Absolute Norms on $\mathbb{C}^n$

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It is known that for every absolute normalized norm on $\mathbb{C}^2$ there corresponds a unique convex function on $[0, 1]$ satisfying certain suitable conditions (see Bonsall-Duncan [1], also [2]). Recently the authors [3] extended this result to the $n$-dimensional case. In this note we shall present a brief introduction of our result. We first recall the 2-dimensional case, and then we treat 3- and $n$-dimensional cases, where we focus our discussion on the 3-dimensional case which will illustrate the $n$-dimensional situation. We shall also present a characterization of the strict convexity of these norms, which extends our previous result in [4].

A norm $\| \cdot \|$ on $\mathbb{C}^n$ is called \textit{absolute} if

\begin{equation}
(1) \quad \|(|x_1|, \ldots, |x_n|)\| = \|(x_1, \ldots, x_n)\| \quad \forall (x_1, \ldots, x_n) \in \mathbb{C}^n,
\end{equation}

and is called \textit{normalized} if

\begin{equation}
(2) \quad \|(1, 0, \ldots, 0)\| = \|(0, 1, 0, \ldots, 0)\| = \cdots = \|(0, \ldots, 0, 1)\| = 1.
\end{equation}

The $\ell_p$-norms $\| \cdot \|_p$ are such examples:

$$
\| (x_1, x_2, \ldots, x_n) \|_p = \begin{cases} 
( |x_1|^p + \cdots + |x_n|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\
\max(|x_1|, \ldots, |x_n|) & \text{if } p = \infty.
\end{cases}
$$

Let $AN_n$ be the set of all absolute normalized norms on $\mathbb{C}^n$.

We recall the 2-dimensional case. For any $\| \cdot \| \in AN_2$ let

\begin{equation}
(3) \quad \psi(t) = \|(1-t, t)\| \quad (0 \leq t \leq 1).
\end{equation}

Then $\psi$ is convex continuous on $[0, 1]$ and

\begin{equation}
(4) \quad \psi(0) = \psi(1) = 1, \quad \max\{1-t, t\} \leq \psi(t) \leq 1.
\end{equation}

Let $\Psi_2$ denote the set of all convex continuous functions on $[0, 1]$ which satisfies (4). Then the converse is valid.
Theorem 1 (Bonsall-Duncan [1]). The sets \( AN_2 \) and \( \Psi_2 \) are in one-to-one correspondence under (3). That is, for any \( \psi \in \Psi_2 \) let

\[
\|(z, w)\|_\psi = \begin{cases} 
(|z| + |w|) \psi \left( \frac{|w|}{|z|+|w|} \right) & \text{if } (z, w) \neq (0, 0), \\
0 & \text{if } (z, w) = (0, 0).
\end{cases}
\]

Then \( \| \cdot \|_\psi \in N_2 \) and

\[
\psi(t) = \|(1-t, t)\|_\psi \quad (0 \leq t \leq 1).
\]

1. Absolute Norms on \( \mathbb{C}^3 \)

Lemma 2. Let \( \| \cdot \| \in AN_3 \). Then

\[
\|(0, y, z)\| \leq \|(x, y, z)\|,
\]

\[
\|(x, 0, z)\| \leq \|(x, y, z)\|,
\]

\[
\|(x, y, 0)\| \leq \|(x, y, z)\|.
\]

In particular

\[
\| \cdot \|_{\infty} \leq \| \cdot \| \leq \| \cdot \|_1.
\]

Proof. For any \( (x, y, z) \in \mathbb{C}^3 \), we have

\[
\|(x, y, 0)\| = \frac{1}{2} \|(x, y, z) + (x, y, -z)\| 
\]

\[
\leq \frac{1}{2} (\|(x, y, z)\| + \|(x, y, -z)\|) = \|(x, y, z)\|.
\]

Similarly we have the other inequalities.

Lemma 3. Let \( \| \cdot \| \in AN_3 \). If \( |x_1| \leq |x_2|, |y_1| \leq |y_2| \) and \( |z_1| \leq |z_2| \), then

\[
\|(x_1, y_1, z_1)\| \leq \|(x_2, y_2, z_2)\|.
\]

Let

\[
\Delta_3 = \{(s, t) : 0 \leq s + t \leq 1, s, t \geq 0\}.
\]

For any \( \| \cdot \| \in AN_3 \), we put

\[
\psi(s, t) = \|(1-s-t, s, t)\| \quad \text{for } (s, t) \in \Delta_3.
\]

Then \( \psi \) is a continuous convex on \( \Delta_3 \) and satisfies that

\[
\psi(0, 0) = \psi(1, 0) = \psi(0, 1) = 1.
\]
Further, by Lemma 2, we have
\[
\psi(s, t) = \| (1 - s - t, s, t) \| \geq \| (0, s, t) \| \\
= (s + t) \left\| \left( 0, \frac{s}{s + t}, \frac{t}{s + t} \right) \right\| = (s + t) \psi \left( \frac{s}{s + t}, \frac{t}{s + t} \right),
\]
\[
\psi(s, t) = \| (1 - s - t, s, t) \| \geq \| (1 - s - t, 0, t) \| \\
= (1 - s) \left\| \left( 1 - \frac{t}{1-s}, 0, \frac{t}{1-s} \right) \right\| = (1 - s) \psi \left( 0, \frac{t}{1-s} \right),
\]
and similarly
\[
\psi(s, t) \geq (1 - t) \psi \left( \frac{s}{1-t}, 0 \right).
\]

Let \( \Psi_3 \) denote the family of all continuous convex functions on \( \Delta_3 \) satisfying the following conditions:

(7) \[ \psi(0, 0) = \psi(1, 0) = \psi(0, 1) = 1, \]

(8) \[ \psi(s, t) \geq (s + t) \psi \left( \frac{s}{s + t}, \frac{t}{s + t} \right), \]

(9) \[ \psi(s, t) \geq (1 - s) \psi \left( 0, \frac{t}{1-s} \right), \]

(10) \[ \psi(s, t) \geq (1 - t) \psi \left( \frac{s}{1-t}, 0 \right). \]

**Lemma 4.** Let \( \psi \in \Psi_3 \). Then
\[
1 \geq \psi(s, t) \geq \psi_\infty(s, t) \geq \frac{1}{3} \quad \text{for all } (s, t) \in \Delta_3.
\]

**Remark 5.** We consider the following function \( \psi \) on \( \Delta_3 \):
\[
\psi(s, t) = \max \{ 1 - 2s, 1 - 2t, 2s + 2t - 1 \}.
\]

Then \( \psi \) is continuous convex on \( \Delta_3 \) and satisfies (7), but not in \( \Psi_3 \). Indeed, suppose that \( \psi \) is in \( \Psi_3 \). Then since \( \psi \left( \frac{s}{s+t}, \frac{t}{s+t} \right) = \psi \left( 0, \frac{t}{1-s} \right) = 1 \) for all \( (s, t) \in \Delta_3 \), we have
\[
\psi(s, t) \geq \max \{ s + t, 1 - s, 1 - t \} \geq \frac{2}{3}
\]
by (8)-(10), whereas \( \psi \left( \frac{1}{3}, \frac{1}{3} \right) = \frac{1}{3} \), which is a contradiction.
Theorem 6. For any $\| \cdot \| \in AN_{3f}$, put

$$
\psi(s, t) = \|(1 - s - t, s, t)\| \quad \text{for} \ (s, t) \in \Delta_{3}.
$$

Then $\psi \in \Psi_{3}$. Conversely, for any $\psi \in \Psi_{3}$ define

$$
\|(x, y, z)\|_{\psi} = \begin{cases} 
(|x| + |y| + |z|) \psi \left( \frac{|y|}{|x| + |y| + |z|}, \frac{|z|}{|x| + |y| + |z|} \right) & \text{if} \ (x, y, z) \neq (0, 0, 0), \\
0 & \text{if} \ (x, y, z) = (0, 0, 0).
\end{cases}
$$

Then $\| \cdot \|_{\psi}$ is in $AN_{3}$ and $\| \cdot \|_{\psi}$ satisfies (11).

We see the converse assertion. Let $\psi \in \Psi_{3}$, and define $\| \cdot \|_{\psi}$ by (12). We only see the triangular inequality

$$
\|(x_{1}, y_{1}, z_{1}) + (x_{2}, y_{2}, z_{2})\|_{\psi} \leq \|(x_{1}, y_{1}, z_{1})\|_{\psi} + \|(x_{2}, y_{2}, z_{2})\|_{\psi}.
$$

To do this, we show that, if $0 \leq p \leq a$, $0 \leq q \leq b$ and $0 \leq r \leq c$, then

$$
\|(p, q, r)\|_{\psi} \leq \|(a, b, c)\|_{\psi},
$$

that is,

$$
(p + q + r) \psi \left( \frac{q}{p + q + r}, \frac{r}{p + q + r} \right) \leq (a + b + c) \psi \left( \frac{b}{a + b + c}, \frac{c}{a + b + c} \right).
$$

At first, we show that, if $0 \leq p < a$, then

$$
(p + q + r) \psi \left( \frac{q}{p + q + r}, \frac{r}{p + q + r} \right) \leq (a + q + r) \psi \left( \frac{q}{a + q + r}, \frac{r}{a + q + r} \right).
$$

Take any $(s, t) \in \Delta_{3}$ such that $0 < s + t < 1$. We consider the line segment $[(s, t), (\frac{s}{s+t}, \frac{t}{s+t})]$ in $\Delta_{3}$. For any real number $\lambda$ such that $1 < \lambda \leq \frac{1}{s+t}$, we put $s' = \lambda s$ and $t' = \lambda t$. Then $(s', t')$ is in $[(s, t), (\frac{s}{s+t}, \frac{t}{s+t})]$. Since

$$(s', t') = (\lambda s, \lambda t) = \frac{(s + t)(\lambda - 1)}{1 - s - t} \left( \frac{s}{s + t'}, s + t \right) + \frac{1 - \lambda(s + t)}{1 - s - t} (s, t),$$

we have

$$
\psi(s', t') \leq \frac{(s + t)(\lambda - 1)}{1 - s - t} \psi \left( \frac{s}{s + t'}, \frac{t}{s + t} \right) + \frac{1 - \lambda(s + t)}{1 - s - t} \psi(s, t)
$$

by the convexity of $\psi$. Therefore we have

$$
\frac{\psi(s, t)}{s} - \frac{\psi(s', t')}{s'} \geq \frac{\psi(s, t)}{s} - \frac{1}{\lambda s} \left( \frac{(s + t)(\lambda - 1)}{1 - s - t} \psi \left( \frac{s}{s + t'}, \frac{t}{s + t} \right) + \frac{1 - \lambda(s + t)}{1 - s - t} \psi(s, t) \right) \geq 0.
$$
Thus, \( \frac{\psi(s,t)}{s} \geq \frac{\psi(s',t')}{s'} \). Put

\[
s' = \frac{q}{p + q + r}, \quad t' = \frac{r}{p + q + r}, \quad s = \frac{q}{a + q + r}, \quad \text{and} \quad t = \frac{r}{a + q + r},
\]

respectively. Since \( \frac{s'}{s} = \frac{t'}{t} = \frac{a + q + r}{p + q + r} > 1 \), we have

\[
\psi\left(\frac{q}{p + q + r}, \frac{r}{p + q + r}\right) \leq \psi\left(\frac{q}{a + q + r}, \frac{r}{a + q + r}\right).
\]

This implies (14). Repeating a similar discussion, we obtain (13).

Now, let \((x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{C}^3\). Then

\[
\|(x_1, y_1, z_1) + (x_2, y_2, z_2)\|_\psi = \|(x_1 + x_2, y_1 + y_2, z_1 + z_2)\|_\psi \\
= \|(x_1 + x_2, |y_1 + y_2|, |z_1 + z_2|)\|_\psi \\
\leq \|(|x_1| + |x_2|, |y_1| + |y_2|, |z_1| + |z_2|)\|_\psi \\
= K\psi\left(\frac{|y_1| + |y_2|}{K}, \frac{|z_1| + |z_2|}{K}\right)
\]

\[(K := |x_1| + |x_2| + |y_1| + |y_2| + |z_1| + |z_2|)\]

\[\leq K\left\{\frac{|x_1| + |y_1| + |z_1|}{K}\psi\left(\frac{|y_1|}{|x_1| + |y_1| + |z_1|}, \frac{|z_1|}{|x_1| + |y_1| + |z_1|}\right) + \frac{|x_2| + |y_2| + |z_2|}{K}\psi\left(\frac{|y_2|}{|x_2| + |y_2| + |z_2|}, \frac{|z_2|}{|x_2| + |y_2| + |z_2|}\right)\right\}
\]

\[= \|(x_1, y_1, z_1)\|_\psi + \|(x_2, y_2, z_2)\|_\psi.
\]

2. Absolute Norms on \( \mathbb{C}^n \)

Lemma 7. Let \( \| \cdot \| \in AN_n \). Then

\[(B_1) \quad \|(0, x_2, x_3, \ldots, x_n)\| \leq \|(x_1, \ldots, x_n)\|,
\]

\[(B_2) \quad \|(x_1, 0, x_3, \ldots, x_n)\| \leq \|(x_1, \ldots, x_n)\|,
\]

\[\vdots
\]

\[(B_n) \quad \|(x_1, x_2, \ldots, x_{n-1}, 0)\| \leq \|(x_1, \ldots, x_n)\|.
\]

In particular,

\[\| \cdot \|_\infty \leq \| \cdot \| \leq \| \cdot \|_1.
\]
Now let

$$\triangle_n = \{(s_1, s_2, \cdots, s_{n-1}) : s_1 + s_2 + \cdots + s_{n-1} \leq 1, s_i \geq 0 \ (\forall i)\}.$$

Take any $\| \cdot \| \in AN_n$ and let

(16) $\psi(s) = ||(1 - s_1 - s_2 - \cdots - s_{n-1}, s_1, \cdots, s_{n-1})||$ for $s = (s_1, \cdots, s_{n-1}) \in \triangle_n$

Then $\psi$ is continuous and convex on $\triangle_n$, and

(A0) $\psi(0, \cdots, 0) = \psi(1, 0, \cdots, 0) = \cdots = \psi(0, \cdots, 0, 1) = 1$,

(A1) $\psi(s_1, \cdots, s_{n-1}) \geq (s_1 + \cdots + s_{n-1})\psi\left(\frac{s_1}{s_1 + \cdots + s_{n-1}}, \cdots, \frac{s_{n-1}}{s_1 + \cdots + s_{n-1}}\right)$,

(A2) $\psi(s_1, \cdots, s_{n-1}) \geq (1 - s_1)\psi(0, \frac{s_2}{1 - s_1}, \cdots, \frac{s_{n-1}}{1 - s_1})$,

(A_n) $\psi(s_1, \cdots, s_{n-1}) \geq (1 - s_{n-1})\psi(\frac{s_1}{1 - s_{n-1}}, \cdots, \frac{s_{n-2}}{1 - s_{n-2}}, 0)$.

Let $\Psi_n$ be the family of all continuous convex functions on $\triangle_n$ satisfying (A0), (A1), \ldots, (A_n). The functions corresponding to $\ell_p$-norms are

$$\psi_p(s_1, s_2, \cdots, s_{n-1}) = \begin{cases} (\left(1 - \sum_{i=1}^{n-1} s_i\right)^p + s_1^p + \cdots + s_{n-1}^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max(1 - \sum_{i=1}^{n-1} s_i, s_1, \cdots, s_{n-1}) & \text{if } p = \infty. \end{cases}$$

Lemma 8. Let $\psi \in \Psi_n$. Then

$$\frac{1}{n} \leq \psi_\infty(s_1, s_2, \cdots, s_{n-1}) \leq \psi(s_1, s_2, \cdots, s_{n-1}) \leq 1$$

for all $(s_1, s_2, \cdots, s_{n-1}) \in \triangle_n$.

Theorem 9. The sets $AN_n$ and $\Psi_n$ are in one-to-one correspondence with (16). That is, for any $\| \cdot \| \in AN_n$ the function $\psi$ defined by

(16) $\psi(s_1, \cdots, s_{n-1}) = ||(1 - s_1 - \cdots - s_{n-1}, s_1, \cdots, s_{n-1})||$ for $(s_1, \cdots, s_{n-1}) \in \triangle_n$

is in $\Psi_n$. Conversely, for any $\psi \in \Psi_n$ define

$$\|(x_1, \cdots, x_n)\|_\psi = \begin{cases} (|x_1| + \cdots + |x_n|)\psi\left(\frac{|x_2|}{|x_1| + \cdots + |x_n|}, \cdots, \frac{|x_n|}{|x_1| + \cdots + |x_n|}\right) & \text{if } (x_1, \cdots, x_n) \neq (0, \cdots, 0), \\ 0 & \text{if } (x_1, \cdots, x_n) = (0, \cdots, 0). \end{cases}$$

Then $\| \cdot \|_\psi \in AN_n$ and $\| \cdot \|_\psi$ satisfies (16).
3. Strict Convexity of Absolute Norms on $\mathbb{C}^n$

A Banach space $X$ is called strictly convex if for all $x, y \in X$ ($x \neq y$, $\|x\| = \|y\| = 1$) we have $\|\frac{x+y}{2}\| < 1$. A function $\psi$ on $\Delta_n$ is called strictly convex if for all $s, t \in \Delta_n$ ($s \neq t$) we have $\psi\left(\frac{1}{2}(s + t)\right) < \frac{1}{2}(\psi(s) + \psi(t))$. 

For $x = (x_1, \cdots, x_n) \in \mathbb{C}^n$, we define $|x|$ by $|x| = (|x_1|, \cdots, |x_n|)$. We say that $|x| \leq |y|$ if $|x_j| \leq |y_j|$ for $1 \leq j \leq n$. Further, we say that $|x| < |y|$ if $|x| \leq |y|$ and $|x_j| < |y_j|$ for some $j$. Then we have the following lemma.

**Lemma 10 ([3]).** Let $\psi \in \Psi_n$. Let $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbb{C}^n$. Then

(i) If $|x| \leq |y|$, then $\|x\|_{\psi} \leq \|y\|_{\psi}$.

(ii) If $\psi$ is strictly convex and $|x| < |y|$, then $\|x\|_{\psi} < \|y\|_{\psi}$.

**Theorem 11 ([3]).** Let $\psi \in \Psi_n$. Then $\|\cdot\|_{\psi}$ is strictly convex if and only if $\psi$ is strictly convex on $\Delta_n$.

We see the proof in brief. Let $\|\cdot\|_{\psi}$ is strictly convex. Suppose that $\psi$ is not strictly convex. Then there exist $s = (s_1, s_2, \cdots, s_{n-1})$, $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$ ($s \neq t$) such that $\psi\left(\frac{1}{2}(s + t)\right) = \frac{1}{2}(\psi(s) + \psi(t))$. Put $x = (1-s_1 - \cdots - s_{n-1}, s_1, \cdots, s_{n-1})$ and $y = (1 - t_1 - \cdots - t_{n-1}, t_1, \cdots, t_{n-1})$. Then a direct calculation shows that $\|x + y\|_{\psi} = \|x\|_{\psi} + \|y\|_{\psi}$. Since $\|\cdot\|_{\psi}$ is strictly convex, $x$ and $y$ are colinear, that is, there exists a $k > 0$ such that $x = ky$. This implies that $k = 1$, and so $x = y$. Hence we have $s = t$, a contradiction.

Conversely, suppose that $\psi$ is strictly convex on $\Delta_n$. Take any $x = (x_1, \cdots, x_{n-1})$, $y = (y_1, \cdots, y_{n-1}) \in \mathbb{C}^n$, $x \neq y$, such that $\|x\| = \|y\| = 1$. Put

$$s = \left(\frac{|x_2|}{|x_1| + \cdots + |x_n|}, \cdots, \frac{|x_n|}{|x_1| + \cdots + |x_n|}\right) \quad \text{and} \quad t = \left(\frac{|y_2|}{|y_1| + \cdots + |y_n|}, \cdots, \frac{|y_n|}{|y_1| + \cdots + |y_n|}\right).$$

Then, $s, t \in \Delta_n$. If $s \neq t$, in the same way as (15), we have

$$\|x + y\|_{\psi} < \|x\|_{\psi} + \|y\|_{\psi} = 2$$

by Lemma 10 since $\psi$ is strictly convex. In case of $s = t$, we have $|x_j| = |y_j|$ ($1 \leq j \leq n$). So, there exists a positive number $\theta_j$ ($0 \leq \theta_j < 2\pi$) such that $x_j = e^{i\theta_j}y_j$. Since $x \neq y$, there exists a $j_0$ such that $x_{j_0} \neq y_{j_0}$, whence $0 < \theta_{j_0} < 2\pi$. Put $c = |1 + e^{i\theta_{j_0}}|/2$. Then $0 < c < 1$, and we have

$$\|x + y\|_{\psi} = \|(|x_1 + y_{j_0}|, \cdots, |x_n + y_{j_0}|)\|_{\psi}$$

$$= \|(1 + e^{i\theta_1}|y_1|, \cdots, 1 + e^{i\theta_0}|y_{j_0}|, \cdots, 1 + e^{i\theta_n}|y_n|)\|_{\psi}$$

$$\leq 2\|(|y_1|, \cdots, 2|c||y_{j_0}|, \cdots, 2|y_n|)\|_{\psi}$$

$$= 2\|(|y_1|, \cdots, c|y_{j_0}|, \cdots, |y_n|)\|_{\psi}$$

$$< 2\|(|y_1|, \cdots, |y_{j_0}|, \cdots, |y_n|)\|_{\psi} = 2$$

by Lemma 10, as is desired.
References


