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ITERATIVE SCHEMES FOR APPROXIMATING SOLUTIONS OF RELATIONS INVOLVING ACCRETIVE OPERATORS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce two iterative schemes for approximating solutions of the relation $0 \in Av$, where A is an accretive operator satisfying the range condition.

1. INTRODUCTION

Let E be a real Banach space, let $A \subset E \times E$ be an m-accretive operator and let $J_r = (I + rA)^{-1}$ be the resolvent of A for $r > 0$. In this paper, we shall study iterative schemes for solving the relation $0 \in Av$. A well-known method is the following: $x_0 = x \in E$,

$$x_{n+1} = J_{r_n} x_n, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $\{r_n\}$ is a sequence of positive real numbers. The convergence of (1.1) has been studied by Rockafellar [15], Brézis and Lions [1], Lions [7], Pazy [11], Bruck and Reich [4], Reich [12, 13], Nevanlinna and Reich [9], Bruck and Passty [3], Jung and Takahashi [6] etc. On the other hand, Halpern [5] and Mann [8] introduced the following iterative schemes for approximating fixed points of nonexpansive mappings T of E into itself:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

respectively, where $x_0 = x \in E$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. The iterative schemes (1.2) and (1.3) have been studied extensively. See, for example, Takahashi [18, 19] and the references therein.

In this paper, motivated by (1.1), (1.2) and (1.3), we study two iterative schemes to solve the relation $0 \in Av$, where A is an accretive operator satisfying the range condition, that is, $\overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$. Let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Then correspondence to (1.2) is

$$x_{n+1} = P(\alpha_n x + (1 - \alpha_n) J_{r_n} x_n + f_n), \quad n = 0, 1, 2, \ldots$$

and that to (1.3) is

$$x_{n+1} = P(\alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n + f_n), \quad n = 0, 1, 2, \ldots,$$

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where P is a nonexpansive retraction of E onto C and f_n is the term showing a computational error.

2. PRELIMINARIES

Throughout this paper, we denote the set of all nonnegative integers by $\mathbb{N}$. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. The modulus of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If E is uniformly convex, then δ satisfies that

$$\left\|\frac{x+y}{2}\right\| \le r\left(1 - \delta\left(\frac{\epsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \le r$, $\|y\| \le r$ and $\|x-y\| \ge \epsilon$. Let $U = \{x \in E : \|x\| = 1\}$. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for every $x \in E$. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E. A Banach space E is said to satisfy Opial's condition [10] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies

$$\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - z\|$$

for all $z \in E$ with $z \ne y$.

Let C be a closed convex subset of E. A mapping $T \colon C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A closed convex subset C of E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C into itself has a fixed point in D. Let D be a subset of C. We denote the closure of the convex hull of D by $\overline{co}D$. A mapping P of D into itself is said to be a retraction if $P^2 = P$. A subset D of C is said to be a nonexpansive retract of C if there exists a nonexpansive retraction of C onto D.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \ne \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \ge 0$. If A is accretive, then we have $\|x_1 - x_2\| \le \|x_1 - x_2 + r(y_1 - y_2)\|$ for all $x_i \in D(A), y_i \in Ax_i, i = 1, 2$ and $r > 0$. An accretive operator A is said to satisfy the range condition if $\overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r\colon R(I+rA) \to D(A)$ by $J_r = (I+rA)^{-1}$. It is called the resolvent of A. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I+rA)$ and $\|A_r x\| \le \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I+rA)$. We also know that for an accretive operator A satisfying the range condition, $A^{-1}0 = F(J_r)$ for all $r > 0$. An accretive operator A is said to be m-accretive if $R(I+rA) = E$ for all $r > 0$.

In the sequel, unless stated otherwise, we assume that $A \subset E \times E$ is an accretive operator satisfying the range condition and that J_r is the resolvent of A for $r > 0$.

3. Strong convergence theorem

In this section, we study the strong convergence of Halpern's type iteration. We need the following result for the proof of our theorem.

Theorem 1 (Takahashi and Ueda [21]). *Let E be a reflexive Banach space whose norm is uniformly Gâteaux differentiable. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I+rA)$. If $A^{-1}0 \ne \emptyset$, then the strong $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$ for all $x \in C$.*

See also Reich [14]. Using this result, we prove the following theorem. The proof is mainly due to Wittmann [22] and Shioji and Takahashi [16].

Theorem 2. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex nonexpansive retract of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I+rA)$ and let P be a nonexpansive retraction of E onto C. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P(\alpha_n x + (1-\alpha_n) J_{r_n} x_n + f_n), \quad n \in \mathbb{N},$$

where $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{f_n\} \subset E$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} r_n = \infty$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $A^{-1}0 \ne \emptyset$, then $\{x_n\}$ converges strongly to an element of $A^{-1}0$.

Proof. Let $y_n = J_{r_n} x_n$, $v_n = \alpha_n x + (1-\alpha_n) y_n + f_n$ and $u \in A^{-1}0$. Then we have

$$\begin{aligned}
\|x_1 - u\| &= \|P(\alpha_0 x + (1-\alpha_0) y_0 + f_0) - Pu\| \\
&\le \|\alpha_0 x + (1-\alpha_0) y_0 + f_0 - u\| \\
&\le \alpha_0 \|x - u\| + (1-\alpha_0)\|y_0 - u\| + \|f_0\| \\
&\le \alpha_0 \|x - u\| + (1-\alpha_0)\|x_0 - u\| + \|f_0\| \\
&= \|x - u\| + \|f_0\|.
\end{aligned}$$

If $\|x_n - u\| \le \|x - u\| + \sum_{i=0}^{n-1} \|f_i\|$ for some $n \in \mathbb{N} \setminus \{0\}$, then we can similarly show that $\|x_{n+1} - u\| \le \|x - u\| + \sum_{i=0}^{n} \|f_i\|$. Therefore, by induction, we obtain $\|x_{n+1} - u\| \le \|x - u\| + \sum_{i=0}^{n} \|f_i\|$ for all $n \in \mathbb{N}$ and hence $\{x_n\}$ is bounded because $\sum_{n=0}^{\infty} \|f_n\| < \infty$. Then $\{y_n\}$ and $\{v_n\}$ are also bounded. Next we shall show that

$$\limsup_{n\to\infty} \langle x - z, J(v_n - z) \rangle \le 0. \tag{3.1}$$

Since $(x - J_t x)/t \in AJ_t x$, $A_{r_n} x_n \in Ay_n$ and A is accretive, we have

$$\left\langle A_{r_n} x_n - \frac{x - J_t x}{t}, J(y_n - J_t x) \right\rangle \ge 0$$

and hence

$$\langle x - J_t x, J(y_n - J_t x)\rangle \le t\langle A_{r_n} x_n, J(y_n - J_t x)\rangle$$

for all $n \in \mathbb{N}$ and $t > 0$. Then, from $A_{r_n} x_n = (x_n - y_n)/r_n \to 0$ as $n \to \infty$, we obtain

$$\limsup_{n\to\infty}\langle x - J_t x, J(y_n - J_t x)\rangle \le 0 \tag{3.2}$$

for all $t > 0$. It follows from Theorem 1 that $J_t x \to z \in A^{-1}0$ as $t \to \infty$. Then, since the norm of E is uniformly Gâteaux differentiable, for any $\epsilon > 0$, there exists $t_0 > 0$ such that

$$|\langle z - J_t x, J(y_n - J_t x)\rangle| \le \frac{\epsilon}{2} \quad \text{and} \quad |\langle x - z, J(y_n - J_t x) - J(y_n - z)\rangle| \le \frac{\epsilon}{2}$$

for all $t \ge t_0$ and $n \in \mathbb{N}$. Then it follows that

$$\begin{aligned}
&|\langle x - J_t x, J(y_n - J_t x)\rangle - \langle x - z, J(y_n - z)\rangle| \\
&\quad \le |\langle x - J_t x, J(y_n - J_t x)\rangle - \langle x - z, J(y_n - J_t x)\rangle| \\
&\qquad + |\langle x - z, J(y_n - J_t x)\rangle - \langle x - z, J(y_n - z)\rangle| \\
&\quad = |\langle z - J_t x, J(y_n - J_t x)\rangle| + |\langle x - z, J(y_n - J_t x) - J(y_n - z)\rangle| \\
&\quad \le \epsilon
\end{aligned} \tag{3.3}$$

for all $t \ge t_0$ and $n \in \mathbb{N}$. Therefore it follows from (3.2) and (3.3) that

$$\limsup_{n\to\infty}\langle x - z, J(y_n - z)\rangle \le \limsup_{n\to\infty}\langle x - J_t x, J(y_n - J_t x)\rangle + \epsilon \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\limsup_{n\to\infty}\langle x - z, J(y_n - z)\rangle \le 0. \tag{3.4}$$

On the other hand, since $v_n - y_n = \alpha_n(x - y_n) + f_n \to 0$ as $n \to \infty$ and the norm of E is uniformly Gâteaux differentiable, we have

$$\lim_{n\to\infty} |\langle x - z, J(v_n - z)\rangle - \langle x - z, J(y_n - z)\rangle| = 0. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain (3.1).

From $(1 - \alpha_n)(y_n - z) = (v_n - z) - \alpha_n(x - z) - f_n$, we have

$$(1 - \alpha_n)^2\|y_n - z\|^2 \ge \|v_n - z\|^2 - 2\langle \alpha_n(x - z) + f_n, J(v_n - z)\rangle$$

and hence

$$\begin{aligned}
\|x_{n+1} - z\|^2 = \|Pv_n - Pz\|^2 &\le \|v_n - z\|^2 \\
&\le (1 - \alpha_n)^2\|y_n - z\|^2 + 2\langle \alpha_n(x - z) + f_n, J(v_n - z)\rangle \\
&\le (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle x - z, J(v_n - z)\rangle + M\|f_n\|
\end{aligned}$$

for all $n \in \mathbb{N}$, where $M = 2\sup_{n\in\mathbb{N}}\|v_n - z\|$. By (3.1) and $\sum_{n=0}^{\infty}\|f_n\| < \infty$, for any $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$M\sum_{i=m}^{\infty}\|f_i\| \le \frac{\epsilon}{2} \quad \text{and} \quad \langle x - z, J(v_n - z)\rangle \le \frac{\epsilon}{2}$$

for all $n \ge m$. Hence

$$\|x_{n+m+1} - z\|^2 \le (1 - \alpha_{n+m})\|x_{n+m} - z\|^2 + \alpha_{n+m}\frac{\epsilon}{2} + M\|f_{n+m}\|$$

for all $n \in \mathbb{N}$. Then, by induction, we obtain

$$\|x_{n+m+1} - z\|^2 \leq \|x_m - z\|^2 \prod_{i=m}^{n+m} (1-\alpha_i) + \left\{1 - \prod_{i=m}^{n+m} (1-\alpha_i)\right\} \frac{\epsilon}{2} + M \sum_{i=m}^{n+m} \|f_i\|$$
$$\leq \|x_m - z\|^2 \exp\left(-\sum_{i=m}^{n+m} \alpha_i\right) + \frac{\epsilon}{2} + M \sum_{i=m}^{n+m} \|f_i\|$$

for all $n \in \mathbb{N}$. Therefore it follows from $\sum_{n=0}^{\infty} \alpha_n = \infty$ that

$$\limsup_{n\to\infty} \|x_n - z\|^2 = \limsup_{n\to\infty} \|x_{n+m+1} - z\|^2 \leq \frac{\epsilon}{2} + M \sum_{i=m}^{\infty} \|f_i\| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\{x_n\}$ converges strongly to $z \in A^{-1}0$. □

Let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself. Then $A = I - T$ is an accretive operator which satisfies $C = \overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$ and $A^{-1}0 = F(T)$; see Takahashi [17]. Then, putting $A = I - T$ in Theorem 2, we obtain the following result.

Corollary 3. *Let C be a nonempty closed convex nonexpansive retract of a reflexive Banach space E whose norm is a uniformly Gâteaux differentiable, let P be a nonexpansive retraction of E onto C and let T be a nonexpansive mapping from C into itself. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \dfrac{1}{1+r_n} x_n + \dfrac{r_n}{1+r_n} T y_n, \\ x_{n+1} = P(\alpha_n x + (1-\alpha_n) y_n + f_n), \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{f_n\} \subset E$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} r_n = \infty$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly in $F(T)$.

In the case where A is an m-accretive operator, we obtain the following result.

Corollary 4. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let $A \subset E \times E$ be an m-accretive operator. Let $x_0 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) J_{r_n} x_n + f_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{f_n\} \subset E$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} r_n = \infty$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to an element of $A^{-1}0$.

4. Weak convergence theorem

In this section, we prove a weak convergence theorem for Mann's type iteration. Before proving the theorem, we need the following two lemmas.

Lemma 5 (Browder [2]). *Let C be a closed bounded convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $Tz = z$.*

Lemma 6 (Reich [13]). *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable norm, let C be a nonempty closed convex subset of E and let $\{T_0, T_1, T_2, \dots\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and $S_n = T_n T_{n-1} \cdots T_0$ for all $n \in \mathbb{N}$. Then the set $\bigcap_{n=0}^{\infty} \overline{\mathrm{co}}\{S_m x : m \geq n\} \cap U$ consists of at most one point, where $U = \bigcap_{n=0}^{\infty} F(T_n)$.*

For the proof of Lemma 6, see Takahashi and Kim [20]. Now we can prove the following weak convergence theorem.

Theorem 7. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition, let C be a nonempty closed convex nonexpansive retract of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ and let P be a nonexpansive retraction of E onto C. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P(\alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n + f_n), \quad n \in \mathbb{N}, \tag{4.1}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{f_n\} \subset E$ satisfy $\limsup_{n\to\infty} \alpha_n < 1$, $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Proof. First we prove the theorem in the case of $f_n \equiv 0$. Let u be an element of $A^{-1}0$ and $y_n = J_{r_n} x_n$. Then for $l = \|x - u\|$, the set $D = C \cap \{z \in E : \|z - u\| \leq l\}$ is a nonempty closed bounded convex subset of E which is invariant under J_s for $s > 0$. Then we may assume that C is bounded. From

$$\begin{aligned} \|x_{n+1} - u\| &= \|\alpha_n x_n + (1 - \alpha_n) y_n - u\| \\ &\leq \alpha_n \|x_n - u\| + (1 - \alpha_n) \|y_n - u\| \\ &\leq \|x_n - u\|, \end{aligned}$$

$\lim_{n\to\infty} \|x_n - u\|$ exists. We may assume that $\lim_{n\to\infty} \|x_n - u\| \neq 0$ without loss of generality. Since A is accretive and E is uniformly convex, it follows that

$$\begin{aligned} \|y_n - u\| &\leq \left\| y_n - u + \frac{r_n}{2}(A_{r_n} x_n - 0) \right\| \\ &= \left\| y_n - u + \frac{1}{2}(x_n - y_n) \right\| \\ &= \left\| \frac{x_n + y_n}{2} - u \right\| \\ &\leq \|x_n - u\| \left\{ 1 - \delta\left(\frac{\|x_n - y_n\|}{\|x - u\|} \right) \right\} \end{aligned}$$

and hence

$$\begin{aligned} &(1 - \alpha_n) \|x_n - u\| \delta\left(\frac{\|x_n - y_n\|}{\|x - u\|} \right) \\ &\leq (1 - \alpha_n) \{ \|x_n - u\| - \|y_n - u\| \} \\ &= \|x_n - u\| - \alpha_n \|x_n - u\| - (1 - \alpha_n) \|y_n - u\| \\ &\leq \|x_n - u\| - \|x_{n+1} - u\| \end{aligned}$$

for all $n \in \mathbb{N}$. Then, by $\limsup_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \|x_n - u\| \neq 0$, we obtain $\delta(\|x_n - y_n\| / \|x - u\|) \to 0$. This implies $x_n - y_n \to 0$. Let $v \in E$ be a weak subsequential limit of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. Then it follows that $y_{n_i} \rightharpoonup v$. Further, from

$$\begin{aligned}\|y_n - J_1 y_n\| &= \|(I - J_1)y_n\| = \|A_1 y_n\| \le \inf\{\|z\| : z \in Ay_n\}\\ &\le \|A_{r_n} x_n\| = \left\|\frac{x_n - y_n}{r_n}\right\|\end{aligned}$$

and $\liminf_{n\to\infty} r_n > 0$, we have $y_n - J_1 y_n \to 0$. Therefore it follows from Lemma 5 that $v \in F(J_1) = A^{-1}0$.

We assume that E has a Fréchet differentiable norm. Putting $T_n = \alpha_n I + (1 - \alpha_n)J_{r_n}$ and $S_n = T_n T_{n-1} \cdots T_0$, we have $\bigcap_{n=0}^{\infty} F(T_n) = A^{-1}0$ and $\{v\} = \bigcap_{n=0}^{\infty} \overline{\text{co}}\{x_m : m \ge n\} \cap A^{-1}0$ by Lemma 6. Therefore $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Next we assume that E satisfies Opial's condition. Let v_1 and v_2 be two weak subsequential limits of the sequence $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. As above, we have $v_1, v_2 \in A^{-1}0$. We claim that $v_1 = v_2$. If not, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - v_1\| &= \lim_{i\to\infty} \|x_{n_i} - v_1\| < \lim_{i\to\infty} \|x_{n_i} - v_2\| = \lim_{n\to\infty} \|x_n - v_2\|\\ &= \lim_{j\to\infty} \|x_{n_j} - v_2\| < \lim_{j\to\infty} \|x_{n_j} - v_1\| = \lim_{n\to\infty} \|x_n - v_1\|.\end{aligned}$$

This is a contradiction. Hence we have $v_1 = v_2$. This implies that $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Finally we prove the theorem in the case of $f_n \not\equiv 0$. Let $U_n z = T_n z + f_n$ for all $z \in E$ and $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by (4.1) satisfies $x_{n+1} = PU_n x_n$. We define, for every $m \in \mathbb{N}$, the sequence $\{z_n(m)\}$ by $z_0(m) = x_m$ and $z_{n+1}(m) = T_{n+m} z_n(m)$, $n \in \mathbb{N}$. Then, from the above discussion, we know that $\{z_n(m)\}$ converges weakly to some $z(m) \in A^{-1}0$ as $n \to \infty$. By definition, we have

$$\begin{aligned}&\|z_n(m+1) - z_{n+1}(m)\|\\ &\quad= \|T_{n+m}T_{n+m-1}\cdots T_{m+1}x_{m+1} - T_{n+m}T_{n+m-1}\cdots T_m x_m\|\\ &\quad\le \|x_{m+1} - T_m x_m\|\\ &\quad= \|f_m\|\end{aligned}$$

for all $n, m \in \mathbb{N}$. This implies that $\|z(m+1) - z(m)\| \le \|f_m\|$ for all $m \in \mathbb{N}$. Then, from $\sum_{n=0}^{\infty} \|f_n\| < \infty$, $\{z(m)\}$ is a Cauchy sequence and hence $\{z(m)\}$ converges strongly to some $a \in A^{-1}0$ as $m \to \infty$. Now we have

$$\begin{aligned}\|x_{n+m+1} - z_{n+1}(m)\| &= \|PU_{n+m}x_{n+m} - PT_{n+m}z_n(m)\|\\ &\le \|U_{n+m}x_{n+m} - T_{n+m}z_n(m)\|\\ &\le \|T_{n+m}x_{n+m} - T_{n+m}z_n(m)\| + \|f_{n+m}\|\\ &\le \|x_{n+m} - z_n(m)\| + \|f_{n+m}\|\\ &\ \ \vdots\\ &\le \sum_{i=m}^{n+m} \|f_i\|\end{aligned}$$

for all $n, m \in \mathbb{N}$. Therefore

$$\begin{aligned}|\langle x_{n+m+1} - a, h\rangle| &\le |\langle x_{n+m+1} - z_{n+1}(m), h\rangle| + |\langle z_{n+1}(m) - z(m), h\rangle| \\ &\quad + |\langle z(m) - a, h\rangle| \\ &\le \left(\sum_{i=m}^{n+m} \|f_i\| + \|z(m) - a\|\right) \|h\| + |\langle z_{n+1}(m) - z(m), h\rangle|\end{aligned}$$

for all $h \in E^*$ and $n, m \in \mathbb{N}$. This implies

$$\begin{aligned}\limsup_{n\to\infty} |\langle x_n - a, h\rangle| &= \limsup_{n\to\infty} |\langle x_{n+m+1} - a, h\rangle| \\ &\le \left(\sum_{i=m}^{\infty} \|f_i\| + \|z(m) - a\|\right) \|h\|\end{aligned}$$

for all $h \in E^*$ and $m \in \mathbb{N}$. Letting m to ∞, we have $\langle x_n - a, h\rangle \to 0$ for all $h \in E^*$ and hence $\{x_n\}$ converges weakly to $a \in A^{-1}0$. □

As direct consequences of Theorem 7, we obtain the following two results.

Corollary 8. *Let C be a nonempty closed convex nonexpansive retract of a uniformly convex Banach space E whose norm is Fréchet differentiable or which satisfies Opial's condition, let P be a nonexpansive retraction of E onto C and let T be a nonexpansive mapping of C into itself. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \dfrac{1}{1+r_n} x_n + \dfrac{r_n}{1+r_n} T y_n, \\ x_{n+1} = P(\alpha_n x_n + (1-\alpha_n) y_n + f_n), \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{f_n\} \subset E$ satisfy $\limsup_{n\to\infty} \alpha_n < 1$, $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly in $F(T)$.

Corollary 9. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition and let $A \subset E \times E$ be an m-accretive operator. Let $x_0 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n + f_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{f_n\} \subset E$ satisfy $\limsup_{n\to\infty} \alpha_n < 1$, $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

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