数理解析研究所講究録
1187 巻 2001 年 131-142

Recurrent Dimensions of Quasi-Periodic Orbits with Frequencies Given by Weak Liouville Numbers

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1. Introduction

In our previous papers ([4], [5]) we estimated correlation dimensions of quasi-periodic orbits according to algebraic properties, rational (badly) approximable properties, of the irrational frequencies. We introduced a class of irrational numbers, quasi Roth numbers, which contains the class of Roth numbers. These irrational numbers are classified according to badness levels of approximable properties by rational numbers. On the contrary, Liouville numbers are well known as the irrational numbers which have extremely good approximable properties by rational numbers. In [7] we introduced a new class of irrational numbers which contains the class of Liouville numbers and we called them α-order Liouville numbers or quasi Liouville numbers, specifying goodness levels of rational approximations by the order values. In this paper we consider a class of irrational numbers, which have weaker goodness levels of rational approximations than the α-order Liouville numbers, and call them α-order weak Liouville numbers.

In [4] we estimated the correlation dimensions of discrete quasi-periodic orbits from below, using the badness levels of rational approximations for the irrational frequencies which are α-order quasi Roth numbers. In this paper first we introduce definitions of recurrent or periodically recurrent dimensions and we give the relations between correlation dimensions and recurrent dimensions. Then we estimate lower and upper dimensions of quasi-periodic orbits of a nonlinear discrete dynamical system by using the goodness levels of rational approximations for the irrational frequencies which are α-order Liouville or α-order weak Liouville numbers.

Our plan of this paper is as follows: In section 2 we introduce definitions of recurrent dimensions and give inequality relations with correlation dimensions. In section 3 we estimate these dimensions, from below and upper, of quasi-periodic orbits with frequencies given by quasi Roth or weak Liouville numbers.

2. Recurrent dimension

Let T be a nonlinear operator on a Banach space X. For an element $x \in X$ we consider a discrete dynamical system given by

$$x_n = T^n x, \quad n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$$

and its orbit is denoted by

$$\Sigma_x = \{T^n x : \quad n \in \mathbf{N}_0\}.$$

For a small $\varepsilon > 0$, define upper and lower first ε-recurrent times by

$$\overline{M}_\varepsilon = \sup_{n \in \mathbf{N}_0} \min\{m : T^{m+n}x \in V_\varepsilon(T^n x), \quad m \in \mathbf{N}\},$$
$$\underline{M}_\varepsilon = \inf_{n \in \mathbf{N}_0} \min\{m : T^{m+n}x \in V_\varepsilon(T^n x), \quad m \in \mathbf{N}\},$$

respectively, where $V_\varepsilon(z) = \{y \in X : \|y - z\| < \varepsilon\}$. Then upper and lower recurrent dimensions are defined as follows:

$$\overline{D}_r(\Sigma_x) = \limsup_{\varepsilon \to 0} \frac{\log \overline{M}_\varepsilon}{-\log \varepsilon},$$
$$\underline{D}_r(\Sigma_x) = \liminf_{\varepsilon \to 0} \frac{\log \underline{M}_\varepsilon}{-\log \varepsilon}.$$

If $\overline{M}_\varepsilon = \underline{M}_\varepsilon$ and the limit exits as $\varepsilon \to 0$, we denote $D_r(\Sigma_x) = \overline{D}_r(\Sigma_x) = \underline{D}_r(\Sigma_x)$.

The recurrent properties are essential for almost periodic dynamical systems. Next we define periodically recurrent dimensions of almost periodic orbits. Let the operator T be invertible and consider the almost periodic orbit $\tilde{\Sigma}_x = \{T^m x : m \in \mathbf{Z}\}$:

For each $\varepsilon > 0$ there exists a number $l_\varepsilon > 0$ such that for every $m \in \mathbf{Z}$ there exists an integer $\mu \in [m, m + l_\varepsilon] \cap \mathbf{Z}$ with the property

$$|T^{(\mu+n)}x - T^n x| \leq \varepsilon \quad \text{for all} \quad n \in \mathbf{Z}. \tag{2.1}$$

Here the point μ is called an ε-almost period and l_ε is called an inclusion length for ε-almost period.

By using the inclusion length we can define periodically recurrent dimensions as follows:

$$\overline{D}_p(\tilde{\Sigma}_x) = \limsup_{\varepsilon \to 0} \frac{\log l_\varepsilon}{-\log \varepsilon},$$
$$\underline{D}_p(\tilde{\Sigma}_x) = \liminf_{\varepsilon \to 0} \frac{\log l_\varepsilon}{-\log \varepsilon}.$$

If the limit exists as $\varepsilon \to 0$, we put $D_p(\tilde{\Sigma}_x) = \overline{D}_p(\tilde{\Sigma}_x) = \underline{D}_p(\tilde{\Sigma}_x)$.

From the definitions it is obvious that

$$D_p(\tilde{\Sigma}_x) \geq D_r(\Sigma_x),$$

since $l_\varepsilon \geq M_\varepsilon$.

Correlation dimensions are most popular and studied in various dynamical systems or in fractal geometry. Let $S = \{x_1, x_2, ..., x_n, ...\}$ be an infinite sequence of elements in X and, for a small number $\varepsilon > 0$, define

$$N(\varepsilon) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} H(\varepsilon - \|x_i - x_j\|)$$

where $H(\cdot)$ is a Heaviside function:

$$H(a) = \begin{cases} 1 & \text{if } a \geq 0 \\ 0 & \text{if } a < 0. \end{cases}$$

The upper and lower correlation dimension of S, $\overline{D}_c(S), \underline{D}_c(S)$, are defined as follows:

$$\begin{aligned} \overline{D}_c(S) &= \limsup_{\varepsilon \downarrow 0} \frac{\log N(\varepsilon)}{\log \varepsilon}, \\ \underline{D}_c(S) &= \liminf_{\varepsilon \downarrow 0} \frac{\log N(\varepsilon)}{\log \varepsilon}. \end{aligned}$$

If $\overline{D}_c = \underline{D}_c$, we say that S has the correlation dimension $D_c(S) = \overline{D}_c = \underline{D}_c$.

Theorem 1. *Let X be a Banach space and consider a nonlinear operator T on X and its orbits $\Sigma_x = \{T^n x : n \in \mathbf{N}_0\}$ for some $x \in X$. Then*

$$\underline{D}_c(\Sigma_x) \geq \underline{D}_r(\Sigma_x). \tag{2.2}$$

Proof. From the definition, for every $\delta > 0$, there exists a constant $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$,

$$\underline{D}_r \leq \frac{\log \underline{M}_\varepsilon}{-\log \varepsilon} + \delta.$$

It follows that

$$\varepsilon^{-\underline{D}_r + \delta} \leq \underline{M}_\varepsilon.$$

For a large integer $n \in \mathbf{N}$, let $1 \leq l, m \leq n$. If $0 \leq m - l < \varepsilon^{-\underline{D}_r + \delta}$, we have

$$\|T^m x - T^l x\| = \|T^{m-l} T^l x - T^l x\| \geq \varepsilon,$$

and also, if $0 \leq l - m < \varepsilon^{-\underline{D}_r + \delta}$, we have

$$\|T^l x - T^m x\| = \|T^{l-m} T^m x - T^m x\| \geq \varepsilon.$$

Let $M_n(\varepsilon)$ be a number of elements $T^m x,\ 1 \le m \le n$ in the ε-neighborhood of $T^l x,\ 1 \le l \le n$;

$$M_n(\varepsilon) = \#\{m \in \mathbf{N} : T^m x \in V_\varepsilon(T^l x),\ \ 1 \le m \le n\}.$$

Then we have

$$M_n(\varepsilon/2) \le n\varepsilon^{\underline{D}_r - \delta}$$

and it follows that

$$\frac{1}{n^2}\sum_{l,m=1}^{n} H(\frac{\varepsilon}{2} - \|T^l x - T^m x\|) \le \frac{1}{n^2} n \cdot n\varepsilon^{\underline{D}_r-\delta} = \varepsilon^{\underline{D}_r-\delta}.$$

Thus we obtain

$$\underline{D}_c = \liminf_{\varepsilon\to 0} \frac{\log N(\frac{\varepsilon}{2})}{\log\frac{\varepsilon}{2}} \ge \liminf_{\varepsilon\to 0} \frac{\log \varepsilon^{\underline{D}_r-\delta}}{\log\frac{\varepsilon}{2}} = \underline{D}_r - \delta$$

for every $\delta > 0$, which yields (2.2). $\square$

On the other hand, we can estimate the upper bound of the correlation dimensions by using the periodically recurrent dimension by using the similar argument to the proof of Theorem 1.

Theorem 2. *Let $T^m x, m \in \mathbf{Z}$, be almost periodic and denote*

$$\tilde{\Sigma}_x = \{T^m x : \quad m \in \mathbf{Z}\}.$$

Then we have

$$\overline{D}_p(\tilde{\Sigma}_x) \ge \overline{D}_c(\tilde{\Sigma}_x). \tag{2.3}$$

3. Dimensions of quasi-periodic orbits

Let $S(t),\ t \ge 0$, be a semigroup of continous (generally nonlinear) operators on a Banach space X. For each $x \in X$, assume that $S(t+1)x = S(t)x,\ t \ge 0$ and consider the following Hölder conditions:

(G1) There exists a constants $\delta_1 :\ 0 < \delta_1 \le 1$ and a monotone increasing function $k_1 : \mathbf{R}^+ \to \mathbf{R}^+$, which satisfies

$$\|S(t)x - S(s)x\| \le k_1(\|x\|)|t-s|^{\delta_1}, \quad t, s \ge 0, |t-s| \le \varepsilon_0$$

for a small constant $\varepsilon_0 > 0$.

(G2) There exists a constant $\delta_2 :\ 0 < \delta_2 \leq 1$ and a monotone increasing function $k_2 : \mathbf{R}^+ \to \mathbf{R}^+$, which satisfies

$$\|S(t)x - S(s)x\| \geq k_2(\|x\|)|t-s|^{\delta_2}, \quad t,s \geq 0, |t-s| \leq \frac{1}{2}.$$

For an irrational number $\tau : 0 < \tau < 1$, define a quasi-periodic dynamical system by

$$T^n x = S(\tau n)x, \quad n \in \mathbf{N}_0,$$

then our purpose is to estimate the recurrent dimension under the following algebraic conditions on the frequency τ.

(i) Constant type; there exists a constant $c_0 > 0$ such that

$$|\tau - \frac{r}{q}| \geq \frac{c_0}{q^2} \tag{3.1}$$

for every positive integers r, q.
(ii) Roth number type; for every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ which satisfies

$$|\tau - \frac{r}{q}| \geq \frac{c_\varepsilon}{q^{2+\varepsilon}} \tag{3.2}$$

for every positive integers r, q.
(iii) α_0-order quasi Roth number type; there exist a constant $\alpha_0 > 0$ such that for every $\alpha \geq \alpha_0$ there exists a constant $c_\alpha > 0$ which satisfies

$$|\tau - \frac{r}{q}| \geq \frac{c_\alpha}{q^{2+\alpha}} \tag{3.3}$$

for every positive integers r, q.

These above conditions are classified by the rational approximable properties of the irrational number τ:

Consider the continued fraction of the number $\tau = [a_1 a_2 \cdots a_n \cdots]$. and take the rational approximation as follows. Let $m_0 = 1, n_0 = 0, m_{-1} = 0, n_{-1} = 1$ and define the pair of sequences of natural numbers

$$m_i = a_i m_{i-1} + m_{i-2}, \tag{3.4}$$

$$n_i = a_i n_{i-1} + n_{i-2}, \quad i \geq 1, \tag{3.5}$$

then the elementary number theory gives the Diophantine approximation

$$\frac{1}{m_i(m_{i+1}+m_i)} < |\tau - \frac{n_i}{m_i}| < \frac{1}{m_i m_{i+1}} < \frac{1}{m_i^2} \tag{3.6}$$

where the sequence $\{n_i/m_i\}$ is the best approximation in the sense that

$$|\tau - \frac{n_i}{m_i}| \leq |\tau - \frac{r}{l}|$$

holds for every rational $r/l: \ l \leq m_i$

An irrational number τ, which has extremely good approximable property by rational numbers, is called a Liouville number if

$$|\tau - \frac{n_i}{m_i}| \leq \frac{1}{m_i^i}, \quad \forall i.$$

Here we introduce a class of irrational numbers which contains Liouville numbers as follows. We state that an irrational number τ is an α_1-order Liouville number, or a quasi Liouville number with its order α_1 if
(iv) there exist constants $c, \alpha_1 > 0$ such that

$$|\tau - \frac{n_i}{m_i}| \leq \frac{c}{m_i^{2+\alpha_1}}, \quad \forall i. \tag{3.7}$$

Furthermore, considering some subsequece of the Diophantine approximation, we define α_1-order weak Liouville numbers as follows:
(v) There exists a subsequence $\{m_{k_j}\} \subset \{m_j\}$: which satisfies

$$|\tau - \frac{n_{k_j}}{m_{k_j}}| < \frac{c}{m_{k_j}^{2+\alpha_1}}$$

for some constant $c > 0$.

For the case of the constant type (i), it is well known (cf. [8]) that the uniform boundedness of the sequence $\{a_j\}$ is equivalent to the property (1.1). For the quasi Roth numbers and the weak Liouville numbers we can show the equivalent or sufficient conditions to the rationally approximable properties of these numbers by using the growth rate of some subsequences $\{m_{k_j}\}$.

(R1) There exists a subsequence $\{m_{k_j}\}$ which satisfies

$$m_{k_{j+1}} \leq K m_{k_j}^{1+\beta}, \quad \forall j. \tag{3.8}$$

for some constants $\beta, K > 0$: We can obtain the following two lemmas.

Lemma 1. *If Hypothesis* **(R1)** *is satisfied for an irrational number τ, then τ is a quasi Roth number with its order*

$$\alpha_0 = \beta(\beta + 3). \tag{3.9}$$

On the other hand, in [5] we have already proved the following Lemma.

Lemma 2.([5]) *If τ is a quasi Roth number with its order α_0, then for every $\beta \geq \alpha_0$, there exists $K_\beta > 0$ which satisfies*

$$m_{j+1} \leq K_\beta m_j^{1+\beta}, \quad \forall j. \tag{3.10}$$

For the α-order Liouville numbers we have given the equivalent condition in [5]:

(L1) There exist constants $\alpha_1, L > 0$:

$$m_{j+1} \geq L m_j^{1+\alpha_1}, \quad \forall j. \tag{3.11}$$

Lemma 3.([5]) *τ is a quasi Liouville number with its order α_1 if and only if τ satisfies the condition* **(L1)**.

Obviously, **(L1)** is equivalent to the following condition on the partial quotients in the continued fraction expansion of τ.

(L2) There exist constants $\alpha_1, L' > 0$:

$$a_{j+1} \geq L' m_j^{\alpha_1}, \quad \forall j. \tag{3.12}$$

In [5] we have given a sufficient condition for a quasi Roth number, using the partial quotients of the continued fraction expansion.

Lemma 4.([5]) Let $\{a_j\}$ be the partial quotients in the continued fraction expansion of τ. Assume that, for a given constant $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$, which satisfies

$$a_{j+1} a_j^2 \leq C_\varepsilon (a_{j-1} a_{j-2} \cdots a_1)^\varepsilon, \quad \forall j.$$

Then we have

$$|\tau - \frac{r}{q}| \geq \frac{c_\varepsilon}{q^{2+\varepsilon}}, \quad \forall q, r \in N$$

where $c_\varepsilon = 1/(16 C_\varepsilon)$.

For a α-order Liouville number we can show the following lemma.

Lemma 5. If the partial quotients in the continued fraction expansion of τ satisfies

$$a_{j+1} \geq L_0 a_j^{\beta+1}, \quad \forall j$$

for some $\beta > 0$ and $L_0 \geq 2^{\beta+1}$, then τ is a quasi Liouville number with its order β.

For the weak Liouville numbers we can show the equivalent condition:

(WL1) There exist constants $\alpha_1, L > 0$:

$$m_{k_j+1} \geq L m_{k_j}^{1+\alpha_1}, \quad \forall j. \tag{3.13}$$

Lemma 6. *τ is a weak Liouville numbers with its order α_1 if and only if τ satisfies the condition* **(WL1)**.

Obviously, **(WL1)** is equivalent to the following condition on the partial quotients in the continued fraction expansion of τ.

(WL2) There exist constants $\alpha_1, L' > 0$:

$$a_{k_j+1} \geq L' m_{k_j}^{\alpha_1}, \quad \forall j. \tag{3.14}$$

For a sufficient condition for a quasi-Roth number, instead of Lemma 4, we can show the following lemma.

Lemma 7. Let $\{a_j\}$ be the partial quotients in the continued fraction expansion of τ. Assume that there exists a subsequence $\{a_{k_j}\}$, which satisfies that, for a given constant $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$(a_{k_j+1}+1)(a_{k_j}+1)^2(a_{k_j-1}+1)^2 \cdots (a_{k_{j-1}+2}+1)^2(a_{k_{j-1}+1}+1)^2 \leq C_\varepsilon (a_{k_{j-1}} a_{k_{j-1}-1} \cdots a_1)^\varepsilon, \quad \forall j.$$

Then we have

$$|\tau - \frac{r}{l}| \geq \frac{c_\varepsilon}{l^{2+\varepsilon}}, \quad \forall l, r \in N.$$

On the other hand, for a weak Liouville number, we can show the following lemma.

Lemma 8. Assume that the partial quotients $\{a_j\}$ in the continued fraction expansion of τ has a subsequence $\{a_{k_j}\}$, which satisfies

$$a_{k_{j+1}} \geq (a_{k_{j+1}-1}+1)^\beta (a_{k_{j+1}-2}+1)^\beta \cdots (a_{k_j+1}+1)^\beta (a_{k_j}+1)^\beta a_{k_j} \tag{3.15}$$

for some $\beta > 0$, then τ is a weak Liouville number with its order β.

Example 1. For some positive numbers $\kappa, M > 1$, let

$$a_j \sim M^{\kappa^j},$$

that is, there exist constants $d_1 > d_2 > 0$:

$$d_1 M^{\kappa^j} \geq a_j \geq d_2 M^{\kappa^j}. \tag{3.16}$$

Assume that

$$M^{\kappa^2-\kappa} \geq \frac{2d_1}{d_2}, \tag{3.17}$$

then τ is a quasi Liouville number with its order β:

$$\beta \leq \frac{\log d_2 + \kappa^2 \log M}{\log 2d_1 + \kappa \log M} - 1. \tag{3.18}$$

Example 2. Let $\{k_j\}$ be a sequence of integers which is increasing and goes to infinity such that

$$k_j - k_{j-1} \leq C\kappa^j \tag{3.19}$$

for some $C > 0$ and $\kappa > 1$. For constants $M, M' > 1$, to simplify the argument, let

$$a_{k_j} = M^{\kappa^j}, \qquad a_l \leq M', \quad l \notin \{k_j : j \in \mathrm{N}\}. \tag{3.20}$$

Then the irrational number, which has the partial quotients above, is a weak Liouville number with its order β, which satisfies

$$\beta \leq \frac{\kappa - 1}{1 + \frac{C\kappa \log(M'+1)}{\log M} + \frac{\kappa^{-1}\log 2}{\log M}}. \tag{3.21}$$

The number, which satisfies (3.19) and (3.20), is also a quasi Roth number with its order α such that

$$\alpha \geq (\kappa - 1)(2C\kappa \frac{\log(M'+1)}{\log M} + 2).$$

In fact, for every $\varepsilon > 0$, which saisfies

$$\varepsilon \frac{\kappa}{\kappa - 1} \geq \frac{\log(M'+1)}{\log M} 2C\kappa^2 + 2\kappa,$$

there exists a constant C_ε such that

$$\begin{aligned} C_\varepsilon M^{\frac{\kappa(\kappa^{j-1}-1)}{\kappa-1}\varepsilon} &\geq (M'+1) M^{\frac{\log(M'+1)}{\log M} 2C\kappa^{j+1}} 2^2 M^{2\kappa^j} \\ &\geq (M'+1) M^{\frac{\log(M'+1)}{\log M} 2(k_{j+1}-k_j-1)} (M^{\kappa^j}+1)^2. \end{aligned}$$

Thus we have

$$C_\varepsilon M^{(\kappa^{j-1}+\kappa^{j-2}+\cdots+\kappa^1)\varepsilon} \geq (M'+1)(M'+1)^{2(k_{j+1}-k_j-1)}(M^{\kappa^j}+1)^2,$$

which implies the condition in Lemma 7.

Since the correlation dimensions are estimated by the recurrent dimensions, here we give upper bounds and lower bounds of the recurrent dimension of the quasi-periodic orbits

$$\Sigma_x = \{S(\tau n)x : n \in \mathbf{N}_0\}, \quad x \in X$$

when the frequency τ is (iii) a quasi Roth number and (v) a weak Liouville number, respectively.

Theorem 3. *Under the assumption* **(G2)**, *assume that there exists a constant $K_2 > 0$ such that*

$$\inf_{n \in \mathbf{N}_0} k_2(\|S(\tau n)x\|) \geq K_2$$

and assume that the frequency τ is a quasi Roth number with its order α_0. Then the recurrent dimension of the quasi-periodic orbit Σ_x satisfies

$$\underline{D}_r(\Sigma_x) \geq \frac{1}{\delta_2(1+\alpha_0)}. \tag{3.22}$$

Proof. Put

$$\varphi(m) = S(\tau m)x, \quad m \in \mathbf{N}_0,$$

then, since we can find an integer n':

$$|m\tau - n'| < \frac{1}{2},$$

it follows from (iii), **(G2)** and Hypothesis that

$$\begin{aligned} \|\varphi(m+n) - \varphi(n)\| &= \|S(\tau(m+n))x - S(\tau n)x\| \\ &= \|S(\tau(m+n))x - S(\tau n + n')x\| \\ &\geq k_2(\|S(\tau n)x\|)|\tau m - n'|^{\delta_2} \\ &\geq K_2\left(\frac{c_\alpha}{m^{1+\alpha}}\right)^{\delta_2}, \quad \forall \alpha \geq \alpha_0, \end{aligned}$$

for all $m \in \mathbf{N}$ and for all $n \in \mathbf{N}_0$. For every $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$K_2(\frac{c_\alpha}{(m+1)^{1+\alpha}})^{\delta_2} \leq \varepsilon < K_2(\frac{c_\alpha}{m^{1+\alpha}})^{\delta_2}$$

and as $\varepsilon \to 0$, $m \to +\infty$. Thus we can obtain

$$\begin{aligned} \underline{D}_r &= \liminf_{\varepsilon \to 0} \frac{\log M_\varepsilon}{-\log \varepsilon} \\ &\geq \liminf_{\varepsilon \to 0} \frac{\log m}{-\log \varepsilon} \\ &\geq \lim_{m \to \infty} \frac{\log m}{\delta_2(1+\alpha)\log(m+1) - \log K_2 c_\alpha^{\delta_2}} \\ &= \frac{1}{\delta_2(1+\alpha)} \end{aligned}$$

for all $\alpha \geq \alpha_0$. □

Theorem 4. *Under the assumption* **(G1)**, *assume that the frequency τ is a weak Liouville number with its order α_1. Then the recurrent dimension of the quasi-periodic orbit Σ_x satisfies*

$$\underline{D}_r(\Sigma_x) \leq \frac{1}{\delta_1(1+\alpha_1)}. \tag{3.23}$$

Proof. Put

$$\varepsilon_j = \frac{k_1(\|S(\tau n_0)x\|)c^{\delta_1}}{m_{k_j}^{\delta_1(1+\alpha_1)}}$$

for some positive integer n_0 and $x \in X$. It follows from Hypothesis that we have

$$|m_{k_j}\tau - n_{k_j}| < \frac{c}{m_{k_j}^{\alpha_1+1}}. \tag{3.24}$$

By the above estimate and **(G1)** we have

$$\begin{aligned} \|\varphi(m_{k_j}+n_0) - \varphi(n_0)\| &= \|S(\tau(m_{k_j}+n_0)x - S(\tau n_0)x\| \\ &= \|S(\tau(m_{k_j}+n_0)x - S(\tau n_0 + n_{k_j})x\| \\ &\leq k_1(\|S(\tau n_0)x\|)|m_{k_j}\tau - n_{k_j}|^{\delta_1} \\ &\leq \frac{k_1(\|S(\tau n_0)x\|)c^{\delta_1}}{m_{k_j}^{\delta_1(1+\alpha_1)}} = \varepsilon_j. \end{aligned}$$

Thus we can obtain

$$\begin{aligned}
\underline{D}_r &= \lim_{\varepsilon_0 \to 0} \inf_{0<\varepsilon<\varepsilon_0} \frac{\log \underline{M}_\varepsilon}{-\log \varepsilon} \\
&\le \lim_{j\to\infty} \frac{\log \underline{M}_{\varepsilon_j}}{-\log \varepsilon_j} \\
&\le \lim_{j\to\infty} \frac{\log m_{k_j}}{\delta_1(1+\alpha_1)\log m_{k_j} - \log k_1(\|S(\tau n_0)x\|)c^{\delta_1}} \\
&= \frac{1}{\delta_1(1+\alpha_1)}. \qquad \square
\end{aligned}$$

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