Minimax Theorems for Vector-Valued Multifunctions

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Georgiev, Pando Gr.; Tanaka, Tamaki

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Minimax Theorems for Vector-Valued Multifunctions *

PANDO GR. GEORGIEV† and TAMAKI TANAKA (田中 憲)‡

1 Introduction

We present a Ky Fan type inequality of mixed kind for vector-valued multifunctions. We use it for proving our first type minimax theorem for vector-valued multifunctions. It is a generalization of the classical Sion minimax theorem for scalar functions (in the compact case), as well as, a generalization of a theorem of Tanaka for vector-valued functions.

We use a vector-valued variant for multifunctions of Ky Fan type inequality, described in the another presentation of us in this volume, in order to derive our second type minimax theorems for vector-valued multifunctions, which is stronger than the first one and uses a special notion of convexity for multifunctions.

The theory of vector optimization has been intensively developed in recent years, as currently the interest is focused on vector-valued multifunctions. Important parts of this theory are the minimax problems and saddle point problems, which have their one specific features with respect to the real-valued case. For a development of such vector-valued problems we refer to [T1-T5] and references therein. The vector-valued, set-valued case proposes more possibilities for definitions of saddle points. In this paper we prove also a Nash equilibrium theorem for vector-valued multifunctions using scalarization and Ky Fan's inequality. As a corollary we obtain a loose saddle point theorem for convex-concave multifunctions (with respect to a specified definition). An advantage in our loose saddle point theorems with respect to the existing ones in the literature (see [K-K], [L-V]) is that our conditions are explicit.

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†Department of Mathematics and Informatics, Sofia University “St. Kl. Ohridski,” 5 James Bourchier Blvd., 1126 Sofia, Bulgaria(ブルガリア・ソフィア大学 数理情報学部), E-mail: pandogg@fmi.uni-sofia.bg, Current address: Laboratory for Advanced Brain Signal Processing, Brain Science Institute, The Institute of Physical and Chemical Research (RIKEN), 2-1, Hirosawa, Wako, Saitama, 351-0198, Japan. Current E-mail: georgiev@bsp.brain.riken.go.jp

‡Department of Mathematical System Science, Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan(千歳大学理工学部数理システム科学科) E-mail: sltana@cc.hirosaki-u.ac.jp
2 Scalar and vector-valued Ky Fan type inequality of mixed kind

Proposition 2.1 (Scalar Ky Fan type inequality of mixed kind). Assume that the functions \( f, g : K \times K \to \mathbb{R} \), where \( K \) is a compact convex nonempty subset of topological vector space, satisfy the properties:

(i) \( f(\cdot, y), g(\cdot) \) are lower semicontinuous for every \( x, y \in K \);
(ii) \( f(x, \cdot), g(\cdot, y) \) are quasi-concave for every \( x, y \in K \).
(iii) \( \min\{f(x, y), g(x, y)\} \leq 0 \) \( \forall x, y \in K \).

Then there exist \( x_0, y_0 \in K \) such that

\[
\min\{\sup_{y \in K} f(x_0, y), \sup_{x \in K} g(x, y_0)\} \leq 0.
\]

Proof. Define the function

\[
h(\tilde{x}, \tilde{y}, x, y) = \min\{f(\tilde{x}, y), g(x, \tilde{y})\}.
\]

It is easy to see that \( h(\cdot, \cdot, x, y) \) is lower semicontinuous on \( K \times K \) and \( h(\tilde{x}, \tilde{y}, \cdot, \cdot) \) is quasiconvex on \( K \times K \). Applying the classical scalar Ky Fan’s inequality (see for instance [A-E]), we obtain the result.

Let \( Y \) be a Banach space, \( C \subset Y \) a closed convex cone with nonempty interior and \( E \) a topological vector space.

Definition 2.2 The multivalued mapping \( F : E \to 2^Y \) is called \( C \)-properly quasiconvex if for every two points \( x_1, x_2 \in X \) and every \( \lambda \in [0, 1] \) we have either

\[
F(\lambda x_1 + (1 - \lambda)x_2) \subseteq F(x_1) - C \quad \text{or} \quad F(\lambda x_1 + (1 - \lambda)x_2) \subseteq F(x_2) - C.
\]

If \( -F \) is \( C \)-properly quasiconvex, then \( F \) is called \( C \)-properly quasiconcave, which is equivalent to \((-C)\)-proper quasi-convex mapping.

Definition 2.3 We shall say that the multifunction \( F : E \to 2^Y \) is \( C \)-lower semicontinuous at \( x_0 \), if for every \( y \in F(x_0) \) and every open \( V \ni 0 \) there exists an open \( U \ni x_0 \) such that \( (y + V + C) \cap F(x) = \emptyset \) for every \( x \in U \).

Definition 2.4 The multifunction \( F \) is called \( C \)-upper semicontinuous at \( x_0 \), if for every \( y \in C \cup (-C) \) such that \( F(x_0) \subset y + \text{int}C \), there exists an open \( U \ni x_0 \) such that \( F(x) \subset y + \text{int}C \) for every \( x \in U \).

Theorem 2.5 (Ky Fan type inequality of mixed kind for multifunctions). Suppose that \( E_1 \) and \( E_2 \) are topological vector spaces, \( X \subset E_1 \) is a nonempty convex compact subset, \( K \subset E_2 \) is a nonempty convex compact subset, \( C \) is closed convex strongly pointed cone with nonempty interior in a Banach space \( Y \) and \( F, G : X \times K \to 2^Y \) are multifunctions satisfying the following conditions:

(i) \( G(x, \cdot) \) is \( C \)-quasiconvex for every \( x \in X \), and \( F(\cdot, y) \) is \( C \)-properly quasiconvex for every \( y \in K \);
(ii) $G(\cdot, y)$ is $-C$-lower semicontinuous for every $y \in K$, and $F(x, \cdot)$ is $-C$-upper semicontinuous for every $x \in X$.

(iii) for every $x \in X, y \in K$ we have: either $G(x, y) \cap (-\text{int}C) = \emptyset$ or $F(x, y) \not\subset -\text{int}C$.

Then there exist $x_0 \in X, y_0 \in K$ such that for every $x \in X, y \in K$ we have: either $G(x_0, y) \cap (-\text{int}C) = \emptyset$ or $F(x, y_0) \not\subset -\text{int}C$.

Proof. Define

$$\varphi((x, y), (x', y')) := \inf\{f(x, y'), g(x', y)\},$$

where

$$f(x, y) = -\inf_{k \in B} \sup_{z \in F(x, y)} h(k, x, z),$$

$$g(x, y) = -\inf_{k \in B} \inf_{z \in G(x, y)} h(k, x, z)$$

and $B$ is an open base of $C$. Using Lemmas 3.1, 3.3 of [G-T1] we obtain that $\varphi((\cdot, \cdot), (x', y'))$ is lower semicontinuous for every $x', y' \in K$, and by Lemmas 3.2, 3.4 in [G-T1], $\varphi((x, y), (\cdot, \cdot))$ is quasi-concave for every $x \in X, y \in K$. We have also $\varphi((x, y), (x, y)) \leq 0$ for every $x, y \in K$. Applying Proposition 2.1 we obtain the result.

We shall denote by sup $A$ (resp. inf $A$), where $A \subset Y$, the set of all efficient points of the set $\overline{A}$ (the norm closure of $A$) with respect to $C$ (resp. with respect to $-C$), i.e.

$$\sup A = \{a \in \overline{A} : (a + C) \cap A = \{a\}\};$$

$$\inf A = \{a \in \overline{A} : (a - C) \cap A = \{a\}\}.$$

Recall that $A$ is bounded with respect to $C$, if the set $(a + C) \cap A$ is bounded for every $a \in A$. A classical lemma of R. Phelps [Ph], which is equivalent to Ekeland's variational principle and which we shall use in the sequel, states that sup $A \neq \emptyset$ (resp. inf $A \neq \emptyset$), if $A$ is bounded with respect to $C$ (resp. with respect to $-C$).

We shall say that the multivalued mapping $F : X \rightarrow 2^Y$, where $X$ is topological space, is bounded with respect to $C$, if for every $x \in X$ and every $y \in F(x)$ the set $(y + C) \cap F(x)$ is bounded.

3 Minimax theorems

**Theorem 3.1 (Minimax theorem 1).** Suppose that $E_1$ and $E_2$ are topological vector spaces, $X \subset E_1$ is nonempty convex compact subset, $K \subset E_2$ is a nonempty convex compact subset, $C$ is closed convex strongly pointed cone with nonempty interior in a Banach space $Y$ and $F, G : X \times K \rightarrow 2^Y$ are multifunctions, bounded with respect to $C$ and $-C$ respectively, and satisfying the following conditions:

(i) $G(x, \cdot)$ is $C$-quasiconvex for every $x \in X$, and $-F(\cdot, y)$ is $C$-properly quasiconvex for every $y \in K$;

(ii) $G(\cdot, y)$ is $-C$-lower semicontinuous for every $y \in K$, and $F(x, \cdot)$ is $C$-upper semicontinuous for every $x \in X$.

(iii) for every $x \in X, y \in K$ and every two vectors $z_1, z_2 \in Y$ satisfying $z_1 - z_2 \notin C$, we have
either $[G(x, y) - z_1] \cap (-\text{int}C) = \emptyset$, or $z_2 - F(x, y) \not\subset -\text{int}C$.

Then for every $z_1$ such that
\[(a) \quad z_1 - \text{int}C \supset \sup \cup_{x \in X} \inf \cup_{y \in K} G(x, y),\]
and for every $z_2$ such that
\[(b) \quad z_2 + \text{int}C \supset \inf \cup_{y \in K} \sup \cup_{x \in X} F(x, y),\]
we have $z_1 - z_2 \in C$.

**Proof.** Assume the contrary. By (ii) it follows that $G(\cdot, y) - z_1$ is $-C$-lower semicontinuous and $z_2 - F(x, \cdot)$ is $-C$-upper semicontinuous. By (i) it follows that $G(x, \cdot) - z_1$ is $C$-quasiconvex and $z_2 - F(\cdot, y)$ is $C$-properly quasiconvex. So, using (iii) we apply Theorem 2.5 and obtain that there exist points $x_0, y_0$ such that for every $x \in X, y \in K$ we have:

\[\text{either } (G(x_0, y) - z_1) \cap (-\text{int}C) = \emptyset\]
\[\text{or } z_2 - F(x, y_0) \not\subset -\text{int}C.\]

Assume that there exists $x \in X$ such that
\[z_2 - F(x, y_0) \subset -\text{int}C.\]

Then
\[(G(x_0, y) - z_1) \cap (-\text{int}C) = \emptyset \quad \forall y \in K.\]

This implies
\[(\inf \cup_{y \in K} G(x_0, y)) \cap (z_1 - \text{int}C) = \emptyset.\] (1)

It is easy to see, using Phelps lemma (see [Ph]) that for any set $S$ which is bounded with respect to $C$, we have
\[S \subset \sup S - C \quad \text{(2)}\]

So, for $S = \inf \cup_{y \in K} G(x_0, y)$, by (2) we have (using (a))
\[
\begin{align*}
\inf \cup_{y \in K} G(x_0, y) & \subset \sup \cup_{x \in X} \inf \cup_{y \in K} G(x, y) - C \\
& \subset z_1 - \text{int}C - C \\
& = z_1 - \text{int}C,
\end{align*}
\]

which is a contradiction with (1). Therefore
\[z_2 - F(x, y_0) \not\subset -\text{int}C \quad \forall x \in X.\]

This implies
\[\sup \cup_{x \in X} F(x, y_0) \not\subset z_2 + \text{int}C \quad \text{(3)}\]

By (b) and (2) we obtain
\[z_2 + \text{int}C = z_2 + \text{int}C + C \supset \inf \cup_{y \in K} \sup \cup_{x \in X} F(x, y) + C \sup \cup_{x \in X} F(x, y_0),\]

which is a contradiction with (3). \(\blacksquare\)

**Definition 3.2** A multifunction $F : E \to 2^Y$ is called **(in the sense of [K-T-H, Definition 3.6])**
(a) type-(v) $C$-properly quasiconvex if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$ we have either $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_1) - C$ or $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) - C$.

(b) type-(iii) $C$-properly quasiconvex if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$ we have either $F(x_1) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C$ or $F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C$.

If $-F$ is type-(v) [resp. type-(iii)] $C$-properly quasiconvex, then $F$ is said be type-(v) [resp. type-(iii)] $C$-properly quasiconcave, which is equivalent to type-(v) [resp. type-(iii)] $(-C)$-properly quasiconvex mapping.

The following theorem is a generalization (in the compact case) of a scalar two-function result of Simon [S, Theorem 1.4], which in turn is a generalization of Sion's minimax theorem [Si].

**Theorem 3.3 (Minimax theorem II).** Suppose that $E_1$ and $E_2$ are topological vector spaces, $X \subset E_1$ is a nonempty convex compact subset, $K \subset E_2$ is a nonempty convex compact subset, $C$ is closed convex strongly pointed cone with nonempty interior in a Banach space $Y$ and $F, G : X \times K \to 2^Y$ are multifunctions, bounded with respect to $C$ and $-C$ respectively, such that the set $\cup_{y \in K} \cup_{x \in X} F(x, y)$ is bounded with respect to $-C$ and the set $\cup_{x \in X} \inf_{y \in K} G(x, y)$ is bounded with respect to $C$. Suppose that $F$ and $G$ satisfy the following conditions:

(i) $G(\cdot, \cdot)$ is type-(iii) $C$-properly quasiconvex on $K$ for every $x \in X$;

and $F(\cdot, \cdot)$ is type-(iii) $C$-properly quasiconcave on $K$ for every $y \in K$;

(ii) $G(\cdot, y)$ is $-C$-lower semicontinuous for every $y \in K$, and $F(x, \cdot)$ is $C$-lower semicontinuous for every $x \in X$.

(iii) $F(x, y) - G(x, y) \subset -C$ for every $x \in X, y \in K$.

Then there exist two points

$$z_1 \in \sup_{x \in X} \inf_{y \in K} G(x, y)$$

and

$$z_2 \in \inf_{y \in K} \sup_{x \in X} F(x, y)$$

such that $z_1 - z_2 \in C$.

For the proof of this theorem we need the following result.

**Theorem 3.4 ([G-T] Theorem 4.4).** Let $K$ be a nonempty convex subset of a topological vector space $E$, $Y$ a Banach space, and $F : K \times K \to 2^Y$ a multifunction. Assume that

1. $C : K \to 2^Y$ is a multifunction with a closed graph such that $C(x)$ is closed convex cone with compact base $B(x) = (2B_Y \setminus B_Y) \cap C(x)$ for every $x$;

2. for every $x, y \in K$, $F(\cdot, y)$ is $C(x)$-lower semicontinuous and locally bounded;

3. there exists a multifunction $G : K \times K \to 2^Y$ such that

   (a) for every $x \in K, G(x, x) \subset -C(x)$,

   (b) $F(x, y) \not\subset -C(x)$ implies $G(x, y) \not\subset -C(x)$,
(c) $G(x, \cdot)$ is type-(iii) $C(x)$-properly quasiconcave on $K$ for every $x \in K$;

4. there exists a nonempty compact convex subset $D$ of $K$ such that for every $x \in K \setminus D$, there exists $y \in D$ with $F(x, y) \not\subset -C(x)$.

Then, the solutions set

$$S = \{x \in K : F(x, y) \subset -C(x), \text{ for all } y \in K\}$$

is a nonempty and compact subset of $D$.

Proof of Theorem 3.3. Define the mapping $H : X \times K \times X \times K \to 2^Y$ by

$$H(\tilde{x}, \tilde{y}, x, y) = F(x, \tilde{y}) - \tilde{c}(\tilde{x}, y).$$

Applying Theorem 3.4 for $H$ we obtain that there exists $x_0, y_0$ such that

$$H(x_0, y_0, x, y) \subset -C \quad \forall x \in X, \forall y \in K,$$

whence

$$\sup \bigcup_{x \in X} F(x, y_0) - \inf \bigcup_{y \in K} G(x_0, y) \subset -C. \quad (4)$$

By (2) we obtain

$$\sup \bigcup_{x \in X} F(x, y_0) \subset \inf \bigcup_{y \in K} \sup \bigcup_{x \in X} F(x, y) + C$$

and

$$\inf \bigcup_{y \in K} G(x_0, y) \subset \sup \bigcup_{x \in X} \inf \bigcup_{y \in K} G(x, y) - C.$$ 

Therefore, by (4) there exist

$$z_1 \in \sup \bigcup_{x \in X} \inf \bigcup_{y \in K} G(x, y), c_1 \in C$$

and

$$z_2 \in \inf \bigcup_{y \in K} \sup \bigcup_{x \in X} F(x, y), c_2 \in C$$

such that

$$z_2 + c_2 - (z_1 - c_1) \in -C,$$

which implies

$$z_1 - z_2 \in C + c_1 + C \subset C.$$

4 Nash equilibrium and loose saddle point theorems

Definition 4.1 The multifunction $F : E \supset X \to 2^Z$, where $X$ is a convex nonempty subset, is called $C$-convex, if for every $x, y \in X$, $\lambda \in [0, 1]$, $u \in \lambda F(x) + (1 - \lambda)F(y)$ there exists $v \in F(\lambda x + (1 - \lambda)y)$ such that $u - v \in C$. If $F$ is $-C$-convex, then $F$ is called $C$-concave.

Let $k^0 \in \text{int}C$ be fixed. Define the functions

$$h(x) = \inf \{t \in \mathbb{R} : x \in tk^0 - C\},$$

$$\varphi(x) = \inf h(F(x)),$$

$$\psi(x) = \sup h(F(x)).$$
It is easy to see that $h$ is continuous and sublinear (see [Tam1], [Tam2]).

**Lemma 4.2** Let the multifunction $F : E \supset X \to 2^Z$ be $C$-convex. Then the function $\varphi$ is convex.

**Proof.** Let $x_1, x_2 \in X$. By definition of $\varphi$ and $h$, for every $\varepsilon > 0$ there exist $z_i \in F(x_i), t_i \in \mathbb{R}, i = 1, 2$ such that

$$z_i - t_i k^0 \in -C$$

and

$$t_i < \varphi(x_i) + \varepsilon.$$ 

By definition of $C$-convex multifunction,

$$\exists v \in F(\lambda x_1 + (1 - \lambda)x_2) : \lambda z_1 + (1 - \lambda)z_2 \in v + C. \quad (6)$$

By (5) we have

$$-C \ni \lambda(z_1 - t_1 k^0) + (1 - \lambda)(z_2 - t_2 k^0) = \lambda z_1 + (1 - \lambda)z_2 - (\lambda t_1 + (1 - \lambda)t_2)k^0. \quad (7)$$

By (6) and (7) we have

$$v \in \lambda z_1 + (1 - \lambda)z_2 - C$$

$$\subseteq (\lambda t_1 + (1 - \lambda)t_2)k^0 - C - C$$

$$= (\lambda t_1 + (1 - \lambda)t_2)k^0 - C.$$ 

Hence

$$h(v) \leq \lambda t_1 + (1 - \lambda)t_2$$

$$< \lambda \varphi(x_1) + (1 - \lambda)\varphi(x_2) + 2\varepsilon.$$ 

Therefore

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) := \inf_{z \in F(\lambda x_1 + (1 - \lambda)x_2)} h(z) \leq \lambda \varphi(x_1) + (1 - \lambda)\varphi(x_2) + 2\varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrarily small, we obtain

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda \varphi(x_1) + (1 - \lambda)\varphi(x_2).$$

**Definition 4.3** The multifunction $F : E \to 2^Z$ will be called $(C, k^0)$-upper semicontinuous at $x_0$, if for every $\varepsilon > 0$ there exists an open $U \ni x_0$ such that

$$[(\varphi(x_0) - \varepsilon)k^0 - C] \cap F(x) = \emptyset \quad \forall x \in U.$$ 

**Lemma 4.4** If $F$ is $-C$-lower semicontinuous, then $\varphi$ is upper semicontinuous.

**Proof.** Let $x_0 \in E, \varepsilon > 0$ be fixed and $y_0 \in F(x_0)$ be such that

$$h(y_0) < \inf h(F(x_0)) + \varepsilon.$$
By continuity of $h$, there exists an open $V \ni 0$ such that

$$h(v) < \varepsilon \quad \forall v \in V.$$ 

By definition of $-C$-lower semicontinuity, there exists an open $U \ni x_0$ such that

$$F(x) \cap (y_0 + V - C) \neq \emptyset \quad \forall x \in U.$$ 

Let $y \in F(x) \cap (y_0 + V - C)$. Then $y = y_0 + v - c$ for some $v \in V, c \in C$ and we can write

$$\varphi(x) = \inf_{y \in F(x)} h(y')$$

$$\leq h(y)$$

$$\leq h(y_0) + h(v) + h(-c) \quad \text{(by sublinearity of } h)$$

$$\leq \varphi(x_0) + 2\varepsilon.$$ 

**Lemma 4.5** If $F$ is $(C, k^0)$-upper semicontinuous, then $\varphi$ is lower semicontinuous.

**Proof.** Let $x_0 \in E, y \in F(x_0)$ and $x \in U$, where $U$ is given by the definition of $(C, k^0)$-upper semicontinuity of $F$ at $x_0$. Let $z \in F(x)$. Then by definition we have:

$$0 \leq \inf \{t : z - tk^0 \in (\varphi(x_0) - \varepsilon)k^0 - C \}$$

$$= \inf \{t : z - t(\varphi(x_0) - \varepsilon)k^0 \in -C \}$$

$$= \varepsilon - \varphi(x_0) + \inf \{t : z - tk^0 \in -C \}$$

$$\leq \varepsilon - \varphi(x_0) + h(X_0).$$

Hence $\varphi(x_0) \leq h(z) + \varepsilon$, and $z \in F(x)$ is arbitrary, this implies $\varphi(x_0) \leq \varphi(x) + \varepsilon$. \blacksquare

Below we prove a Nash equilibrium type theorem and a loose saddle point theorem. The proofs are based on scalarization via the previous lemmas and on the Ky Fan inequality.

Let $E_1, E_2$ be topological vector spaces, $Z$ be a Banach space, $X \subset E_1, Y \subset E_2$ be convex compact nonempty subsets and $C_i \subset Z$ be closed convex cones with nonempty interiors, $k^0_i \in \text{int} C_i, i = 1, 2$.

**Theorem 4.6** (Nash equilibrium). Let the multifunctions $F_i : X \times Y \rightarrow 2^Z$ be $(C_i, k^0_i)$-upper semicontinuous. Assume that $F_1(\cdot, y)$ is $C_1$-convex for every $y \in Y$, $F_1(x, \cdot)$ is $-C_1$-lower semicontinuous for every $x \in X$, $F_2(x, \cdot)$ is $C_2$-convex for every $x \in X$ and $F_2(\cdot, y)$ is $-C_2$-lower semicontinuous for every $y \in Y$. Then there exists a Nash equilibrium, $(x_0, y_0) \in X \times Y$, which means

$$F_1(x, y_0) \cap [\text{inf } h(F_1(x_0, y_0))k^0_1 - \text{int } C_1] = \emptyset \quad \forall x \in X,$$

$$F_2(x_0, y) \cap [\text{inf } h(F_2(x_0, y_0))k^0_2 - \text{int } C_2] = \emptyset \quad \forall y \in Y.$$ 

**Proof.** Define

$$f(x, y, \bar{x}, \bar{y}) = \text{inf } h(F_1(x, y)) - \text{inf } h(F_1(\bar{x}, y)) + \text{inf } h(F_2(x, y)) - \text{inf } h(F_2(\bar{x}, \bar{y}))$$

By Lemma 4.2, $f(x, y, \cdot, \cdot)$ is concave for every $x \in X, y \in Y$ and by Lemmas 4.4, 4.5, $f(\cdot, \cdot, \bar{x}, \bar{y})$ is lower semicontinuous for every $\bar{x} \in X, \bar{y} \in Y$. By Ky Fan's inequality (see [A-E, Theorem 6.3.5]) there exists $(x_0, y_0) \in X \times Y$ such that

$$\sup_{(\bar{x}, \bar{y}) \in X \times Y} f(x_0, y_0, \bar{x}, \bar{y}) \leq 0$$
Putting $\overline{y} = y_0$ we obtain
\[ \inf h(F_1(x_0, y_0)) \leq \inf h(F_1(x, y_0)) \quad \forall x \in X, \tag{8} \]
and putting $\overline{x} = x_0$ we obtain
\[ \inf h(F_2(x_0, y_0)) \leq \inf h(F_2(x, y_0)) \quad \forall y \in Y. \tag{9} \]
But (8) implies
\[ F_1(x, y_0) \cap [\inf h(F_1(x_0, y_0)) k_1^0 - \text{int}C_1] = \emptyset \]
and (9) implies
\[ F_2(x_0, y) \cap [\inf h(F_2(x_0, y_0)) k_2^0 - \text{int}C_2] = \emptyset, \]
which finishes the proof. \(\blacksquare\)

In the special case when $F_1 = -F_2$ and $C_1 = C_2 = C, k_1^0 = k_2^0 = k^0$, we obtain the following loose saddle point theorem.

**Theorem 4.7 (Loose saddle point theorem).** Suppose that the multifunction $F : X \times Y \to 2^Z$ have compact images and is $(C, k^0)$-lower semicontinuous and $(-C, -k^0)$-lower semicontinuous, $F(\cdot, y), y \in Y$ is $C$-convex and $C$-lower semicontinuous, $F(x, \cdot), x \in X$ is $C$-concave and $-C$-lower semicontinuous. Then there exists a loose saddle point $(x_0, y_0) \in X \times Y$, namely there exist $z_1, z_2 \in F(x_0, y_0)$, such that
\[ (z_1 - \text{int}C) \cap F(x, y_0) = \emptyset \quad \forall x \in X, \]
\[ (z_2 + \text{int}C) \cap F(x_0, y) = \emptyset \quad \forall y \in Y. \]

**REFERENCES**


