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Absolute Norms on $\mathbb{C}^n$

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It is known that for every absolute normalized norm on $\mathbb{C}^2$ there corresponds a unique convex function on $[0,1]$ satisfying certain suitable conditions (see Bonsall-Duncan [1], also [2]). Recently the authors [3] extended this result to the n-dimensional case. In this note we shall present a brief introduction of our result. We first recall the 2-dimensional case, and then we treat 3- and n-dimensional cases, where we focus our discussion on the 3-dimensional case which will illustrate the n-dimensional situation. We shall also present a characterization of the strict convexity of these norms, which extends our previous result in [4].

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is called *absolute* if

(1) $$\|(|x_1|,\cdots,|x_n|)\| = \|(x_1,\cdots,x_n)\| \quad \forall(x_1,\cdots,x_n)\in\mathbb{C}^n,$$

and is called *normalized* if

(2) $$\|(1,0,\cdots,0)\| = \|(0,1,0,\cdots,0)\| = \cdots = \|(0,\cdots,0,1)\| = 1.$$

The ℓ_p-norms $\|\cdot\|_p$ are such examples:

$$\|(x_1,x_2,\cdots,x_n)\|_p = \begin{cases} (|x_1|^p+\cdots+|x_n|^p)^{1/p} & \text{if } 1\le p<\infty, \\ \max(|x_1|,\cdots,|x_n|) & \text{if } p=\infty. \end{cases}$$

Let AN_n be the set of all absolute normalized norms on $\mathbb{C}^n$.

We recall the 2-dimensional case. For any $\|\cdot\|\in AN_2$ let

(3) $$\psi(t) = \|(1-t,t)\| \quad (0\le t\le 1).$$

Then ψ is convex continuous on $[0,1]$ and

(4) $$\psi(0)=\psi(1)=1, \quad \max\{1-t,t\}\le\psi(t)\le 1.$$

Let Ψ_2 denote the set of all convex continuous functions on $[0,1]$ which satisfies (4). Then the converse is valid.

Theorem 1 (Bonsall-Duncan [1]). *The sets AN_2 and Ψ_2 are in one-to-one correspondence under* (3). *That is, for any $\psi \in \Psi_2$ let*

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{5}$$

Then $\|\cdot\|_\psi \in N_2$ and

$$\psi(t) = \|(1-t,t)\|_\psi \quad (0 \le t \le 1).$$

1. Absolute Norms on $\mathbb{C}^3$

Lemma 2. *Let $\|\cdot\| \in AN_3$. Then*

$$\begin{aligned} \|(0,y,z)\| &\le \|(x,y,z)\|, \\ \|(x,0,z)\| &\le \|(x,y,z)\|, \\ \|(x,y,0)\| &\le \|(x,y,z)\|. \end{aligned}$$

In particular

$$\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1.$$

Proof. For any $(x,y,z) \in \mathbb{C}^3$, we have

$$\begin{aligned} \|(x,y,0)\| &= \frac{1}{2}\|(x,y,z)+(x,y,-z)\| \\ &\le \frac{1}{2}(\|(x,y,z)\| + \|(x,y,-z)\|) = \|(x,y,z)\|. \end{aligned}$$

Similarly we have the other inequalities.

Lemma 3. *Let $\|\cdot\| \in AN_3$. If $|x_1| \le |x_2|$, $|y_1| \le |y_2|$ and $|z_1| \le |z_2|$, then $\|(x_1,y_1,z_1)\| \le \|(x_2,y_2,z_2)\|$.*

Let

$$\Delta_3 = \{(s,t) : 0 \le s+t \le 1, s,t \ge 0\}.$$

For any $\|\cdot\| \in AN_3$, we put

$$\psi(s,t) = \|(1-s-t,s,t)\| \quad \text{for } (s,t) \in \Delta_3. \tag{6}$$

Then ψ is a continuous convex on Δ_3 and satisfies that

$$\psi(0,0) = \psi(1,0) = \psi(0,1) = 1.$$

Further, by Lemma 2, we have

$$\begin{aligned}
\psi(s,t) &= \|(1-s-t,s,t)\| \ \geq \ \|(0,s,t)\| \\
&= (s+t)\left\|\left(0,\frac{s}{s+t},\frac{t}{s+t}\right)\right\| = (s+t)\psi\left(\frac{s}{s+t},\frac{t}{s+t}\right), \\
\psi(s,t) &= \|(1-s-t,s,t)\| \geq \|(1-s-t,0,t)\| \\
&= (1-s)\left\|\left(1-\frac{t}{1-s},\ 0,\ \frac{t}{1-s}\right)\right\| = (1-s)\psi\left(0,\frac{t}{1-s}\right),
\end{aligned}$$

and similarly

$$\psi(s,t) \geq (1-t)\psi\left(\frac{s}{1-t},0\right).$$

Let Ψ_3 denote the family of all continuous convex functions on Δ_3 satisfying the following conditions:

$$\psi(0,0) \ = \ \psi(1,0) \ = \ \psi(0,1) \ = \ 1, \tag{7}$$

$$\psi(s,t) \ \geq \ (s+t)\psi\left(\frac{s}{s+t},\frac{t}{s+t}\right), \tag{8}$$

$$\psi(s,t) \ \geq \ (1-s)\psi\left(0,\frac{t}{1-s}\right), \tag{9}$$

$$\psi(s,t) \ \geq \ (1-t)\psi\left(\frac{s}{1-t},0\right). \tag{10}$$

Lemma 4. *Let $\psi \in \Psi_3$. Then*

$$1 \geq \psi(s,t) \geq \psi_\infty(s,t) \geq \frac{1}{3} \quad \textit{for all } (s,t) \in \Delta_3.$$

Remark 5. We consider the following function ψ on Δ_3:

$$\psi(s,t) = \max\{1-2s,\ 1-2t,\ 2s+2t-1\}.$$

Then ψ is continuous convex on Δ_3 and satisfies (7), but not in Ψ_3. Indeed, suppose that ψ is in Ψ_3. Then since $\psi\left(\frac{s}{s+t},\frac{t}{s+t}\right) = \psi\left(0,\frac{t}{1-s}\right) = \psi\left(\frac{s}{1-t},0\right) = 1$ for all $(s,t) \in \Delta_3$, we have

$$\psi(s,t) \geq \max\{s+t, 1-s, 1-t\} \geq \frac{2}{3}$$

by (8)-(10), whereas $\psi(\frac{1}{3},\frac{1}{3}) = \frac{1}{3}$, which is a contradiction.

Theorem 6. *For any* $\|\cdot\| \in AN_3$, *put*

(11) $$\psi(s,t) = \|(1-s-t, s, t)\| \quad \textit{for } (s,t) \in \Delta_3.$$

Then $\psi \in \Psi_3$. *Conversely, for any* $\psi \in \Psi_3$ *define*

(12)
$$\|(x,y,z)\|_\psi = \begin{cases} (|x|+|y|+|z|)\psi\left(\frac{|y|}{|x|+|y|+|z|}, \frac{|z|}{|x|+|y|+|z|}\right) & \textit{if } (x,y,z) \neq (0,0,0), \\ 0 & \textit{if } (x,y,z) = (0,0,0). \end{cases}$$

Then $\|\cdot\|_\psi$ *is in* AN_3 *and* $\|\cdot\|_\psi$ *satisfies* (11).

We see the converse assertion. Let $\psi \in \Psi_3$, and define $\|\cdot\|_\psi$ by (12). We only see the triangular inequality

$$\|(x_1,y_1,z_1) + (x_2,y_2,z_2)\|_\psi \le \|(x_1,y_1,z_1)\|_\psi + \|(x_2,y_2,z_2)\|_\psi.$$

To do this, we show that, if $0 \le p \le a$, $0 \le q \le b$ and $0 \le r \le c$, then

$$\|(p,q,r)\|_\psi \le \|(a,b,c)\|_\psi,$$

that is,

(13) $$(p+q+r)\psi\left(\frac{q}{p+q+r}, \frac{r}{p+q+r}\right) \le (a+b+c)\psi\left(\frac{b}{a+b+c}, \frac{c}{a+b+c}\right).$$

At first, we show that, if $0 \le p < a$, then

(14) $$(p+q+r)\psi\left(\frac{q}{p+q+r}, \frac{r}{p+q+r}\right) \le (a+q+r)\psi\left(\frac{q}{a+q+r}, \frac{r}{a+q+r}\right).$$

Take any $(s,t) \in \Delta_3$ such that $0 < s+t < 1$. We consider the line segment $[(s,t), (\frac{s}{s+t}, \frac{t}{s+t})]$ in Δ_3. For any real number λ such that $1 < \lambda \le \frac{1}{s+t}$, we put $s' = \lambda s$ and $t' = \lambda t$. Then (s',t') is in $[(s,t), (\frac{s}{s+t}, \frac{t}{s+t})]$. Since

$$(s',t') = (\lambda s, \lambda t) = \frac{(s+t)(\lambda-1)}{1-s-t}\left(\frac{s}{s+t}, \frac{t}{s+t}\right) + \frac{1-\lambda(s+t)}{1-s-t}(s,t),$$

we have

$$\psi(s',t') \le \frac{(s+t)(\lambda-1)}{1-s-t}\psi\left(\frac{s}{s+t}, \frac{t}{s+t}\right) + \frac{1-\lambda(s+t)}{1-s-t}\psi(s,t)$$

by the convexity of ψ. Therefore we have

$$\begin{aligned} & \frac{\psi(s,t)}{s} - \frac{\psi(s',t')}{s'} \\ \ge\ & \frac{\psi(s,t)}{s} - \frac{1}{\lambda s}\left\{\frac{(s+t)(\lambda-1)}{1-s-t}\psi\left(\frac{s}{s+t}, \frac{t}{s+t}\right) + \frac{1-\lambda(s+t)}{1-s-t}\psi(s,t)\right\} \\ =\ & \frac{\lambda-1}{\lambda s(1-s-t)}\left\{\psi(s,t) - (s+t)\psi\left(\frac{s}{s+t}, \frac{t}{s+t}\right)\right\} \ge 0. \end{aligned}$$

Thus, $\frac{\psi(s,t)}{s} \geq \frac{\psi(s',t')}{s'}$. Put

$$s' = \frac{q}{p+q+r}, \ t' = \frac{r}{p+q+r}, \ s = \frac{q}{a+q+r}, \text{ and } t = \frac{r}{a+q+r},$$

respectively. Since $\frac{s'}{s} = \frac{t'}{t} = \frac{a+q+r}{p+q+r} > 1$, we have

$$\frac{\psi(\frac{q}{p+q+r}, \frac{r}{p+q+r})}{\frac{q}{p+q+r}} \leq \frac{\psi(\frac{q}{a+q+r}, \frac{r}{a+q+r})}{\frac{q}{a+q+r}}.$$

This implies (14). Repeating a similar discussion, we obtain (13).

Now, let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{C}^3$. Then

$$\begin{aligned}
(15) \qquad & \|(x_1, y_1, z_1) + (x_2, y_2, z_2)\|_\psi \\
= \ & \|(x_1 + x_2, y_1 + y_2, z_1 + z_2)\|_\psi \\
= \ & \|(|x_1 + x_2|, |y_1 + y_2|, |z_1 + z_2|)\|_\psi \\
\leq \ & \|(|x_1| + |x_2|, |y_1| + |y_2|, |z_1| + |z_2|)\|_\psi \\
= \ & K\psi\left(\frac{|y_1| + |y_2|}{K}, \frac{|z_1| + |z_2|}{K}\right) \\
& \qquad (K := |x_1| + |x_2| + |y_1| + |y_2| + |z_1| + |z_2|) \\
\leq \ & K\left\{\frac{|x_1| + |y_1| + |z_1|}{K}\psi\left(\frac{|y_1|}{|x_1| + |y_1| + |z_1|}, \frac{|z_1|}{|x_1| + |y_1| + |z_1|}\right)\right. \\
& \left. + \frac{|x_2| + |y_2| + |z_2|}{K}\psi\left(\frac{|y_2|}{|x_2| + |y_2| + |z_2|}, \frac{|z_2|}{|x_2| + |y_2| + |z_2|}\right)\right\} \\
= \ & \|(x_1, y_1, z_1)\|_\psi + \|(x_2, y_2, z_2)\|_\psi.
\end{aligned}$$

2. Absolute Norms on $\mathbb{C}^n$

Lemma 7. *Let* $\|\cdot\| \in AN_n$. *Then*

$$\begin{aligned}
(B_1) \qquad & \|(0, x_2, x_3, \cdots, x_n)\| \leq \|(x_1, \cdots, x_n)\|, \\
(B_2) \qquad & \|(x_1, 0, x_3, \cdots, x_n)\| \leq \|(x_1, \cdots, x_n)\|, \\
& \vdots \\
(B_n) \qquad & \|(x_1, x_2, \cdots, x_{n-1}, 0)\| \leq \|(x_1, \cdots, x_n)\|.
\end{aligned}$$

In particular,

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1.$$

Now let

$$\Delta_n = \{(s_1, s_2, \cdots, s_{n-1}) : s_1 + s_2 + \cdots + s_{n-1} \leq 1,\ s_i \geq 0\ (\forall i)\}.$$

Take any $\|\cdot\| \in AN_n$ and let

(16) $\psi(s) = \|(1 - s_1 - s_2 - \cdots - s_{n-1}, s_1, \cdots, s_{n-1})\|$ for $s = (s_1, \cdots, s_{n-1}) \in \Delta_n$

Then ψ is continuous and convex on Δ_n, and

$$\begin{array}{llcl}
(A_0) & \psi(0, \cdots, 0) & = & \psi(1, 0, \cdots, 0) = \cdots = \psi(0, \cdots, 0, 1) = 1, \\
(A_1) & \psi(s_1, \cdots, s_{n-1}) & \geq & (s_1 + \cdots + s_{n-1})\psi(\frac{s_1}{s_1 + \cdots + s_n}, \cdots, \frac{s_{n-1}}{s_1 + \cdots + s_{n-1}}). \\
(A_2) & \psi(s_1, \cdots, s_{n-1}) & \geq & (1 - s_1)\psi(0, \frac{s_2}{1 - s_1}, \cdots, \frac{s_{n-1}}{1 - s_1}), \\
 & \cdots\cdots\cdots & & \\
(A_n) & \psi(s_1, \cdots, s_{n-1}) & \geq & (1 - s_{n-1})\psi(\frac{s_1}{1 - s_{n-1}}, \cdots, \frac{s_{n-2}}{1 - s_{n-2}}, 0).
\end{array}$$

Let Ψ_n be the family of all continuous convex functions on Δ_n satisfying (A_0), (A_1), $\cdots$, (A_n). The functions corresponding to ℓ_p-norms are

$$\psi_p(s_1, s_2, \cdots, s_{n-1}) = \begin{cases} ((1 - \sum_{i=1}^{n-1} s_i)^p + s_1^p + \cdots + s_{n-1}^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max(1 - \sum_{i=1}^{n-1} s_i, s_1, \cdots, s_{n-1}) & \text{if } p = \infty. \end{cases}$$

Lemma 8. *Let $\psi \in \Psi_n$. Then*

$$\frac{1}{n} \leq \psi_\infty(s_1, s_2, \cdots, s_{n-1}) \leq \psi(s_1, s_2, \cdots, s_{n-1}) \leq 1$$

for all $(s_1, s_2, \cdots, s_{n-1}) \in \Delta_n$.

Theorem 9. *The sets AN_n and Ψ_n are in one-to-one correspondence with* (16). *That is, for any $\|\cdot\| \in AN_n$ the function ψ defined by*

(16) $\psi(s_1, \cdots, s_{n-1}) = \|(1 - s_1 - \cdots - s_{n-1}, s_1, \cdots, s_{n-1})\|$ *for* $(s_1, \cdots, s_{n-1}) \in \Delta_n$

is in Ψ_n. Conversely, for any $\psi \in \Psi_n$ define

$$\|(x_1, \cdots, x_n)\|_\psi = \begin{cases} (|x_1| + \cdots + |x_n|)\psi\left(\frac{|x_2|}{|x_1| + \cdots + |x_n|}, \cdots, \frac{|x_n|}{|x_1| + \cdots + |x_n|}\right) & \\ & \textit{if } (x_1, \cdots, x_n) \neq (0, \cdots, 0), \\ 0 & \textit{if } (x_1, \cdots, x_n) = (0, \cdots, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_n$ and $\|\cdot\|_\psi$ satisfies (16).

3. Strict Convexity of Absolute Norms on $\mathbb{C}^n$

A Banach space X is called *strictly convex* if for all $x, y \in X$ $(x \neq y,\ \|x\| = \|y\| = 1)$ we have $\|\frac{x+y}{2}\| < 1$. A function ψ on Δ_n is called *strictly convex* if for all $s, t \in \Delta_n$ $(s \neq t)$ we have $\psi(\frac{1}{2}(s+t)) < \frac{1}{2}(\psi(s) + \psi(t))$.

For $x = (x_1, \cdots, x_n) \in \mathbb{C}^n$, we define $|x|$ by $|x| = (|x_1|, \cdots, |x_n|)$. We say that $|x| \leq |y|$ if $|x_j| \leq |y_j|$ for $1 \leq j \leq n$. Further, we say that $|x| < |y|$ if $|x| \leq |y|$ and $|x_j| < |y_j|$ for some j. Then we have the following lemma.

Lemma 10 ([3]). *Let $\psi \in \Psi_n$. Let $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbb{C}^n$. Then*
(i) *If $|x| \leq |y|$, then $\|x\|_\psi \leq \|y\|_\psi$.*
(ii) *If ψ is strictly convex and $|x| < |y|$, then $\|x\|_\psi < \|y\|_\psi$.*

Theorem 11 ([3]) *Let $\psi \in \Psi_n$. Then $\|\cdot\|_\psi$ is strictly convex if and only if ψ is strictly convex on Δ_n.*

We see the proof in brief. Let $\|\cdot\|_\psi$ is strictly convex. Suppose that ψ is not strictly convex. Then there exist $s = (s_1, s_2, \cdots, s_{n-1})$, $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$ $(s \neq t)$ such that $\psi(\frac{1}{2}(s+t)) = \frac{1}{2}(\psi(s) + \psi(t))$. Put $x = (1 - s_1 - \cdots - s_{n-1}, s_1, \cdots, s_{n-1})$ and $y = (1 - t_1 - \cdots - t_{n-1}, t_1, \cdots, t_{n-1})$. Then a direct calculation shows that $\|x+y\|_\psi = \|x\|_\psi + \|y\|_\psi$. Since $\|\cdot\|_\psi$ is strictly convex, x and y are colinear, that is, there exists a $k > 0$ such that $x = ky$. This implies that $k = 1$, and so $x = y$. Hence we have $s = t$, a contradiction.

Conversely, suppose that ψ is strictly convex on Δ_n. Take any $x = (x_1, \cdots, x_{n-1})$, $y = (y_1, \cdots, y_{n-1}) \in \mathbb{C}^n, x \neq y$, such that $\|x\| = \|y\| = 1$. Put

$$s = \left(\frac{|x_2|}{|x_1| + \cdots + |x_n|}, \cdots, \frac{|x_n|}{|x_1| + \cdots + |x_n|}\right) \text{ and } t = \left(\frac{|y_2|}{|y_1| + \cdots + |y_n|}, \cdots, \frac{|y_n|}{|y_1| + \cdots + |y_n|}\right).$$

Then, s, $t \in \Delta_n$. If $s \neq t$, in the same way as (15), we have

$$\|x+y\|_\psi < \|x\|_\psi + \|y\|_\psi = 2$$

by Lemma 10 since ψ is strictly convex. In case of $s = t$, we have $|x_j| = |y_j|$ $(1 \leq j \leq n)$. So, there exists a positive number θ_j $(0 \leq \theta_j < 2\pi)$ such that $x_j = e^{i\theta_j} y_j$. Since $x \neq y$, there exists a j_0 such that $x_{j_0} \neq y_{j_0}$, whence $0 < \theta_{j_0} < 2\pi$. Put $c = |1 + e^{i\theta_{j_0}}|/2$. Then $0 < c < 1$, and we have

$$\begin{aligned}
\|x+y\|_\psi &= \|(|x_1+y_1|, \cdots, |x_n+y_n|)\|_\psi \\
&= \|(|1+e^{i\theta_1}||y_1|, \cdots, |1+e^{i\theta_{j_0}}||y_{j_0}|, \cdots, |1+e^{i\theta_n}||y_n|)\|_\psi \\
&\leq \|(2|y_1|, \cdots, 2c|y_{j_0}|, \cdots, 2|y_n|)\|_\psi \\
&= 2\|(|y_1|, \cdots, c|y_{j_0}|, \cdots, |y_n|)\|_\psi \\
&< 2\|(|y_1|, \cdots, |y_{j_0}|, \cdots, |y_n|)\|_\psi = 2
\end{aligned}$$

by Lemma 10, as is desired.

References

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