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NONLINEAR STRONG ERGODIC THEOREMS WITH COMPACT DOMAINS

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1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Banach space E. Then a mapping $T: C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. For any $x \in C$, the ω-limit set of x is defined by

$$\omega(x) = \{z \in C : z = \lim_{i \to \infty} T^{n_i} x \quad \text{with} \quad n_i \to \infty \quad \text{as} \quad i \to \infty\}.$$

Similarly, the ω-limit set of x for a one-parameter semigroup $\mathcal{S}$ on C is defined by

$$\omega(\mathcal{S}, x) = \{z \in C : z = \lim_{i \to \infty} T(s_i) x \quad \text{with} \quad s_i \to \infty \quad \text{as} \quad i \to \infty\}.$$

Edelstein [10] obtained the following nonlinear ergodic theorem for nonexpansive mappings with compact domains in a strictly convex Banach space:

Theorem 1.1 (Edelstein). Let C be a nonempty compact convex subset of a strictly convex Banach space and let T be a nonexpansive mapping of C into itself. Let $x \in C$. Then, for any $\xi \in \overline{\mathrm{co}}\omega(x)$, the Cesàro mean $S_n(\xi) = (1/n)\sum_{k=0}^{n-1} T^k \xi$ converges strongly to a fixed point of T, where $\overline{\mathrm{co}}A$ is the closure of the convex hull of A.

Dafermos and Slemrod [9] also obtained the following theorem:

Theorem 1.2 (Dafermos and Slemrod). Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on C. Let $x \in C$. Then, for any $\xi \in \overline{\mathrm{co}}\omega(\mathcal{S}, x)$, $(1/t)\int_0^t T(s)\xi ds$ converges strongly to a common fixed point of $T(t), t \in \mathbb{R}^+$.

On the other hand, the first nonlinear weak ergodic theorem for nonexpansive mappings with bounded domains was established in the framework of a Hilbert space by Baillon [5]. Bruck [7] extended Baillon's theorem in [5] to a uniformly convex Banach space whose norm is Fréchet differentiable. Brézis and Browder [6] also proved a nonlinear strong ergodic theorem for nonexpansive mappings of odd-type in a Hilbert space (see also Reich [15]).

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The purpose of this paper is to study nonlinear strong ergodic theorems for families of nonexpansive mappings with compact domains in a strictly convex Banach space. In Section 2, we give an improved result of Edelstein's theorem in [10] by using Bruck [7, 8] and [1, 2]. In Section 3, we give a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup. In Section 4, we study nonlinear strong ergodic properties for commutative semigroups of nonexpansive mappings in a strictly convex Banach space.

2. Theorem for nonexpansive mappings

Throughout this paper, we assume that a Banach space E is real. We denote by E^* the dual space of E and by $\mathbb{N}$ the set of all positive integers. In addition, we denote by $\mathbb{R}$ and $\mathbb{R}^+$ the sets of all real numbers and all nonnegative real numbers, respectively. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, $\overline{A}$, co$A$ and $\overline{\text{co}}A$ mean the closer of A, the convex hull of A and the closure of the convex hull of A, respectively. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors converges strongly to x.

A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. Throughout this paper, we assume that E is a strictly convex Banach space.

In this section, we give a nonlinear strong ergodic theorem for nonexpansive mappings with compact domains in a strictly convex Banach space. The following Lemma will be useful for us.

Lemma 2.1 ([2]). Let C be a nonempty compact convex subset of E and let T be a nonexpansive mapping of C into itself. Let $x \in C$ and $n \in \mathbb{N}$. Then, for any $\varepsilon > 0$, there exists $l_0 = l_0(n, \varepsilon) \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^{l+k+m} x - T^k \left(\frac{1}{n} \sum_{l=0}^{n-1} T^{l+m} x \right) \right\| < \varepsilon$$

for every $m \geq l_0$.

Using Lemma 2.1, we can prove the following lemma.

Lemma 2.2 ([2]). Let C be a nonempty compact convex subset of E and let T be a nonexpansive mapping of C into itself. Let $x \in C$. Then, there exists a sequence $\{i_n\}$ in $\mathbb{N}$ such that for each $z \in F(T)$,

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\|$$

exists.

Remark 2.3 ([2]). In Lemma 2.2, take a sequence $\{i_n'\}$ in $\mathbb{N}$ such that $i_n' \geq i_n$ for each $n \in \mathbb{N}$. Then, we can see that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\| = \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n'} x - z \right\|.$$

for every $z \in F(T)$.

The following lemma plays an important role in the proof of Theorem 2.5.

Lemma 2.4 ([2]). Let C be a nonempty compact convex subset of E. Then,

$$\lim_{n\to\infty} \sup_{\substack{y\in C \\ T\in N(C)}} \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i y - T\left(\frac{1}{n} \sum_{i=0}^{n-1} T^i y \right) \right\| = 0,$$

where $N(C)$ denotes the set of all nonexpansive mappings of C into itself.

Using Lemma 2.2, 2.4 and Remark 2.3, we can prove a nonlinear strong ergodic theorem for nonexpansive mappings (see [2]).

Theorem 2.5 ([2]). Let X be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of X into itself such that $T(X) \subset K$ for some compact subset K of X and let $x \in X$. Then, $(1/n)\sum_{i=0}^{n-1} T^{i+h}x$ converges strongly to a fixed point of T uniformly in $h \in \mathbb{N} \cup \{0\}$. In this case, if $Qx = \lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} T^i x$ for each $x \in X$, then Q is a nonexpansive mapping of X onto $F(T)$ such that $QT^k = T^k Q = Q$ for every $k \in \mathbb{N}$ and $Qx \in \overline{\mathrm{co}}\{T^k x : k \in \mathbb{N}\}$ for every $x \in X$.

3. Theorem for a one-parameter nonexpansive semigroup

In this section, we give a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup with compact domains in a strictly convex Banach space.

A family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$;
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in \mathbb{R}^+$;
(iv) for each $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of common fixed points of $T(t), t \in \mathbb{R}^+$, that is, $F(\mathcal{S}) = \bigcap_{0\leq t<\infty} F(T(t))$.

The following lemma will be useful for us.

Lemma 3.1 ([3]). Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C. Let $x \in C$ and $t > 0$.

Then, for any $\varepsilon > 0$, there exists $p_t = p_t(\varepsilon) \in \mathbb{R}^+$ such that

$$\sup_{h\in\mathbb{R}^+} \left\| \frac{1}{t}\int_0^t T(h+p+\tau)xd\tau - T(h)\left(\frac{1}{t}\int_0^t T(p+\tau)xd\tau\right)\right\| < \varepsilon$$

for every $p \geq p_t$.

Using Lemma 3.1, we can show the following lemma.

Lemma 3.2 ([3]). Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C. Let $x \in C$. Then, there exists a net $\{p_t\}$ in $\mathbb{R}^+$ such that for each $z \in F(\mathcal{S})$,

$$\lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t T(\tau + p_t)xd\tau - z\right\|$$

exists.

Remark 3.3 ([3]). In Lemma 3.2, take a net $\{p_t{}'\}$ in $\mathbb{R}^+$ such that $p_t{}' \geq p_t$ for each $t > 0$. Then, we can see

$$\lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t T(\tau + p_t)xd\tau - z\right\| = \lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t T(\tau + p_t{}')xd\tau - z\right\|$$

for every $z \in F(\mathcal{S})$.

The following lemma plays an important role in the proof of Theorem 3.5.

Lemma 3.4 ([3]). Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on C. Then, for any $h \in \mathbb{R}^+$,

$$\lim_{t\to\infty} \sup_{y\in C} \left\| \frac{1}{t}\int_0^t T(s)yds - T(h)\left(\frac{1}{t}\int_0^t T(s)yds\right)\right\| = 0.$$

Using Lemmas 3.2,3.4 and Remark 3.3, we can show a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup (see [3]).

Theorem 3.5 ([3]). Let C be a nonempty compact convex subset of E. Let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on C and let $x \in C$. Then, $(1/t)\int_0^t T(\tau+h)xd\tau$ converges strongly to a common fixed point of $T(t), t \in \mathbb{R}^+$ uniformly in $h \in \mathbb{R}^+$. In this case, if $Qx = \lim_{t\to\infty}(1/t)\int_0^t T(\tau)xd\tau$ for each $x \in C$, then Q is a nonexpansive mapping of C onto $F(\mathcal{S})$ such that $QT(q) = T(q)Q = Q$ for every $q \in \mathbb{R}^+$ and $Qx \in \overline{\text{co}}\{T(s)x : 0 \leq s < \infty\}$ for every $x \in C$.

4. Theorem for commutative semigroups

In this section, we establish our main strong mean ergodic theorem for commutative semigroups with compact domains in a strictly convex Banach space. Throughout the rest of this paper, we assume that S is a commutative semigroup with identity unless other specified. In this case, $(S, \leq)$ is a directed system when the binary relation $\leq$ on S is defined by $a \leq b$ if and only if there is $c \in S$ with $a + c = b$.

Let $B(S)$ be the Banach space of all bounded real-valued functions on S with the supremum norm. Then, for each $s \in S$ and $g \in B(S)$, we can define $r_s g \in B(S)$ by $(r_s g)(t) = g(t+s)$ for all $t \in S$. We also denote by r_s^* the conjugate operator of r_s. Let D be a subspace of $B(S)$ and let μ be an element of D^*. Then, we denote by $\mu(g)$ the value of μ at $g \in D$. Sometimes, $\mu(g)$ will be also denoted by $\mu_t(g(t))$ or $\int g(t)d\mu(t)$. When D contains 1, a linear functional μ on D is called a mean on D if $\|\mu\| = \mu(1) = 1$. Further, let D be r_s-invariant, i.e., $r_s(D) \subset D$ for every $s \in S$. Then, a mean μ on D is said to be invariant if $\mu(r_s g) = \mu(g)$ for all $s \in S$ and $g \in D$. For $s \in S$, we can define the point evaluation δ_s by $\delta_s(g) = g(s)$ for every $g \in B(S)$. A convex combination of point evaluations is called a finite mean on S. A finite mean μ on S is also a mean on any subspace D of $B(S)$ containing 1.

The following definition which was introduced by Takahashi [17] is crucial in the nonlinear ergodic theory for abstract semigroups (see also [11]). Let f be a function of S into E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact. Let D be a subspace of $B(S)$ containing 1 and r_s-invariant for every $s \in S$. Assume that for each $x^* \in E^*$, the function $t \mapsto \langle f(t), x^* \rangle$ is in D. Then, for any $\mu \in D^*$ there exists a unique element $f_\mu \in E$ such that

$$\langle f_\mu, x^* \rangle = \int \langle f(t), x^* \rangle d\mu(t)$$

for all $x^* \in E^*$. If μ is a mean on D, then f_μ is contained in $\overline{\text{co}}\{f(t) : t \in S\}$ (for example, see [12, 13, 17]). Sometimes, f_μ will be denoted by $\int f(t)d\mu(t)$.

Let C be a subset of a Banach space E. Then, a family $\mathcal{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(s+t) = T(s)T(t)$ for all $s, t \in S$;
(ii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $T(t), t \in S$, that is, $F(\mathcal{S}) = \bigcap_{t \in S} F(T(t))$. If C is a compact convex subset of strictly convex Banach space E and S is commutative, then we know that $F(\mathcal{S})$ is nonempty. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that for each $x \in C$, $\{T(t)x : t \in S\}$ is contained in a weakly compact, convex subset of C. Let D be a subspace of $B(S)$ containing 1 with the property that the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of D for each $x \in C$ and $x^* \in E^*$, and let μ be a mean on D. Following [16], we also write $T_\mu x$ instead of $\int T(t)x \, d\mu(t)$ for $x \in C$. We remark that T_μ is a nonexpansive mapping of C onto itself and $T_\mu x = x$ for each $x \in F(\mathcal{S})$.

The following lemma will be useful for us (see Lemmas 2.1 and 3.1).

Lemma 4.1 ([4]). Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let $x \in C$. Then, for any finite mean μ on S

and $\varepsilon > 0$, there exists $w_0 = w_0(\mu, \varepsilon) \in S$ such that

$$\left\| \int T(h+s+w)x d\mu(s) - T(h)\Big(\int T(s+w)x d\mu(s) \Big) \right\| < \varepsilon$$

for every $h \in S$ and $w \geq w_0$.

Using Lemma 4.1, we can prove the following lemma (see Lemmas 2.2 and 3.2).

Lemma 4.2 ([4]). Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let $x \in C$ and let $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ be nets of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \quad \text{and} \quad \lim_\beta \|\lambda_\beta - r_t^* \lambda_\beta\| = 0 \quad \text{for every } t \in S. \tag{$*$}$$

Then, there exist nets $\{p_\alpha : \alpha \in I\}$ and $\{q_\beta : \beta \in J\}$ in S such that for any $z \in F(\mathcal{S})$,

$$\lim_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int T(q_\beta + t)x d\lambda_\beta(t) - z \right\|. \tag{1}$$

Remark 4.3 ([4]). In Lemma 4.2, take nets $\{p_\alpha{}'\}$ and $\{q_\beta{}'\}$ in S such that $p_\alpha{}' \geq p_\alpha$ and $q_\beta{}' \geq q_\beta$. Then, we can see

$$\lim_\alpha \left\| \int T(p_\alpha{}' + t)x d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int T(q_\beta{}' + t)x d\lambda_\beta(t) - z \right\|$$

for every $z \in F(\mathcal{S})$.

The following lemma plays an important role in the proof of Lemma 4.5 (see Lemmas 2.4 and 3.4).

Lemma 4.4 ([4]). Let C be a nonempty compact convex subset of E, let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C and let $x \in C$. Let $\{\mu_\alpha : \alpha \in I\}$ be a net of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \quad \text{for every} \quad t \in S. \tag{$*$}$$

Then, for any $\varepsilon > 0$ and $t \in S$, there exists $\alpha_0(\varepsilon, t) \in I$ such that

$$\left\| \int T(s+p)x d\mu_\alpha(s) - T(t)\Big(\int T(s+p)x d\mu_\alpha(s) \Big) \right\| < \varepsilon$$

for all $\alpha \geq \alpha_0(\varepsilon, t)$ and $p \in S$.

Using Lemmas 4.2, 4.4 and Remark 4.3, we can show the following lemma which is crucial to prove the main theorem (Theorem 4.6).

Lemma 4.5 ([4]). Let X be a nonempty closed convex subset of E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on X. Assume $\bigcup_{t \in S} T(t)(X) \subset K$ for some compact subset K of X. Let D be a subspace of $B(S)$ such that $1 \in D$, D is r_s-invariant for each

$s \in S$ and the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of D for each $x \in X$ and $x^* \in E^*$. Let $\{\mu_\alpha : \alpha \in I\}$ be a net of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_s^* \mu_\alpha\| = 0 \quad \text{for every } s \in S.$$

Then, for any $x \in X$, $\int T(p+t)x d\mu_\alpha(t)$ converges strongly to a common fixed point y_0 of $T(t), t \in S$ uniformly in $p \in S$. Furthermore, y_0 is independent of $\{\mu_\alpha : \alpha \in I\}$ and for any invariant mean μ on D, $y_0 = T_\mu x = \int T(t)x d\mu(t)$.

Sketch of proof. Let $x \in X$. From Mazur's theorem, $C = \overline{\text{co}}\big(\{x\} \cup \bigcup_{t \in S} T(t)(X)\big)$ is a compact subset of X. We see that $C = \overline{\text{co}}\big(\{x\} \cup \bigcup_{t \in S} T(t)(X)\big)$ is convex and invariant under $T(t), t \in S$. Thus, we may assume that $\mathcal{S} = \{T(t) : t \in S\}$ is a nonexpansive semigroup on a compact convex subset of X.

Let $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ be nets of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \quad \text{and} \quad \lim_\beta \|\lambda_\beta - r_t^* \lambda_\beta\| = 0 \tag{$*$}$$

for each $t \in S$. By Lemma 4.2, we can take a net $\{p_\alpha\}$ in S such that for any $z \in F(\mathcal{S})$,

$$\lim_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\| \tag{2}$$

exists. Let $\{\Phi_\alpha\} = \big\{\int T(p_\alpha + t)x d\mu_\alpha(t) : \alpha \in I\big\}$. Then, we first prove that Φ_α converges strongly to a common fixed point of $T(t), t \in S$. From the compactness, $\{\Phi_\alpha\}$ must contain a subnet which converges strongly to a point. So, let $\{\Phi_{\alpha_\gamma}\}$ be a subnet of $\{\Phi_\alpha\}$ such that $\lim_\gamma \Phi_{\alpha_\gamma} = y_0$. Using Lemma 4.4, we can show that y_0 is a common fixed points of $T(t), t \in S$. So, from (2), we have

$$\lim_\alpha \|\Phi_\alpha - y_0\| = \lim_\gamma \|\Phi_{\alpha_\gamma} - y_0\| = 0.$$

This implies that $\Phi_\alpha \to y_0$.

Next we prove that $\int T(h+t)x d\mu_\alpha(t)$ converges strongly to $y_0 \in F(\mathcal{S})$ uniformly in h. In the above argument, take a net $\{p_\alpha' : \alpha \in I\}$ in S such that $p_\alpha' \geq p_\alpha$ for each $\alpha \in I$. Then, repeating the above argument, we see that $\Phi_\alpha' = \int T(p_\alpha' + t)x d\mu_\alpha(t)$ converges strongly to a common fixed point y_1 of $T(t), t \in S$. By Remark 4.3, we can show $y_0 = y_1 \in F(\mathcal{S})$. Since $\{p_\alpha'\}$ is an arbitrary net in S such that $p_\alpha' \geq p_\alpha$ for each $\alpha \in I$, we have that $\int T(h + p_\alpha + t)x d\mu_\alpha(t)$ converges strongly to y_0 uniformly in $h \in S$. Hence, we can show that $\int T(h+t)x d\lambda_\beta(t)$ converges strongly to y_0 uniformly in $h \in S$. Since $\{\lambda_\beta : \beta \in J\}$ and $\{\mu_\alpha : \alpha \in I\}$ are arbitrary nets of finite means on S such that

$$\lim_\beta \|\lambda_\beta - r_t^* \lambda_\beta\| = 0 \quad \text{and} \quad \lim_\beta \|\mu_\alpha - r_t^* \mu_\alpha\| = 0,$$

for every $t \in S$, we see that such an element y_0 of $F(\mathcal{S})$ is independent of $\{\lambda_\beta : \beta \in J\}$ and $\{\mu_\alpha : \alpha \in I\}$. Further, we can prove that for any invariant mean μ on D, $y_0 = T_\mu x$. □

Let D be a subspace of $B(S)$ containing 1 and r_s-invariant for every $s \in S$. Then, a net $\{\mu_\alpha : \alpha \in I\}$ of linear functionals on D is called strongly regular [11] if it satisfies the following conditions:

(a) $\sup_\alpha \|\mu_\alpha\| < +\infty$;

(b) $\lim_\alpha \mu_\alpha(1) = 1$;

(c) $\lim_\alpha \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in S$.

Now, we can show a nonlinear strong ergodic theorem for commutative semigroups.

Theorem 4.6 ([4]). Let X be a nonempty a closed convex subset of E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on X. Assume $\bigcup_{t \in S} T(t)(X) \subset K$ for some compact subset K of X. Let D be a subspace of $B(S)$ such that $1 \in D$, D is r_s-invariant for each $s \in S$ and the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of D for each $x \in X$ and $x^* \in E^*$. Let $\{\lambda_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on D and let $x \in X$. Then, $\int T(h+t)x d\lambda_\alpha(t)$ converges strongly to a common fixed point y_0 of $T(t), t \in S$ uniformly in $h \in S$. Further, such an element y_0 of $F(\mathcal{S})$ is independent of $\{\lambda_\alpha\}$ and for any invariant mean μ on D, $y_0 = T_\mu x = \int T(t)x d\mu(t)$. In this case, putting $Qx = \lim_\alpha \int T(t)x d\lambda_\alpha(t)$ for each $x \in X$, Q is a nonexpansive mapping of X onto $F(\mathcal{S})$ such that $QT(t) = T(t)Q = Q$ for every $t \in S$ and $Qx \in \overline{\text{co}}\{T(s)x : s \in S\}$ for every $x \in X$.

Sketch of proof. Let $\{\lambda_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on D and let $\{\mu_\beta : \beta \in B\}$ be a net of finite means on S such that

$$\lim_\beta \|\mu_\beta - r_t^* \mu_\beta\| = 0 \quad \text{for every } t \in S. \tag{$*$}$$

From Lemma 4.5, we have that $\int T(h+t)x d\mu_\beta(t)$ converges strongly to a common fixed point y_0 of $T(t), t \in S$ uniformly in $h \in S$. Let $\varepsilon > 0$ and let μ be an invariant mean on D. From Lemma 4.5, we also know $y_0 = T_\mu x$. Further, there exists β_1 such that

$$\left\| \int T(h+t)x d\mu_\beta(t) - T_\mu x \right\| < \frac{\varepsilon}{\sup_\alpha \|\lambda_\alpha\|}$$

for all $\beta \geq \beta_1$ and $h \in S$. Suppose

$$\mu_{\beta_1} = \sum_{i=1}^{n} b_i \delta_{t_i} \quad (b_i \geq 0, \ \sum_{i=1}^{n} b_i = 1) \tag{3}$$

and put $\mu_1 = \mu_{\beta_1}$. Then, we have

$$\left\| \int T(h+t)x d\mu_1(t) - T_\mu x \right\| < \frac{\varepsilon}{\sup_\alpha \|\lambda_\alpha\|} \tag{4}$$

for every $h \in S$. Since $\{\lambda_\alpha\}$ is strongly regular, there exists α_0 such that

$$|1 - \lambda_\alpha(1)| < \frac{\varepsilon}{\max\{1, \|T_\mu x\|\}}$$

and

$$\|\lambda_\alpha - r_{t_i}^* \lambda_\alpha\| < \frac{\varepsilon}{\max\{1, M\}} \tag{5}$$

for every $i \in \{1, 2, \cdots, n\}$ and $\alpha \geq \alpha_0$, where $M = \sup_{g \in S} \|T(g)x\|$. Then, we have

$$\left\| T_\mu x - \int T_\mu x d\lambda_\alpha(s) \right\| \leq \sup_{x^* \in S_1(E^*)} \left| \langle T_\mu x, x^* \rangle \right| \cdot |1 - \lambda_\alpha(1)| < \varepsilon$$

for every $\alpha \geq \alpha_0$ and from (4),

$$\left\| \iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s) - \int T_\mu x d\lambda_\alpha(s) \right\| < \varepsilon$$

for every $h \in S$ and $\alpha \in A$. Thus, we obtain

$$\left\| \iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s) - T_\mu x \right\| < \varepsilon + \varepsilon = 2\varepsilon$$

for every $h \in S$ and $\alpha \geq \alpha_0$. On the other hand, from (3) and (5), we have

$$\left\| \int T(h+s)x d\lambda_\alpha(s) - \iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s) \right\| \leq \sum_{i=1}^{n} b_i \|\lambda_\alpha - r_{t_i}^* \lambda_\alpha\| \cdot M < \varepsilon$$

for every $h \in S$ and $\alpha \geq \alpha_0$. Therefore, we obtain

$$\left\| \int T(h+s)x d\lambda_\alpha(s) - T_\mu x \right\| < \varepsilon + 2\varepsilon = 3\varepsilon$$

for every $h \in S$ and $\alpha \geq \alpha_0$. Then, $\int T(h+t)x d\lambda_\alpha(t)$ converges strongly to a common fixed point y_0 of $T(t), t \in S$ uniformly in h. Further, such an element y_0 is independent of $\{\lambda_\alpha\}$ and $y_0 = T_\mu x$ for any invariant mean μ on D. If $Qx = \lim_\alpha \int T(t)x d\lambda_\alpha(t)$ for each $x \in X$, then Q is a nonexpansive mapping of X onto $F(\mathcal{S})$ such that $QT(t) = T(t)Q = Q$ for every $t \in S$ and $Qx \in \overline{co}\{T(s)x : s \in S\}$ for every $x \in X$. □

5. Applications of the main theorem

We now apply Theorem 4.6 to obtain other nonlinear strong ergodic theorems with compact domains.

Theorem 5.1 ([4]). Let X be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of X into itself such that $T(X)$ is relatively compact. Then, for each $x \in X$, $(1-s)\sum_{i=0}^{\infty} s^i T^{i+k} x$ converges strongly to some $y \in F(T)$, as $s \uparrow 1$, uniformly in $k \in \mathbb{Z}^+$.

Let $Q = \{q_{n,m}\}_{n,m \in \mathbb{Z}^+}$ be a matrix satisfying the following conditions:

(a) $\sup_{n\in\mathbb{Z}^+}\sum_{m=0}^{\infty}|q_{n,m}|<\infty;$

(b) $\lim_{n\to\infty}\sum_{m=0}^{\infty}q_{n,m}=1;$

(c) $\lim_{n\to\infty}\sum_{m=0}^{\infty}|q_{n,m+1}-q_{n,m}|=0.$

Then, according to Lorentz [14], Q is called a strongly regular matrix. If Q is a strongly regular matrix, then for each $m\in\mathbb{Z}^+$, we have that $|q_{n,m}|\to 0$, as $n\to\infty$ (see also [11]).

Theorem 5.2 ([4]). Let E,X and T be as in Theorem 5.1. Let $Q=\{q_{n,m}\}_{n,m\in\mathbb{Z}^+}$ be a strongly regular matrix. Then, for any $x\in X$, $\sum_{m=0}^{\infty}q_{n,m}T^{m+k}x$ converges strongly to some $y\in F(T)$, as $n\to\infty$, uniformly in $k\in\mathbb{Z}^+$.

Theorem 5.3 ([4]). Let X be a nonempty closed convex subset of E. Let U and T be nonexpansive mappings of X into itself with $UT=TU$. Assume $\big(U(X)\cup T(X)\big)\subset K$ for some compact subset K of X. Then, for each $x\in X$, $(1/n^2)\sum_{i,j=0}^{n-1}U^{i+k}T^{j+h}x$ converges strongly to some $y\in F(U)\cap F(T)$, as $n\to\infty$, uniformly in $k,h\in\mathbb{Z}^+$.

Theorem 5.4 ([4]). Let X be a nonempty compact convex subset of E and let $\mathcal{S}=\{T(t):t\in\mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on X. Then, for any $x\in X$, $r\int_0^{\infty}e^{-rt}T(t+k)xdt$ converges strongly to some $y\in F(\mathcal{S})$, as $r\downarrow 0$, uniformly in $k\in\mathbb{R}^+$.

Let $Q=\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ be a function satisfying the following conditions:

(a) $\sup_{s\in\mathbb{R}^+}\int_0^{\infty}|Q(s,t)|dt<\infty;$

(b) $\lim_{s\to\infty}\int_0^{\infty}Q(s,t)dt=1;$

(c) $\lim_{s\to\infty}\int_0^{\infty}|Q(s,t+h)-Q(s,t)|dt=0 \quad$ for every $\quad h\in\mathbb{R}^+.$

Then, Q is called a strongly regular kernel.

Theorem 5.5 ([4]). Let $E,X,\mathcal{S}=\{T(t):t\in\mathbb{R}^+\}$ be as in Theorem 5.4. Let $Q:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ be a strongly regular kernel. Then, for any $x\in X$, $\int_0^{\infty}Q(s,t)T(t+h)xdt$ converges strongly to some $y\in F(\mathcal{S})$, as $s\to\infty$, uniformly in $h\in\mathbb{R}^+$.

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