<table>
<thead>
<tr>
<th>Title</th>
<th>APPROXIMATION OF COMMON FIXED POINTS FOR A FAMILY OF NON-LIPSCHITZIAN SELF-MAPPINGS (Nonlinear Analysis and Convex Analysis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kim, Tae Hwa</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1187: 165-175</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64677">http://hdl.handle.net/2433/64677</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
APPROXIMATION OF COMMON FIXED POINTS FOR
A FAMILY OF NON-LIPSCHITZIAN SELF-MAPPINGS

Tae Hwa Kim

Abstract. In the present paper, we first give some examples of self-mappings which are of strongly
asymptotically nonexpansive type, not strictly hemicontractive, but satisfy the property (H). It is
then shown that the modified Mann and Ishikawa iteration processes for a family \( \mathcal{G} = \{ T_n : n \in \mathbb{N} \} \)
of self-mappings \( T_n : K \to K \), defined by \( x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n \) and \( x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n[(1 - \beta_n)x_n + \beta_n T_n x_n] \), respectively, converge strongly to the unique common fixed point of
such a family \( \mathcal{G} \) in general Banach spaces.

1. Preliminaries

Let \( X \) be a real Banach space and \( X^* \) the dual space of \( X \). Let \( U = \{ x \in X : \| x \| = 1 \} \) be
the unit sphere of \( X \). The norm of \( X \) is said to be Gâteaux differentiable (and \( X \) is said to be
smooth) if the limit

\[
\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}
\]

exists for each \( x \) and \( y \) in \( U \). It is said to be uniformly Gâteaux differentiable if for each \( y \in U \),
this limit is attained uniformly for \( x \in U \). The norm is said to be Fréchet differentiable if for
each \( x \in U \), the limit is obtained uniformly for \( y \in U \). Finally, the space \( X \) is said to have
a uniformly Fréchet differentiable norm (and \( X \) is said to be uniformly smooth) if the limit is
attained uniformly for \((x, y) \in U \times U \).

The normalized duality mapping \( J \) from \( X \) into the family of nonempty subset of \( X^* \) is
defined by

\[
J(x) = \{ f \in X^* : \| f \|^2 = \| x \|^2 = \langle x, f \rangle \},
\]

where \( \langle x, f \rangle \) denotes the value of \( f \) at \( x \). It is an immediate consequence of the Hahn-Banach
theorem that \( J(x) \) is nonempty for each \( x \in X \). Moreover, it is known that \( J \) is single valued
if and only if \( X \) is smooth, while if \( X \) is uniformly smooth, then the mapping \( J \) is uniformly continuous on bounded sets.

Let \( X \) be a real Banach space and let \( K \) be a nonempty subset of \( X \) (not necessarily convex)
and \( T : K \to K \) a self mapping of \( K \). There appear in the literature two definitions of an
asymptotically nonexpansive mapping. The weaker definition (cf. Kirk[19]) requires that

\[
\limsup_{n \to \infty} \sup_{y \in K} \left( \| T^n x - T^n y \| - \| x - y \| \right) \leq 0
\]

1991 Mathematics Subject Classification. 47H09, 47H10.

Key words and phrases. strongly asymptotically nonexpansive type, strictly pseudocontractive (or hemi-
contractive), the property (H), common fixed points.

* Supported by Korea Research Foundation Grant (KRF-99-015-DI0014).
for every \( x \in K \) and that \( T^N \) is continuous for some \( N \geq 1 \). The stronger definition (briefly called asymptotically nonexpansive as in [15]) requires each iterate \( T^n \) to be Lipschitzian with Lipschitz constants \( L_n \rightarrow 1 \) as \( n \rightarrow \infty \). For further generalization of an averaging iteration of Schu [25], Bruck et al. [4] introduced a definition somewhere between these two: \( T \) is asymptotically nonexpansive in the intermediate sense provided \( T \) is uniformly continuous and

\[
\limsup_{n \rightarrow \infty} \sup_{x, y \in K} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.
\]  

In this paper, we consider the self mapping of \( K \) satisfying only (1.1) without the assumption of uniform continuity of \( T \). Throughout we shall refer to such a mapping as strongly asymptotically nonexpansive type.

A mapping \( T : K \rightarrow X \) is said to be pseudo-contractive [26] if for all \( x, y \in K \) there exists \( j \in J(x - y) \) such that

\[
\langle Tx - Ty, j \rangle \leq \|x - y\|^2.
\]

In [18], Kato discovered the relationship between pseudocontractive mappings and accretive mappings, proving

**Lemma 1.1** [18]. Let \( x, y \in X \). Then \( \|x\| \leq \|x + \alpha y\| \) for every \( \alpha > 0 \) if and only if there exists \( j \in J(x) \) such that \( \langle y, j \rangle \geq 0 \).

Applying Lemma 1.1, we know that a mapping \( T \) is pseudocontractive if and only if \( (I - T) \) is accretive, i.e., the inequality

\[
\|x - y\| \leq \|x - y + r((I - T)x - (I - T)y)\|
\]

holds for all \( x, y \in K \) and all \( r \geq 0 \).

In the sequel, we need the following two lemmas for the proof of our main results. The first is actually Lemma 1 of Petryshyn [23] and the second is Lemma 2 of Liu [21]. For the first result, Asplund [1] also proved a general result for single-valued duality mappings, which can be used to derive this lemma and more recently this lemma was revisited by Haiyun-Yuting [16].

**Lemma 1.2** [23]. For any \( x, y \in X \) and \( j \in J(x + y) \),

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j \rangle.
\]

**Lemma 1.3** [21]. Let \( \{a_n\} \), \( \{b_n\} \), and \( \{c_n\} \) be three nonnegative real sequences satisfying

\[
a_{n+1} \leq (1 - t_n)a_n + b_n + c_n \]

with \( \{t_n\} \subset [0, 1] \), \( \sum_{n=0}^{\infty} a_n = \infty \), \( b_n = o(t_n) \), and \( \sum_{n=0}^{\infty} c_n < \infty \). Then \( \lim_{n \rightarrow \infty} a_n = 0 \).

A mapping \( T : K \rightarrow X \) is said to be strictly pseudo-contractive [8], [26] (or strong pseudo-contractive [9]) if there exists \( t > 1 \) such that for all \( x, y \in K \) there exists \( j \in J(x - y) \) such that

\[
\text{Re}(Tx - Ty, j) \leq \frac{1}{t}\|x - y\|^2.
\]

Let \( F(T) \) denotes the set of all fixed points of \( T \), i.e., \( F(T) = \{x \in K : Tx = x\} \). If \( F(T) \neq \emptyset \), the mapping \( T : K \rightarrow X \) is said to be strictly hemicontractive [8] if there exists \( t > 1 \) such that for all \( x \in K \) and \( x^* \in F(T) \) there exists \( j \in J(x - x^*) \) such that

\[
\langle Tx - x^*, j \rangle \leq \frac{1}{t}\|x - x^*\|^2.
\]
Using Lemma 1.1, it is easy to check [8] that the strict hemicontractivity of $T$ is equivalent to the following inequality

\[ ||x - x^*|| \leq ||(1 + r)(x - x^*) - rt(Tx - x^*)|| \]

holds for all $x \in K$, $x^* \in F(T)$ and $r > 0$.

For an example of a Lipschitzian self-mapping which is not strictly pseudocontractive but strictly hemicontractive, see [8].

Motivated by the definition of strict hemicontractivity, we can consider a mapping $T : K \to K$ satisfying the following property, i.e., there exists $t > 1$ such that for all $x \in K$ and $x^* \in F(T)(\neq \emptyset)$ there exists $j \in J(x - x^*)$ such that

\[ \limsup_{n \to \infty} \langle T^n x - x^*, j \rangle \leq \frac{1}{t} ||x - x^*||^2. \]  

(H)

Note that any mapping $T : K \to K$ which is both strictly hemicontractive and asymptotically nonexpansive satisfies the property (H). Indeed, since $T$ is strictly hemicontractive and asymptotically nonexpansive, we have

\[ \langle T^n x - x^*, j \rangle \leq \frac{1}{t} ||T^{n-1} x - x^*||^2 \leq \frac{1}{t} L_n^2 ||x - x^*||^2. \]

Taking lim sup on both sides, since $L_n \to 1$ as $n \to \infty$, $T$ satisfies (H).

First we give two examples of the discontinuous self-mappings which are strongly asymptotically nonexpansive type, not strictly hemicontractive, but satisfies the above property (H).

**Example 1.1.** Let $X = \mathbb{R}$ with the usual norm $| \cdot |$ and let $K = [0,1]$. Let $a_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then, construct a discontinuous mapping $T$ as follows. On the each subinterval $[a_{n+1}, a_n]$, the graph of $T$ consists of all rational numbers of the sides of the isosceles triangle with base $[a_{n+1}, a_n]$ and height $a_{n+1}$ and zeros for irrational numbers in $K$. Thus, $T a_n = 0$ and, if $x_n$ denotes the midpoint of $[a_{n+1}, a_n]$, then $Tx_n = a_{n+1}$. If we further define $T : K \to K$ is not continuous but clearly $F(T) = \{0\}$. Since $T^n x \to 0$ uniformly as $n \to \infty$, $T$ is strongly asymptotically nonexpansive type. Obviously, $T$ satisfies the property (H) but is not strictly hemicontractive.

**Example 1.2.** Let $K = [0,1] \subseteq \mathbb{R}$ and define $T x = \frac{1}{4}$ if $x = \frac{1}{4}$, $1$, $T x = 1$ for $x \in [0, \frac{1}{2}] \setminus \{\frac{1}{4}\}$, and $T x = \frac{1}{2}$ for $x \in (\frac{1}{2}, 1]$. Note that for all $x \in K$, $T^n x = \frac{1}{4} \in F(T) = \{\frac{1}{4}\}$ for $n \geq 3$. Then $T : K \to K$ is a discontinuous mapping of strongly asymptotically nonexpansive type which is not nonexpansive. Obviously, $T$ satisfies the property (H). However, $T$ is not strictly hemicontractive.

Here we give an example of the discontinuous self-mapping with the property (H) which is strongly asymptotically nonexpansive type, not asymptotically nonexpansive.

**Example 1.3.** Let $K = [0,1] \subseteq \mathbb{R}$ and let $\varphi$ be the Cantor ternary function. Define $T : K \to C$ by

\[ T(x) = \begin{cases} 
 x/2 & \text{if } 0 \leq x \leq 1/2, \\
 \varphi((1-x)/2) & \text{if } 1/2 < x \leq 1.
\end{cases} \]

Note that $T^n x \to 0$ uniformly on $K$. Therefore, $T$ is a discontinuous mapping of strongly asymptotically nonexpansive type with the property (H). But it is not asymptotically nonexpansive because $\varphi$ is not Lipschitzian continuous on $[0, \frac{1}{2}]$. Note that $T$ is also strictly hemicontractive.
Recall that a mapping $T : K \to X$ is said to be strongly accretive [3] (or [29]) if there exists a positive number $k$ such that for each $x, y \in K$ there is $j \in J(x - y)$ such that

$$\langle Tx - Ty, j \rangle \geq k \|x - y\|^2.$$  

Using Lemma K again this is equivalent to

$$\|x - y\| \leq \|x - y + r\{(I - kI)x - (I - kI)y\}\|,$$

for all $r > 0$, where $I$ denotes the identity mapping of $X$. Without loss of generality, we can assume $k \in (0, 1)$. Then it was known [2] that the similar connection between strict pseudocontractivity and strong accretivity is that a mapping $T : K \to K$ is strictly pseudocontractive if and only if $I - T$ is strongly accretive, i.e., the inequality

$$\|x - y\| \leq \|x - y + r\{(I - T - kI)x - (I - T - kI)y\}\|$$

holds for any $x, y \in K$ and $r > 0$, where $k = \frac{(t-1)}{t} \in (0, 1)$.

It is well known that if $T : K \to X$ is continuous and strictly pseudocontractive, then $T$ has a unique fixed point (see Corollary 1 of Deimling [12]). Furthermore, if $T : X \to X$ is continuous and strongly accretive, then $T$ is surjective, i.e., for a given $f \in X$, the equation $Tx = f$ has a unique solution.

Recently, the convergence problems of Ishikawa and Mann iteration sequences (cf. Ishikawa [17] and Mann [22]) have been studied extensively by many authors (see Chidume [5-8], Chidume and Osilike [9-11], Deng [13], Deng-Ding [14], Haiyun-Yuting [16], Liu [20], Liu [21], Reich [24] and Tan-Xu [27]) for strictly pseudocontractive (or strongly accretive) mappings.

Especially, Liu [20] proved, using the inequality (1.3), that the Mann iteration process converges strongly to the unique fixed point of a Lipschitzian and strictly pseudo-contractive mapping, which extends corresponding results of [5-8], [27] and [29] to the general Banach spaces as follows.

**Theorem 1.1 [20].** Let $K$ be a nonempty closed, convex and bounded subset of a Banach space $X$ and let $T : K \to K$ be Lipschitzian and strictly pseudocontractive mapping. Then the sequence $\{x_n\}_{n=1}^\infty$ generated by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad x_1 \in K,$$

with $\{\alpha_n\} \subset (0, 1]$ satisfying

$$\sum_{n=1}^\infty \alpha_n = \infty, \quad \alpha_n \to 0,$$

converges strongly to $q \in F(T)$ and $F(T)$ is a singleton set.

Subsequently, Haiyun-Yuting [16] proved by using Lemma 1.2 that the Ishikawa iteration process converges strongly to the unique fixed point of a continuous and strictly pseudocontractive map without Lipschitz assumption in a real uniformly smooth Banach space.

**Theorem 1.2 [16].** Let $K$ be a nonempty closed, convex and bounded subset of a real uniformly smooth Banach space $X$. Assume that $T : K \to K$ is a continuous strictly pseudocontractive mapping. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be two real sequences satisfying

(i) $0 < \alpha_n, \beta_n < 1$ and $\alpha_n \to 0, \beta_n \to 0$ as $n \to \infty$;

(ii) $\sum_{n=1}^\infty \alpha_n = \infty$. 
Then the Ishikawa iterative sequence \( \{x_n\}_{n=1}^{\infty} \) generated from an arbitrary \( x_1 \in K \) by
\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nTx_n
\end{align*}
\]
converges strongly to the unique fixed point of \( T \).

On the other hand, Chidume and Osilke [9] proved with the similar method of the proof as in [20] that the Ishikawa iteration process also converges strongly to the unique fixed point of a uniformly continuous and strictly pseudo-contractive mapping in a real Banach space.

**Theorem 1.3 [9].** Let \( K \) be a nonempty closed, convex and bounded subset of a real Banach space \( X \). Suppose \( T : K \to K \) is a uniformly continuous and strictly pseudocontractive mapping. Then, the sequence \( \{x_n\}_{n=1}^{\infty} \) generated from an arbitrary \( x_1 \in K \) by
\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nTx_n
\end{align*}
\]
converges strongly to \( q \in F(T) \) and \( F(T) \) is a singleton set. Here, \( \{\alpha_n\} \) and \( \{\beta_n\} \) are real sequences in \( [0,1] \) satisfying
\[
\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n.
\]

In 1995, Liu [21] introduced the Ishikawa iteration process with errors as follows:
\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n + u_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nTx_n + v_n, \quad n \geq 1,
\end{align*}
\]
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are real sequences in \( [0,1] \) such that (i) \( \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0 \), (ii) \( \{\beta_n\} \) is bounded, (iii) \( \{u_n\} \) and \( \{v_n\} \) are summable sequences in \( X \), and \( T \) is a Lipschitzian strongly accretive mapping in a uniformly smooth Banach space \( X \).

In 1998, Xu [28] introduced the Ishikawa iteration processes emphasizing the randomness of errors as follows:
\[
\begin{align*}
x_{n+1} &= \alpha_n x_n + \beta_nTy_n + \gamma_n u_n, \\
y_n &= \alpha_n x_n + \beta_nTx_n + \gamma_n v_n,
\end{align*}
\]
where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}, \{\gamma_n'\} \) are sequences in \( [0,1] \) such that (i) \( \lim_{n \to \infty} \beta_n = 0 \), (ii) \( \sum_{n=0}^{\infty} \beta_n = 0 \), (i) \( \lim_{n \to \infty} \beta_n = \infty \), (ii) \( \lim_{n \to \infty} \gamma_n = 0 \), (iv) \( \sum_{n=0}^{\infty} \gamma_n < \infty \), (iv) \( \alpha_n + \beta_n + \gamma_n = 1 \), and \( \{u_n\}, \{v_n\} \) are bounded sequences in Banach space \( X \), and \( T \) is a strongly pseudocontractive mapping in uniformly smooth Banach space \( X \).

In these respects, it seems natural to ask whether the above theorems are still valid for a family \( \mathcal{F} = \{T_n : n \in \mathbb{N}\} \) of self-mappings \( T_n : K \to K \) which satisfies the property (H) type (as the definition replaced \( T^n \) in (H) by \( T_n \)). For our affirmative argument, consider the similar iteration process with errors of (1.5) as follows:
\[
\begin{align*}
x_{n+1} &= \alpha_n x_n + \beta_n Ty_n + \gamma_n u_n, \\
y_n &= \alpha_n' x_n + \beta_n'Ty_n + \gamma_n'v_n, \quad n \geq 1,
\end{align*}
\]
where \( \{u_n\} \) and \( \{v_n\} \) are two bounded sequence in \( K \); \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}, \{\gamma_n'\} \) are real sequences in \( [0,1] \) satisfying the conditions
\[
\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = 1,
\]
for all \( n \geq 1 \).
Lemma 1.4. Let $K$ be a nonempty closed and convex subset of a Banach space $X$. Let two iterative sequences $\{x_n\}$ and $\{y_n\}$ be given as in (1.6) for a family $\mathcal{S} = \{T_n : n \in \mathbb{N}\}$ of self-mappings $T_n : K \to K$, $n \in \mathbb{N}$. Put $B := \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\} (\subset K)$, $q \in F(\mathcal{S}) := \cap_{n \in \mathbb{N}} F(T_n)$ and 

$$c_n := \max\{0, \sup_{x \in B}(\|T_n x - q\| - \|x - q\|)\}.$$ 

Then 

$$||x_n - q|| \leq d + 2 \sum_{k=1}^{n-1} c_k, \quad ||y_n - q|| \leq d + 2 \sum_{k=1}^{n-1} c_k + c_n,$$ 

for $n \in \mathbb{N}$, where 

$$d := \max\{\sup_{n \geq 1} \|u_n - q\|, \sup_{n \geq 1} \|v_n - q\|, \|x_1 - q\|\}.$$ 

Proof. The proof employs mathematical induction. Since $\|x_1 - q\| \leq d$ and 

$$\|y_1 - q\| = \|\alpha_1' x_1 + \beta_1' T x_1 + \gamma_1' v_1 - q\| \leq \alpha_1' \|x_1 - q\| + \beta_1' \|T x_1 - q\| + \gamma_1' \|v_1 - q\| \leq \alpha_1' \|x_1 - q\| + \beta_1' (c_1 + \|x_1 - q\|) + \gamma_1' \|v_1 - q\| \leq (\alpha_1' + \beta_1' + \gamma_1') d + \beta_1' c_1 \leq d + c_1,$$

(1.7) holds for $n = 1$. Suppose (1.7) holds for $n = k$, i.e., 

$$\|x_k - q\| \leq d + 2 \sum_{j=1}^{k-1} c_j, \quad \|y_k - q\| \leq d + 2 \sum_{j=1}^{k-1} c_j + c_j.$$ 

Then, for $n = k + 1$, we have 

$$\|x_{k+1} - q\| = \|\alpha_k x_k + \beta_k T_k y_k + \gamma_k u_k - q\| \leq \alpha_k \|x_k - q\| + \beta_k \|T_k y_k - q\| + \gamma_k \|u_k - q\| \leq \alpha_k \|x_k - q\| + \beta_k (c_k + \|y_k - q\|) + \gamma_k \|u_k - q\| \leq \alpha_k (d + 2 \sum_{j=1}^{k-1} c_j) + \beta_k c_k + \beta_k (d + 2 \sum_{j=1}^{k-1} c_j + c_k) + \gamma_k d = d + 2 (\alpha_k + \beta_k) \sum_{j=1}^{k-1} c_j + 2 \beta_k c_k \leq d + 2 \sum_{j=1}^{k} c_j.$$
APPROXIMATION OF COMMON FIXED POINTS

and

\[ \|y_{k+1} - q\| = \|\alpha'_{k+1}x_{k+1} + \beta'_{k+1}T_{k+1}x_{k+1} + \gamma'_{k+1}v_{k+1} - q\| \]
\[ \leq \alpha'_{k+1}\|x_{k+1} - q\| + \beta'_{k+1}\|T_{k+1}x_{k+1} - q\| + \gamma'_{k+1}\|v_{k+1} - q\| \]
\[ \leq \alpha'_{k+1}\|x_{k+1} - q\| + \beta'_{k+1}(c_{k+1} + \|x_{k+1} - q\|) + \gamma'_{k+1}\|v_{k+1} - q\| \]
\[ \leq (\alpha'_{k+1} + \beta'_{k+1})\|x_{k+1} - q\| + \beta'_{k+1}c_{k+1} + \gamma'_{k+1}d \]
\[ \leq (\alpha'_{k+1} + \beta'_{k+1})(d + 2\sum_{j=1}^{k}c_{j}) + \beta'_{k+1}c_{k+1} + \gamma'_{k+1}d \]
\[ \leq d + 2\sum_{j=1}^{k}c_{j} + c_{k+1}. \]

Therefore, by mathematical induction, (1.7) holds for all \( n \in \mathbb{N} \).

2. MAIN RESULTS

We first begin with an easy observation of the property (H) type. The first equivalent is

\[ \lim_{n \to \infty} \inf_{T_{n}} \langle x - T_{n}x, j \rangle \geq \frac{(t - 1)}{t} \|x - x^{*}\|^{2}. \]

Let \( x \neq x^{*} \). For a fixed \( \epsilon \) with \( 0 < \epsilon < \frac{(t - 1)}{t} \), it follows from the property \( (H_{1}) \) that there exists \( n_{0} \in \mathbb{N} \) such that for all \( n \geq n_{0} \),

\[ \langle x - T_{n}x, j \rangle \geq \frac{(t - 1)}{t} - \epsilon \|x - x^{*}\|^{2} = k_{\epsilon}\|x - x^{*}\|^{2}, \]

where \( k_{\epsilon} := \frac{(t - 1)}{t} - \epsilon \in (0, 1) \). This inequality is obviously equivalent to

\[ \langle T_{n}x - x^{*}, j \rangle \leq (1 - k_{\epsilon})\|x - x^{*}\|^{2}, \quad \forall n \geq n_{0}. \]

For employing the method of the proof in [20], we need the following equivalent form of the property \( (H_{2}) \) by virtue of Lemma 1.1:

\[ \|x - x^{*}\| \leq \|x - x^{*} + r\{(I - T_{n} - k_{\epsilon}I)x - (I - T_{n} - k_{\epsilon}I)x^{*}\}\| \]

for all \( n \geq n_{0} \) and all \( r > 0 \).

Using the property \( (H_{3}) \), Lemma 1.3 and 1.4, we are now ready to present the following

**Theorem 2.1.** Let \( K \) be a nonempty closed and convex subset of a Banach space \( X \). Suppose a family \( \mathcal{S} = \{T_{n} : n \in \mathbb{N}\} \) of self-mappings \( T_{n} : K \to K, n \in \mathbb{N} \) satisfies the property \( (H) \) type. Suppose \( F(T) \neq \emptyset \) and put

\[ c_{n} = \max\{0, \sup_{x,y \in K} (\|T_{n}x - T_{n}y\| - \|x - y\|)\}, \]

so that \( \sum_{n=1}^{\infty} c_{n} < \infty \). Then the modified Ishikawa iterative sequence \( \{x_{n}\}_{n=1}^{\infty} \) generated by (1.6) converges strongly to the unique common fixed point of \( \mathcal{S} \) in \( K \), where

\[ \lim_{n \to \infty} \beta_{n} = \lim_{n \to \infty} \beta'_{n} = \lim_{n \to \infty} \gamma'_{n} = 0; \]
\[ \begin{align*}
\sum_{n=1}^{\infty} \beta_n &= \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n < \infty.
\end{align*} \]

**Proof.** Since \( F(T) \neq \emptyset \), take \( q \in F(T) \). Lemma 1.4 immediately gives
\[ \|x_{n+1} - q\| \leq M, \quad \|y_{n+1} - q\| \leq M, \]
for all \( n \in \mathbb{N} \), where \( M := d + 2 \sum_{n=1}^{\infty} c_n < \infty \). Lemma 1.2 and the property \( (H3) \) yields
\[ \|x_{n+1} - q\|^2 = \|\alpha_{n}(x_{n} - q) + \beta_{n}(T_{n}y_{n} - q) + \gamma_{n}(u_{n} - q)\|^2 \]
\[ \leq \alpha_{n}^2 \|x_{n} - q\|^2 + 2\beta_{n}\langle T_{n}y_{n} - q, j_n \rangle + 2\gamma_{n}\langle u_{n} - q, j_n \rangle \]
\[ \leq \alpha_{n}^2 \|x_{n} - q\|^2 + 2\beta_{n}\langle T_{n}x_{n+1} - q, j_n \rangle 
+ 2\beta_{n}\langle T_{n}y_{n} - T_{n}x_{n+1}, j_n \rangle + 2\gamma_{n}\langle u_{n} - q, j_n \rangle \]
\[ \leq \alpha_{n}^2 \|x_{n} - q\|^2 + 2\beta_{n}(1 - k_{\epsilon})\|x_{n+1} - q\|^2 + 2\beta_{n}d_n + 2\gamma_{n}M^2, \]
for \( j_n \in J(x_{n+1} - q) \) and for all \( n \geq n_0 \), where \( d_n := \langle T_{n}y_{n} - T_{n}x_{n+1} \rangle \). We first claim that \( j_n \to 0 \) as \( n \to \infty \). In fact, by the parameter conditions (i) and (ii) we get
\[ \|y_n - x_{n+1}\| = \|(y_n - q) + (q - x_{n+1})\| 
= \|\alpha'_n(x_n - q) + \beta'_n(T_n x_n - q) + \gamma'_n(v_n - q) 
- \alpha_n(x_n - q) - \beta_n(T_n y_n - q) - \gamma_n(u_n - q)\| \]
\[ \leq (|\beta'_n - \beta_n| + |\gamma'_n - \gamma_n|)\|x_n - q\| + \beta'_n\|T_n x_n - q\| 
+ \gamma'_n\|v_n - q\| + \beta_n\|T_n y_n - q\| + \gamma_n\|u_n - q\| \]
\[ \leq (\beta'_n + \beta_n + \gamma'_n + \gamma_n)\|x_n - q\| + \beta_n(c_n + \|x_n - q\|) + \gamma_n\|v_n - q\| 
+ \beta_n(c_n + \|y_n - q\|) + \gamma_n\|u_n - q\| \]
\[ \leq 2(\beta'_n + \beta_n + \gamma'_n + \gamma_n)M + c_n(\beta'_n + \beta_n) \to 0 \quad \text{as} \quad n \to \infty. \]
Therefore, since \( c_n \to 0 \) as \( n \to \infty \), we get
\[ \|T_n y_n - T_n x_{n+1}\| \leq \|T_n y_n - T_n x_{n+1} - \|y_n - x_{n+1}\| + \|y_n - x_{n+1}\| 
\leq c_n + \|y_n - x_{n+1}\|^2 \to 0 \quad \text{as} \quad n \to \infty. \]
Since \( \|j_n\| = \|x_{n+1} - q\| \leq M \), this gives
\[ |d_n| = |\langle T_n y_n - T_n x_{n+1}, j_n \rangle| 
\leq \|T_n y_n - T_n x_{n+1}\| \cdot \|j_n\| \to 0 \quad \text{as} \quad n \to \infty. \]
On the other hand, since \( \sum_{n=1}^{\infty} \beta_n = \infty \) and \( \beta_n \to 0 \) as \( n \to \infty \), we can choose \( n_1 \geq n_0 \) so that \( \beta_n > 0, 1 - 2\beta_n(1 - k_{\epsilon}) > 0 \), and \( 2k_{\epsilon} - \beta_n > 0 \) for all \( n \geq n_1 \). Then, the above inequality (2.1) can be written as follows:
\[ \|x_{n+1} - q\|^2 \]
\[ \leq \frac{\alpha_{n}^2 \|x_{n} - q\|^2}{1 - 2\beta_{n}(1 - k_{\epsilon})} + \frac{2\beta_{n}d_n}{1 - 2\beta_{n}(1 - k_{\epsilon})} + \frac{2\gamma_{n}M^2}{1 - 2\beta_{n}(1 - k_{\epsilon})} \]
\[ \leq \frac{(1 - \beta_n)^2 \|x_{n} - q\|^2}{1 - 2\beta_{n}(1 - k_{\epsilon})} + \frac{2\beta_{n}d_n}{1 - 2\beta_{n}(1 - k_{\epsilon})} + \frac{2\gamma_{n}M^2}{1 - 2\beta_{n}(1 - k_{\epsilon})} \]
APPROXIMATION OF COMMON FIXED POINTS

Since \( \frac{2k_{\epsilon}-\beta_{n}}{1-2\beta_{n}(1-k_{\epsilon})} \to 2k_{\epsilon} \) as \( n \to \infty \) and \( k_{\epsilon} \in (0,1) \), there exists a \( n_{2} \geq n_{1} \) such that

\[
\left| \frac{2k_{\epsilon}-\beta_{n}}{1-2\beta_{n}(1-k_{\epsilon})} - 2k_{\epsilon} \right| \leq k_{\epsilon}
\]

for all \( n \geq n_{2} \). This implies that \( k_{\epsilon} \leq \frac{2k_{\epsilon}-\beta_{n}}{1-2\beta_{n}(1-k_{\epsilon})} \), that is,

\[
\frac{(1-\beta_{n})^{2}}{1-2\beta_{n}(1-k_{\epsilon})} \leq (1-k_{\epsilon}\beta_{n})
\]

for all \( n \geq n_{2} \). The inequality (2.2) can be expressed as follows.

\[
\|x_{n+1}-q\|^{2} \leq (1-k_{\epsilon}\beta_{n})\|x_{n}-q\|^{2} + \frac{2\beta_{n}d_{n}}{1-2\beta_{n}(1-k_{\epsilon})} + \frac{2\gamma_{n}M^{2}}{1-2\beta_{n}(1-k_{\epsilon})}
\]

for all \( n \geq n_{2} \). Then it follows from Lemma 1.3 that the sequence \( \{x_{n}\} \) strongly converges to the unique fixed point \( q \) of \( T \). Finally, we prove that \( F(T) = \{q\} \), a singleton set. If \( p \in F(T) \), by using the property (H), we obtain

\[
\|p-q\|^{2} = \langle p-q,j \rangle = \limsup_{n \to \infty} \langle T_{n}p-q,j \rangle < \underline{1} \|p-q\|^{2},
\]

for \( j \in J(p-q) \). Since \( t > 1 \), we have \( q = p \). \( \square \)

Remark. In view of the examples 1.1 and 1.2, the above theorem is a new approach of the strong convergence problems of iterative sequences to the unique fixed point of discontinuous non-Lipschitzian self-mappings which are not strictly hemicontractive (hence, not strictly pseudocontractive).

Taking \( \beta_{n}' = \gamma_{n}' = 0 \) for all \( n \geq 1 \) in (1.6), as a direct consequence of Theorem 2.1, we have the following

**Corollary 2.1.** Let \( K \) be a nonempty closed convex subset of a Banach space \( X \). Suppose a family \( \mathcal{T} = \{T_{n} : n \in \mathbb{N}\} \) of self-mappings \( T_{n} : K \to K, n \in \mathbb{N} \) satisfies the property (H) type. Suppose \( F(T) \neq \emptyset \) and put

\[
c_n = \max\{0, \sup_{x,y \in K} (\|T_{n}x - T_{n}y\| - \|x - y\|)\},
\]

so that \( \sum_{n=1}^{\infty} c_{n} < \infty \). Then the modified Mann iterative sequence \( \{x_{n}\}_{n=1}^{\infty} \) with errors generated by

\[
x_{n+1} = (1-\alpha_{n})x_{n} + \alpha_{n}T_{n}x_{n}, \quad x_{1} \in K
\]

with \( \{\alpha_{n}\}_{n=1}^{\infty} \subseteq (0,1] \) satisfying

\[
\sum_{n=1}^{\infty} \beta_{n} = \infty, \quad \sum_{n=1}^{\infty} \gamma_{n} < \infty, \quad \text{and} \quad \lim_{n \to \infty} b_{n} = 0,
\]

strongly converges \( q \in F(T) \) and \( F(T) \) is a singleton set.

As a direct consequence of Theorem 2.1, we obtain the following
Theorem 2.2. Let $K$ be a nonempty bounded closed convex subset of a Banach space $X$. Suppose a family $\mathcal{S} = \{T_n : n \in \mathbb{N}\}$ of Lipschitzian self-mappings $T_n : K \to K$, $n \in \mathbb{N}$ satisfies the property (H) type. Suppose $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} (L_n - 1) < \infty$, where $L_n (\geq 1)$ is the Lipschitz constant of $T_n$. Then the modified Ishikawa iterative sequence $\{x_n\}_{n=1}^{\infty}$ with errors generated by (1.6) converges strongly to the unique fixed point of $T$ in $K$, where

\begin{align*}
(i) \quad & \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \beta'_n = \lim_{n \to \infty} \gamma'_n = 0; \\
(ii) \quad & \sum_{n=1}^{\infty} \beta_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n < \infty.
\end{align*}

Proof. Note that

$$c_n = \max\{0, \sup_{x, y \in K} (\|T_n x - T_n y\| - \|x - y\|)\} \leq (L_n - 1) \delta(K),$$

where $\delta(K)$ denotes the diameter of $K$. Note that all assumptions of Theorem 2.1 are fulfilled, $\square$

Taking $\beta'_n = \gamma'_n = 0$ for all $n \geq 1$ in (1.6), as a direct consequence of Theorem 2.2, we have the following

Corollary 2.2. Let $K$ be a nonempty bounded closed convex subset of a Banach space $X$. Suppose a family $\mathcal{S} = \{T_n : n \in \mathbb{N}\}$ of Lipschitzian self-mappings $T_n : K \to K$, $n \in \mathbb{N}$ satisfies the property (H) type. Suppose $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} (L_n - 1) < \infty$, where $L_n (\geq 1)$ is the Lipschitz constant of $T_n$. Then the modified Mann iterative sequence $\{x_n\}_{n=1}^{\infty}$ with errors generated by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad x_1 \in K$$

with $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1]$ satisfying

$$\sum_{n=1}^{\infty} \beta_n = \infty, \quad \sum_{n=1}^{\infty} \gamma_n < \infty, \quad \text{and} \quad \lim_{n \to \infty} b_n = 0,$$

strongly converges $q \in F(T)$ and $F(T)$ is a singleton set.

Remark. Note that if each $T_n : K \to K$ is $L_n$-Lipschitzian with $\limsup_{n \to \infty} L_n < 1$, then $\mathcal{S} = \{T_n : n \in \mathbb{N}\}$ is of (H) type.

References


Division of Mathematical Sciences, Pukyong National University, Pusan 608-737, Korea

E-mail address: taehwa@dolphin.pku.ac.kr