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SOME PROPERTIES OF GENERALIZED SUPREMUM IN PARTIALLY ORDERED LINEAR SPACES

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§1 INTRODUCTION AND BASIC RESULTS

Let E be a linear space over $\mathbb{R}$, and P be a convex cone in E satisfying
(P1) $E=P-P$,
(P2) $P\cap(-P)=\{0\}$.
An order relation in E can be defined by $x\leq y\Longleftrightarrow y-x\in P$. We call a linear space E equipped with such a positive cone P a partially ordered linear space, and denote it by (E,P).

For a subset A of E, the generalized supremum $\operatorname{Sup}A$ is defined to be the set of all minimal elements of $U(A)$, where $U(A)$ is the set of all upper bound of A. In other words, $U(A)=\{x\in E\mid y\leq x,\ \forall y\in A\}$, and $\operatorname{Sup}A=\{a\in E\mid b\leq a,\ \ b\in U(A)\Longrightarrow a=b\}$. The generalized infimum $\operatorname{Inf}A$ can be defined similarly. In order to distinguish this notion from the least upper bound and the greatest lower bound, we denote the latter ones by $\sup A$ and $\inf A$ respectively. If E is order complete, then $\operatorname{Sup}A=\{\sup A\}$ holds whenever the subset A is upper bounded (i.e.,$U(A)\neq\emptyset$). When $E=\mathbb{R}^{n}$ and P is closed and not a lattice cone, $\operatorname{Sup}A$ becomes an infinite set in most cases. However, it is possibly empty, even when A is upper bounded. For the preparation, we recall some basic results of the generalized supremum. The proofs of the following propositions can be found in previous papers([4],[5],[6]).

Proposition 1. *For $a\in E$ and $\lambda>0$, we have*
(1) $\operatorname{Sup}(A+a)=\operatorname{Sup}A+a$,
(2) $\operatorname{Sup}\lambda A=\lambda\operatorname{Sup}A$,
(3) $\operatorname{Sup}A=-\operatorname{Inf}(-A)$.

Proposition 2. *For an arbitrary set $A \subset E$ with $U(A) \neq \emptyset$,*

$$\operatorname{Sup} A = \operatorname{Sup}(coA)$$

holds where coA is the convex hull of A.

Prposition 3. *For $a, b \in E$, $\operatorname{Sup}\{a, b\} \neq \emptyset$ implies $\operatorname{Inf}\{a, b\} \neq \emptyset$ and the converse is also true. Moreover,*

$$a + b - \operatorname{Sup}\{a, b\} = \operatorname{Inf}\{a, b\}$$

holds and in particular we have $a \in a_+ + a_-$ where $a_+ = \operatorname{Sup}\{a, 0\}$ and $a_- = \operatorname{Inf}\{a, 0\}$.

A partially ordered linear space (E, P) is said to be monotone order complete (m.o.c. for short) if every upper bounded totally ordered subset of E has the least upper bound in E. In the case $E = \mathbb{R}^d$, (E, P) is m.o.c. if and only if P is closed. In the case when E is a Banach space with a closed positive cone P satisfying $P^* - P^* = E^*$, (E^*, P^*) is m.o.c. where E^* is the topological dual of E and $P^* = \{x^* \in E^* \mid x^*(x) \geq 0,\ x \in P\}$. The proofs of these facts can be seen in a previous paper [6].

Proposition 4. *Suppose that a partially ordered linear space (E, P) is monotone order complete. Then for every subset A of E,*

$$U(A) = (\operatorname{Sup} A) + P$$

holds. In particular, $\operatorname{Sup}\{a, b\} \neq \emptyset$, $\operatorname{Inf}\{a, b\} \neq \emptyset$ for every $a, b \in E$, and $U(a, b) = (\operatorname{Sup}\{a, b\}) + P$.

Let (E, P) be a partially ordered linear space, and suppose that P is algebraically closed, that is, every straight line of E meets P by a closed interval. A point x of a convex subset $A \subset E$ is called an algebraic interior point of A if for every $z \in E$, there exists $\lambda > 0$ such that $x + \lambda z \in A$. Algebraic exterior points are defined similarly, and we denote the algebraic interior (exterior) of A by $\operatorname{int} A$ $(\operatorname{ext} A)$ respectively. Moreover, $\partial A = (\operatorname{int} A \cup \operatorname{ext} A)^c$ is called the algebraic boundary of A. A convex subset C of P is called an exposed face of P if there exists a supporting hyperplane H of P such that $C = P \cap H$. By $\mathfrak{F}(P)$, we denote the set of all exposed faces of P. For $C \in \mathfrak{F}(P)$, $\dim C$ is defined as the dimension of aff$C$ where aff$C$ denotes the affine hull of C.

Propositon 5. *Suppose that P is algebraically closed and $\operatorname{int} P \neq \emptyset$. If $\dim C < \infty$ for every $C \in \mathfrak{F}(P)$, then*

$$U(A) = (\operatorname{Sup} A) + P$$

holds for every subset $A \subset E$.

Corollary 1. *Suppose that (E, P) satisfies the hypotheses in Proposition 4 or Proposition 5, and let A be a subset of E. If* $\operatorname{Sup} A$ *consists of a single element a, then a is the least upper bound of A.*

Corollary 2. *For every subset A of E, $U(L(U(A))) = U(A)$ holds where $L(U(A))$ denotes the lower bound of $U(A)$. Moreover, if (E, P) satisfies the hypotheses in Proposition 4 or Propositon 5, then we have* $\operatorname{Sup}\operatorname{Inf}\operatorname{Sup} A = \operatorname{Sup} A$.

The proofs of these results can be seen in [4],[5],[6], and [7].

§2 Properties of the set of upper bounds and lower bounds

Through this section, we consider only the case when $E = \mathbb{R}^d$ the finite dimensional Euclidean space and the positive cone P is a closed convex cone satisfying (P1),(P2). Under this assumptions, it is easy to observe that $U(A)$ and $L(A)$ are closed convex sets for every $A \subset \mathbb{R}^d$. Moreover $(\mathbb{R}^d, P)$ is monotone order complete, and by Proposition 4, the formula

$$U(A) = (\operatorname{Sup} A) + P \tag{2.1}$$

always holds. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be the family of all upper bounded subset and lower bounded subset in $\mathbb{R}^d$ respectively, i.e.

$$\mathfrak{B} = \{A \subset \mathbb{R}^d \mid A \neq \emptyset,\ U(A) \neq \emptyset\},$$

$$\mathfrak{B}' = \{B \subset \mathbb{R}^d \mid B \neq \emptyset,\ L(B) \neq \emptyset\}.$$

We define an equivalence relation $\sim$ in $\mathfrak{B}$ by

$$A \sim B \iff U(A) = U(B) \quad (A, B \in \mathfrak{B}).$$

Let X be the quotient set $\mathfrak{B}/\sim\ = \{[A] \mid A \in \mathfrak{B}\}$ where $[A]$ denotes the equivalence class of A.

Proposition 6. $[A] = [L(U(A))] = [L(\operatorname{Sup} A)]$ *holds for every $A \in \mathfrak{B}$ and $[L(B)] = [\operatorname{Inf} B]$ for every $B \in \mathfrak{B}'$. Moreover if $[L(B)] = [A]$ for some $A \in \mathfrak{B}$ and $B \in \mathfrak{B}'$, then $A \subset L(B)$.*

proof. By (2.1) we can easily see that

$$\begin{aligned} U(A) &= U(L(U(A))) \\ &= U(L(\operatorname{Sup} A + P)) \\ &= U(L(\operatorname{Sup} A)). \end{aligned}$$

This directly shows the first formula. Since we also have $L(B) = (\operatorname{Inf} B) - P \quad (B \in \mathfrak{B}')$ by (2.1), the second formula follows similarly. Indeed, $U(\operatorname{Inf} B) = U((\operatorname{Inf} B) - P) = U(L(B))$. The latter statement follows from Corollary 2. Indeed,

$$\begin{aligned} A &\subset L(U(A)) \\ &= L(U(L(B))) \\ &= L(B). \end{aligned}$$

For every $[A] \in X$, two operations $u([A]) = U(A)$ and $l([A]) = L(U(A))$ are well defined. By virtue of (2.1), X can be identified with the set $\{U(A) \mid A \in \mathfrak{B}\}$ or the set $\{\operatorname{Sup} A \mid A \in \mathfrak{B}\}$. We now define an order relation in X by

$$[A] \leq [B] \Longleftrightarrow u([B]) \subset u([A]) \quad [A], [B] \in X.$$

By this definition X becomes a partially ordered set. Moreover, we shall show that X is an order complete lattice and that X has a subset which is order isomorphic to $(\mathbb{R}^d, P)$. Let X_1 be the set of all $[A] \in X$ such that $u([A]) = a + P$ for some $a \in \mathbb{R}^d$. Note that the correspondence which assigns $a \in \mathbb{R}^d$ to $[A] \in X_1$ such that $u([A]) = a + P$ is one to one.

Theorem 1. *X is an order complete lattice with respect to the order '$\leq$'. Moreover, X_1 is order isomorphic to $(\mathbb{R}^d, P)$ by the correspondence $\mathbb{R}^d \ni a \longleftrightarrow [A] \in X_1$ where $u([A]) = a + P$.*

Lemma 1. *Let $\{A_\sigma\}_{\sigma \in \Sigma} \subset \mathfrak{B}$, and $\{B_\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{B}'$, be arbitrary families such that $\cup_{\sigma \in \Sigma} A_\sigma \in \mathfrak{B}$ and $\cup_{\lambda \in \Lambda} B_\lambda \in \mathfrak{B}'$. Then*
(1) $\cap_{\sigma \in \Sigma} u([A_\sigma]) = u([\cup_{\sigma \in \Sigma} A_\sigma])$, $\quad \cap_{\lambda \in \Lambda} l([L(B_\lambda)]) = l([L(\cup_{\lambda \in \Lambda} B_\lambda)])$.
(2) $U(L(\cap_{\sigma \in \Sigma} u([A_\sigma]))) = \cap_{\sigma \in \Sigma} u([A_\sigma])$, $\quad L(U(\cap_{\lambda \in \Lambda} l([L(B_\lambda)]))) = \cap_{\lambda \in \Lambda} l([L(B_\lambda)])$.

proof. (1) can be shown directly by the definitions. Indeed,

$$\begin{aligned} \cap_{\sigma \in \Sigma} u([A_\sigma]) &= \cap_{\sigma \in \Sigma} U(A_\sigma) \\ &= U(\cup_{\sigma \in \Sigma} A_\sigma) \\ &= u([\cup_{\sigma \in \Sigma} A_\sigma]), \end{aligned}$$

and

$$\begin{aligned} \cap_{\lambda \in \Lambda} l([L(B_\lambda)]) &= \cap_{\lambda \in \Lambda} L(U(L(B_\lambda))) \\ &= \cap_{\lambda \in \Lambda} L(B_\lambda) \\ &= L(\cup_{\lambda \in \Lambda} B_\lambda) \\ &= L(U(L(\cup_{\lambda \in \Lambda} B_\lambda))) \\ &= l([L(\cup_{\lambda \in \Lambda} B_\lambda)]). \end{aligned}$$

Moreover, we can see by (1) and Corollary 2 that

$$U(L(\cap_{\sigma\in\Sigma} u([A_\sigma]))) = U(L(u([\cup_{\sigma\in\Sigma} A_\sigma])))$$
$$= u([\cup_{\sigma\in\Sigma} A_\sigma]).$$

The latter formula can be shown similarly.

proof of Theorem 1. Let Y be an upper bounded subset of X. Then there exists a subset $B \in \mathfrak{B}$ such that $U(B) \subset u([A])$ for all $[A] \in Y$. Let

$$C = L(\bigcap_{[A]\in Y} u([A]))$$

Then $C \in \mathfrak{B}$ and by Lemma 1,

$$U(C) = \bigcap_{[A]\in Y} u([A])$$
$$\supset U(B).$$

This means that $[C]$ is the least upper bound of Y. Next we suppose that Y' is a lower bounded subset of X. We put

$$C' = \bigcap_{[A]\in Y'} L(u([A]))$$

Then $C' \in \mathfrak{B}$ and $U(C') \supset U(L(u([A]))) = u([A])$ for every $[A] \in Y'$. Hence $[C']$ is a lower bound of Y'. Let $[B']$ be an arbitrary lower bound of Y' then $u([A]) \subset U(B')$ for every $[A] \in Y'$, and we have $\cap_{[A]\in Y'} L(u([A])) \supset L(U(B'))$. Thus

$$U(C') = U(\bigcap_{[A]\in Y'} L(u([A])))$$
$$\subset U(L(U(B')))$$
$$= u([B']).$$

This means that $[C']$ is the greatest lower bound of Y'. Thus we have proved that X is order complete. To prove that X forms a lattice it is sufficient to show that $\{[A],[B]\}$ is bounded for every pair $[A],[B] \in X$. For $a \in u([A])$ and $b \in u([B])$ we can choose $p, q \in P$ such that $a-b = p-q$ by the condition (P1). Hence $a+q = b+p \in u([A]) \cap u([B])$. Thus $u([A]) \cap u([B])$ and $L(u([A])) \cap L(u([B]))$ are both nonempty, and we put $C_1 = L(u([A]) \cap u([B]))$, and $C_2 = L(u([A])) \cap L(u([B]))$. It is easy to see that $[C_1] \geq [A],[B]$ and $[C_2] \leq [A],[B]$, and this is what we wanted to show. The second statement of this theorem is obvious.

By $[A] \vee [B]$, and $[A] \wedge [B]$ we denote the least upper bound and the greatest lower bound of $\{[A],[B]\}$ in X respectively. Repeating the same arguement of the proof of Theorem 1, we obtain

Proposition 7. *For* $[A], [B] \in X$,

(1) $[A] \vee [B] = [L(u([A]) \cap u([B]))]$,

(2) $[A] \wedge [B] = [L(u([A])) \cap L(u([B]))]$.

For $A \in \mathfrak{B}$ we can characterize $U(A)$ by using the support function of A and the dual cone $P^* = \{x^* \in \mathbb{R}^d \mid \ < x^*, x > \geq 0 \quad x \in P\}$. In the conditions we have assumed, the relation

$$(2.2) \qquad P = P^{**} = \{x \in \mathbb{R}^d \mid \ < x^*, x > \geq 0 \quad x^* \in P^*\}.$$

holds. If $A \in \mathfrak{B}$ then the support function $f_A(x^*) = \sup_{x \in A} < x^*, x >$ is finite on P^*. Indeed if $x_0 \in U(A)$, then $< x^*, x > \leq < x^*, x_0 >$ holds for all $x \in A$.

Theorem 2. *For every* $A \in \mathfrak{B}$,

$$U(A) = \bigcap_{x^* \in \partial P^*} \{x \mid \ < x^*, x > \geq f_A(x^*)\},$$

where ∂P^* *denotes the boundary of* P^*.

It is known that the dual cone P^* satisfies (P1) and (P2), if P is closed in $\mathbb{R}^d$. For the proof of Theorem 2, we prepare a basic lemma.

Lemma 2. *Let* $P \subset \mathbb{R}^d$ *be a closed positive cone satisfying* (P1) *and* (P2). *Then*

(1) *if* $0 \leq b \leq a$ *and* $b \neq 0$, *there exists* $n \in \mathbb{N}$ *such that* $nb \not\leq a$,

(2) *if* a *is an interior point of* P *and* $b \not\leq a$, *then there exists* $t > 0$ *such that* $a + t(a - b) \in \partial P$.

proof. Suppose that $\frac{a}{n} - b \geq 0$ for every $n = 1, 2, 3, \cdots$. Then the closedness of P yields $-b \geq 0$ which contradicts (P1). Hence there exists $n \in \mathbb{N}$ such that $a - nb \not\geq 0$ and (1) follows immediately. Next we suppose that $a + t(a - b) \geq 0$ for every $t > 0$. Then $\frac{t+1}{t} a - b \geq 0 \quad (t > 0)$ and the closedness of P yields $a - b \geq 0$ which contradicts the assumption. Hence we can choose $t_0 = \sup\{t > 0 \mid a + t(a - b) \in P\}$, and $a + t_0(a - b) \in \partial P$.

proof of Theorem 2. Since '$\subset$' is obvious we will prove only the converse. Let x^* be an arbirary element of P^*. By (1) in Lemma 2, we can take $x_1^* \in \partial P^*$ such that $x_1^* \not\leq x^*$. Moreover, by (2) in Lemma 2, there exists $x_2^* \in \partial P^*$ such that $x^* = \lambda x_1^* + (1 - \lambda) x_2^*$ for some $0 < \lambda < 1$. Suppose that $x \in \bigcap_{x^* \in \partial P^*} \{x \mid \ < x^*, x > \geq f_A(x^*)\}$ and $y \in A$, then

$$\begin{aligned} < x^*, x - y > &= \lambda < x_1^*, x - y > + (1 - \lambda) < x_2^*, x - y > \\ &\geq 0. \end{aligned}$$

Since $x^* \in P^*$ and $y \in A$ are arbirary, we can conclude by (2.2) that $x \in U(A)$.

The following is an immediate consequence of this theorem.

Corollary 3. *Let* $A, B \in \mathfrak{B}$ *and suppose that* $f_A(x^*) = f_B(x^*)$ *on* ∂P^*, *then* $[A] = [B]$.

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