\mu_i = 1$ $(1 \le i \le n)$ is allowable here. Then we have the following Linear Relation :

Theorem 1 *Among $\sigma_{n,i}$'s, there is a linear relation*

$$\sum_{k=0}^{n-1} (-1)^k (n-k)\sigma_{n,k} = 0, \tag{1}$$

where we put $\sigma_{n,0} = 1$.

- For the cubic case ($n = 3$),we have $3 - 2\sigma_{3,1} + \sigma_{3,2} = 0$
- For the quartic case ($n = 4$), we have $4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0$

In view of Theorem , we have the natural map Ψ from $\mathbb{M}_n$ to $\mathbb{C}^{n-1}$ corresponding to $\Psi(p) = (\sigma_{n,1}, \sigma_{n,2}, \cdots, \sigma_{n,n-2}, \sigma_{n,n})$.

Let $\Sigma(n)$ be the image $\Psi(\mathbb{M}_n)(\subset \mathbb{C}^{n-1})$.

2.2.2 Characterization of exceptional set

To investigate whether this map Ψ is surjective or not is our main subject: a problem of characterization of the part of $\mathbb{C}^{n-1} \setminus \Sigma(n)$.

We call this set **exceptional set** and denote it by $\mathcal{E}_n = \mathbb{C}^{n-1} \setminus \Sigma(n)$.

our main subject is as follows:
For a given $(s_1, s_2, \cdots, s_{n-2}, s_n) \in \mathbb{C}^{n-1}$, we set s_{n-1} a solution of $\Sigma_{k=0}^{n-1}(-1)^k(n-k)s_k = 0, \quad s_0 = 1$. Then for the point $(s_1, \cdots, s_n) \in \mathbb{C}^{n-1}$, we set a polynomial

$$m(z) = z^n + s_1 z^{n-1} + s_2 z^{n-2} + \cdots + s_{n-1}z + s_n$$

Then w denote the roots of this polynomial by

$$\mu_1, \mu_2, \cdots, \mu_{n-1}, \mu_n,$$

. **Can we obtain a polynomial** $p(z) \in \mathcal{P}_1(n)$ **corresponding to** $(s_1, \cdots, s_n)$ **as** $(\sigma_1, \cdots, \sigma_n)$ **?**

Namely can we find a polynomial satisfying that for fixed points z_i

$$p(z_i) = z_i, \quad (i = 1, \cdots, n) \text{ with } \mu_i = p'(z_i).$$

The case $n = 3$ is nicely solved:Ψ is surjective. (This fact is mentioned in [Mil93] without any details.) We solved this problem for the case $n = 4$: Ψ is no longer surjective.

As for the cases of general n, we expect analogous results.

We have a following result:

Theorem 2 *(M.FUJIMURA)*
If a polynomial $m(z)$ has n roots $\mu_i \neq 1$ satisfying $\sum_i \frac{1}{b_i} = 0 b_i = 1 - \mu_i$, and for any proper subset S of roots, $\sum_{s \in S} \frac{1}{b_s} \neq 0$, then there exists a polynomial $p(z) \in \mathcal{P}_1(n)$ such that

$$p(z_i) = z_i, \quad (i = 1, \cdots, n) \text{ with } \mu_i = p'(z_i).$$

Examples

- For a set $\{\mu, 2 - \mu, \lambda, 2 - \lambda\}, \quad \mu \neq \lambda, \ \mu \neq 1$ a corresponding polynomial exits.
- For a set $\{\mu, 2 - \mu, \mu, 2 - \mu\}$ $\mu \neq 1$, no corresponding polynomial exits.
- For a set $\{\mu, \mu, \mu, \lambda, \lambda\}, \quad \mu \neq 1, \ 5 - 2\mu - 3\lambda = 0$ a corresponding polynomial exits.
- For a set $\{\mu, \mu, \mu, 2 - \mu, \frac{3-\mu}{2}\}, \quad \mu \neq 1$, no corresponding polynomial exits.

2.3 Polynomials of degree 3

2.3.1 Moduli space $\mathbb{M}_3(\mathbb{C})$

Here we abbreviate $\sigma_{3,i}$ as σ_i. These σ_i $i = 1, 2, 3$ are defined on $\mathbb{M}_3(\mathbb{C})$, with the linear relation: $3 - 2\sigma_1 + \sigma_2 = 0$

For the cubic case, we can show that the excetional set is empty: namely for any point $(s_1, s_3) \in \mathbb{C}^2$, we can regard it as a point of $(\sigma_1, \sigma_3) \in \Sigma(3)$ satisfying the above relation $3 - 2\sigma_1 + \sigma_2 = 0$. Therefore, $(s_1, s_3) \in \mathbb{C}^2$ uniquely determines $\langle p \rangle \in \mathbb{M}_3(\mathbb{C})$.

In fact, a map in $\mathrm{Poly}_3(\mathbb{C})$ is conjugate to a normal form $z^3 + az + b$, whose parameter (a, b^2) is unique to the class $\langle p \rangle$. (a, b^2) relates to (σ_1, σ_3) as follows:
Translation Formula for Cubic Polynomials

$$\begin{aligned} \sigma_1 &= -3a + 6, \\ \sigma_3 &= 27b^2 + a(2a-3)^2, \end{aligned} \tag{2}$$

Inverse Formula for Cubic Polynomials

$$\begin{aligned} a &= (6 - \sigma_1)/3, \\ b^2 &= (4\sigma_1^3 - 36\sigma_1^2 + 81\sigma_1 + 27\sigma_3 - 54)/729. \end{aligned} \tag{3}$$

Proposition 1 *(σ_1, σ_3) is a coordinate system of $\mathbb{M}_3(\mathbb{C})$.*

2.4 Polynomials of degree 4

2.4.1 Moduli space $\mathbb{M}_4(\mathbb{C})$

In the case of $\mathrm{Poly}_4(\mathbb{C})$, we can go on further analysis by using a symbolic and algebraic computation systems. Here we write $\sigma_{4,i} = \sigma_i$ $(i = 1, \cdots, 4)$ for brevity. Set $\mathrm{Poly}_4(\mathbb{C}) \ni p(z) = a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$, $\mathcal{P}_1(4) \ni p(z) = z^4 + c_2z^2 + c_1z + c_0$, $\mathbb{M}_4 \ni \langle p \rangle$,

$$\begin{aligned} p &= z^4 + c_2z^2 + c_1z + c_0 \\ &\sim z^4 + \omega c_2 z^2 + c_1 z + \omega^2 c_0 \\ &\sim z^4 + \omega^2 c_2 z^2 + c_1 z + \omega c_0 \\ &\qquad\qquad (\omega^3 = 1) \end{aligned}$$

There are natural projections:

$$\begin{aligned} \Phi &: \mathcal{P}_1(4) \longrightarrow \mathbb{M}_4 \quad \text{three-to-one map} \\ \Psi &: \mathbb{M}_4 \longrightarrow \Sigma(4) \subset \mathbb{C}^{n-1} \quad \text{two-to-one in general.} \end{aligned}$$

2.4.2 Excetional set

For a polynomial $p(z) = a_4z^4+a_3z^3+a_2z^2+a_1z+a_0$, we chose $z^4+c_2z^2+c_1z+c_0 \in <p>$ and set $\Sigma(4) \ni (\sigma_1, \sigma_2, \sigma_4)$. For the quartic case, a linear relation is as follows; $4-3\sigma_1+2\sigma_2-\sigma_3 = 0$. We have a following transformation formula:

$$\begin{aligned}
\sigma_1 &= -8c_1+12 \\
\sigma_2 &= 4c_2^3-16c_0c_2+18c_1^2-60c_1+48 \\
\sigma_4 &= 16c_0c_2^4+(-4c_1^2+8c_1)c_2^3-128c_0^2c_2^2+ \\
&\quad (144c_0c_1^2-288c_0c_1+128c_0)c_2- \\
&\quad 27c_1^4+108c_1^3-144c_1^2+64c_1+256c_0^3
\end{aligned}$$

We have the following result :

Theorem 3 *Exception set is a puncuted curve:*

$$\mathcal{E}_4 = \{(4, s, \frac{s^2}{4}-2s+4),\ s \neq 6,\ \ s \in \mathbb{C}\}$$

2.4.3 On $\mathcal{E}_4$

To a point

$$(\sigma_1, \sigma_2, \sigma_4) = (4, s, \frac{s^2}{4}-2s+4),$$

we set a polynomial

$$m(z) = z^4-\sigma_1z^3+\sigma_2z^2-\sigma_3z+\sigma_4$$

where

$$4-3\sigma_1+2\sigma_2-\sigma_3 = 0.$$

Let roots of this polynomial $m(z)$ be $\mu,\ \mu,\ 2-\mu,\ 2-\mu,$ and $(\sigma_1, \sigma_2, \sigma_4) = (4,\ -2(\mu^2-2\mu-2),\ \mu^4-4\mu^3+4\mu\ ),\ \mu \neq 1$. Then we consider that on the excetional set $\mathcal{E}_4$, quadratic polynomials

$$z^2-\frac{1}{4}\mu^2+\frac{1}{2}\mu$$

are doubled.

参考文献

[FN97] M. Fujimura and K. Nishizawa. Moduli spaces and symmetry loci of polynomial maps. In W. Küchlin, editor, *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation*, pages 342–348. ACM, 1997.

[Mil90] J. Milnor. Dynamics in one complex variables: Introductory lectures. Preprint # 1990/5, SUNY Stony Brook, 1990.

[Mil92] J. Milnor. Remarks on iterated cubic maps. *Experimental Mathematics*, 1:5–24, 1992.

[Mil93] J. Milnor. Geometry and Dynamics of Quadratic Rational Maps. *Experimental Mathematics*, 2(1):37–83, 1993.

[Mil97] J. Milnor. On Rational Maps with Two Critical Points. Preprint ims97-10, SUNY Stony Brook, 1997.

[NF96] K. Nishizawa and M. Fujimura. Moduli spaces of maps with two critical points. *Special Issue No.1, Science Bulletin of Josai Univ.*, pages 99–113, 1997.