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Kyoto University
SOME PROPERTIES OF GENERALIZED SUPREMUM IN PARTIALLY ORDERED LINEAR SPACES

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§1 Introduction and Basic Results

Let $E$ be a linear space over $\mathbb{R}$, and $P$ be a convex cone in $E$ satisfying

(P1) $E = P - P$,
(P2) $P \cap (-P) = \{0\}$.

An order relation in $E$ can be defined by $x \leq y \iff y - x \in P$. We call a linear space $E$ equipped with such a positive cone $P$ a partially ordered linear space, and denote it by $(E, P)$.

For a subset $A$ of $E$, the generalized supremum $\text{Sup} A$ is defined to be the set of all minimal elements of $U(A)$, where $U(A)$ is the set of all upper bound of $A$. In other words, $U(A) = \{x \in E \mid y \leq x, \forall y \in A\}$, and $\text{Sup} A = \{a \in E \mid b \leq a, \ b \in U(A) \implies a = b\}$. The generalized infimum $\text{Inf} A$ can be defined similarly. In order to distinguish this notion from the least upper bound and the greatest lower bound, we denote the latter ones by $\sup A$ and $\inf A$ respectively. If $E$ is order complete, then $\text{Sup} A = \{\sup A\}$ holds whenever the subset $A$ is upper bounded (i.e., $U(A) \neq \emptyset$). When $E = \mathbb{R}^n$ and $P$ is closed and not a lattice cone, $\text{Sup} A$ becomes an infinite set in most cases. However, it is possibly empty, even when $A$ is upper bounded. For the preparation, we recall some basic results of the generalized supremum. The proofs of the following propositions can be found in previous papers([4],[5],[6]).

**Proposition 1.** For $a \in E$ and $\lambda > 0$, we have

(1) $\text{Sup}(A + a) = \text{Sup} A + a$,
(2) $\text{Sup} \lambda A = \lambda \text{Sup} A$,
(3) $\text{Sup} A = -\text{Inf}(-A)$. 
Proposition 2. For an arbitrary set $A \subset E$ with $U(A) \neq \emptyset$,

$$\text{Sup} A = \text{Sup}(\text{co} A)$$

holds where co$A$ is the convex hull of $A$.

Proposition 3. For $a, b \in E$, $\text{Sup}\{a, b\} \neq \emptyset$ implies $\text{Inf}\{a, b\} \neq \emptyset$ and the converse is also true. Moreover,

$$a + b - \text{Sup}\{a, b\} = \text{Inf}\{a, b\}$$

holds and in particular we have $a \in a_+ + a_-$ where $a_+ = \text{Sup}\{a, 0\}$ and $a_- = \text{Inf}\{a, 0\}$.

A partially ordered linear space $(E, P)$ is said to be monotone order complete (m.o.c. for short) if every upper bounded totally ordered subset of $E$ has the least upper bound in $E$. In the case $E = \mathbb{R}^d$, $(E, P)$ is m.o.c. if and only if $P$ is closed. In the case when $E$ is a Banach space with a closed positive cone $P$ satisfying $P^* - P^* = E^*$, $(E^*, P^*)$ is m.o.c. where $E^*$ is the topological dual of $E$ and $P^* = \{x^* \in E^* | x^*(x) \geq 0, x \in P\}$. The proofs of these facts can be seen in a previous paper [6].

Proposition 4. Suppose that a partially ordered linear space $(E, P)$ is monotone order complete. Then for every subset $A$ of $E$,

$$U(A) = (\text{Sup} A) + P$$

holds. In particular, $\text{Sup}\{a, b\} \neq \emptyset$, $\text{Inf}\{a, b\} \neq \emptyset$ for every $a, b \in E$, and $U(a, b) = (\text{Sup}\{a, b\}) + P$.

Let $(E, P)$ be a partially ordered linear space, and suppose that $P$ is algebraically closed, that is, every straight line of $E$ meets $P$ by a closed interval. A point $x$ of a convex subset $A \subset E$ is called an algebraic interior point of $A$ if for every $z \in E$, there exists $\lambda > 0$ such that $x + \lambda z \in A$. Algebraic exterior points are defined similarly, and we denote the algebraic interior (exterior) of $A$ by int$A$ (ext$A$) respectively. Moreover, $\partial A = (\text{int} A \cup \text{ext} A)^c$ is called the algebraic boundary of $A$. A convex subset $C$ of $P$ is called an exposed face of $P$ if there exists a supporting hyperplane $H$ of $P$ such that $C = P \cap H$. By $\mathfrak{F}(P)$, we denote the set of all exposed faces of $P$. For $C \in \mathfrak{F}(P)$, dim $C$ is defined as the dimension of aff$C$ where aff$C$ denotes the affine hull of $C$.

Proposition 5. Suppose that $P$ is algebraically closed and int $P \neq \emptyset$. If dim $C < \infty$ for every $C \in \mathfrak{F}(P)$, then

$$U(A) = (\text{Sup} A) + P$$

holds for every subset $A \subset E$. 
Corollary 1. Suppose that \((E, P)\) satisfies the hypotheses in Proposition 4 or Proposition 5, and let \(A\) be a subset of \(E\). If \(\text{Sup} A\) consists of a single element \(a\), then \(a\) is the least upper bound of \(A\).

Corollary 2. For every subset \(A\) of \(E\), \(U(L(U(A))) = U(A)\) holds where \(L(U(A))\) denotes the lower bound of \(U(A)\). Moreover, if \((E, P)\) satisfies the hypotheses in Proposition 4 or Proposition 5, then we have \(\text{Sup} \text{Inf} \text{Sup} A = \text{Sup} A\).

The proofs of these results can be seen in [4],[5],[6], and [7].

§2 Properties of the set of upper bounds and lower bounds

Through this section, we consider only the case when \(E = \mathbb{R}^d\) the finite dimensional Euclidean space and the positive cone \(P\) is a closed convex cone satisfying \((P1),(P2)\). Under this assumptions, it is easy to observe that \(U(A)\) and \(L(A)\) are closed convex sets for every \(A \subset \mathbb{R}^d\). Moreover \((\mathbb{R}^d, P)\) is monotone order complete, and by Proposition 4, the formula

\[
(2.1) \quad U(A) = (\text{Sup} A) + P
\]

always holds. Let \(\mathfrak{B}\) and \(\mathfrak{B}'\) be the family of all upper bounded subset and lower bounded subset in \(\mathbb{R}^d\) respectively, i.e.

\[
\mathfrak{B} = \{A \subset \mathbb{R}^d \mid A \neq \emptyset, U(A) \neq \emptyset\},
\]

\[
\mathfrak{B}' = \{B \subset \mathbb{R}^d \mid B \neq \emptyset, L(B) \neq \emptyset\}.
\]

We define an equivalence relation \(\sim\) in \(\mathfrak{B}\) by

\[
A \sim B \iff U(A) = U(B) \quad (A, B \in \mathfrak{B}).
\]

Let \(X\) be the quotient set \(\mathfrak{B}/\sim = \{[A] \mid A \in \mathfrak{B}\}\) where \([A]\) denotes the equivalence class of \(A\).

Proposition 6. \([A] = [L(U(A))] = [L(\text{Sup} A)]\) holds for every \(A \in \mathfrak{B}\) and \([L(B)] = [\text{Inf} B]\) for every \(B \in \mathfrak{B}'\). Moreover if \([L(B)] = [A]\) for some \(A \in \mathfrak{B}\) and \(B \in \mathfrak{B}'\), then \(A \subset L(B)\).

proof. By (2.1) we can easily see that

\[
U(A) = U(L(U(A)))
= U(L(\text{Sup} A + P))
= U(L(\text{Sup} A)).
\]
This directly shows the first formula. Since we also have \( L(B) = (\text{Inf } B) - P \) \( (B \in \mathfrak{B}') \) by (2.1), the second formula follows similarly. Indeed, 
\[
U(\text{Inf } B) = U((\text{Inf } B) - P) = U(L(B)).
\]
The latter statement follows from Corollary 2. Indeed,
\[
A \subset L(U(A)) \\
= L(U(L(B))) \\
= L(B).
\]

For every \([A] \in X\), two operations \( u([A]) = U(A) \) and \( l([A]) = L(U(A)) \) are well defined. By virtue of (2.1), \( X \) can be identified with the set \( \{U(A) \mid A \in \mathfrak{B}\} \) or the set \( \{\text{Sup } A \mid A \in \mathfrak{B}\} \). We now define an order relation in \( X \) by
\[
[A] \leq [B] \iff u([B]) \subset u([A]) \quad [A], [B] \in X.
\]
By this definition \( X \) becomes a partially ordered set. Moreover, we shall show that \( X \) is an order complete lattice and that \( X \) has a subset which is order isomorphic to \((\mathbb{R}^d, P)\). Let \( X_1 \) be the set of all \([A] \in X\) such that \( u([A]) = a + P \) for some \( a \in \mathbb{R}^d \). Note that the correspondence which assigns \( a \in \mathbb{R}^d \) to \([A] \in X_1\) such that \( u([A]) = a + P \) is one to one.

**Theorem 1.** \( X \) is an order complete lattice with respect to the order ‘\( \leq \)’. Moreover, \( X_1 \) is order isomorphic to \((\mathbb{R}^d, P)\) by the correspondence \( \mathbb{R}^d \ni a \leftrightarrow [A] \in X_1 \) where \( u([A]) = a + P \).

**Lemma 1.** Let \( \{A_\sigma\}_{\sigma \in \Sigma} \subset \mathfrak{B}, \) and \( \{B_\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{B}' \), be arbitrary families such that \( \cup_{\sigma \in \Sigma} A_\sigma \in \mathfrak{B \ and \ } \cup_{\lambda \in \Lambda} B_\lambda \in \mathfrak{B}' \). Then

1. \( \cap_{\sigma \in \Sigma} u([A_\sigma]) = U(\cap_{\sigma \in \Sigma} A_\sigma) \) \( = U(\cup_{\sigma \in \Sigma} A_\sigma) = u([\cup_{\sigma \in \Sigma} A_\sigma]) \).
2. \( U(L(\cap_{\sigma \in \Sigma} u([A_\sigma]))) = \cap_{\sigma \in \Sigma} u([A_\sigma]) \) \( = U(\cap_{\lambda \in \Lambda} l([L(B_\lambda)])) = \cap_{\lambda \in \Lambda} l([L(B_\lambda)]) \).

**proof.** (1) can be shown directly by the definitions. Indeed,
\[
\cap_{\sigma \in \Sigma} u([A_\sigma]) = U(\cup_{\sigma \in \Sigma} A_\sigma) = u([\cup_{\sigma \in \Sigma} A_\sigma]),
\]
and
\[
\cap_{\lambda \in \Lambda} l([L(B_\lambda)]) = U(\cap_{\lambda \in \Lambda} L(B_\lambda)) = \cap_{\lambda \in \Lambda} L(B_\lambda) = L(\cup_{\lambda \in \Lambda} B_\lambda) = L(U(\cup_{\lambda \in \Lambda} B_\lambda)) = l([L(\cup_{\lambda \in \Lambda} B_\lambda)]).
Moreover, we can see by (1) and Corollary 2 that
\[ U(L(\bigcap_{\sigma\in\Sigma} u([A_{\sigma}]))) = U(L(u([\bigcup_{\sigma\in\Sigma} A_{\sigma}]))) = u([\bigcup_{\sigma\in\Sigma} A_{\sigma}]). \]

The latter formula can be shown similarly.

**proof of Theorem 1.** Let \( Y \) be an upper bounded subset of \( X \). Then there exists a subset \( B \in \mathfrak{B} \) such that \( U(B) \subset u([A]) \) for all \([A] \in Y\). Let
\[ C = L(\bigcap_{[A] \in Y} u([A])) \]
Then \( C \in \mathfrak{B} \) and by Lemma 1,
\[ U(C) = \bigcap_{[A] \in Y} u([A]) \supset U(B). \]
This means that \([C]\) is the least upper bound of \( Y \). Next we suppose that \( Y' \) is a lower bounded subset of \( X \). We put
\[ C' = \bigcap_{[A] \in Y'} L(u([A])) \]
Then \( C' \in \mathfrak{B} \) and \( U(C') \supset U(L(u([A]))) = u([A]) \) for every \([A] \in Y'\). Hence \([C']\) is a lower bound of \( Y' \). Let \([B']\) be an arbitrary lower bound of \( Y' \) then \( u([A]) \subset U(B') \) for every \([A] \in Y'\), and we have \( \bigcap_{[A] \in Y'} L(u([A])) \supset L(U(B')) \). Thus
\[ U(C') = U(\bigcap_{[A] \in Y'} L(u([A]))) \subset U(L(U(B'))) = u([B']). \]
This means that \([C']\) is the greatest lower bound of \( Y' \). Thus we have proved that \( X \) is order complete. To prove that \( X \) forms a lattice it is sufficient to show that \( \{[A], [B]\} \) is bounded for every pair \([A], [B] \in X\). For \( a \in u([A]) \) and \( b \in u([B]) \) we can choose \( p, q \in P \) such that \( a-b = p-q \) by the condition (P1). Hence \( a+q = b+p \in u([A]) \cap u([B]) \). Thus \( u([A]) \cap u([B]) \) and \( L(u([A])) \cap L(u([B])) \) are both nonempty, and we put \( C_1 = L(u([A]) \cap u([B])) \), and \( C_2 = L(u([A])) \cap L(u([B])) \). It is easy to see that \([C_1] \geq [A], [B]\) and \([C_2] \leq [A], [B]\), and this is what we wanted to show. The second statement of this theorem is obvious.

By \([A] \vee [B]\), and \([A] \wedge [B]\) we denote the least upper bound and the greatest lower bound of \( \{[A], [B]\} \) in \( X \) respectively. Repeating the same argument of the proof of Theorem 1, we obtain
Proposition 7. For $[A],[B] \in X$,
(1) $[A] \lor [B] = [L(u([A]) \cap u([B]))]$, 
(2) $[A] \land [B] = [L(u([A])) \cap L(u([B]))]$.

For $A \in \mathfrak{B}$ we can characterize $U(A)$ by using the support function of $A$ and the dual cone $P^* = \{x^* \in \mathbb{R}^d \mid <x^*,x> \geq 0 \quad x \in P\}$. In the conditions we have assumed, the relation

(2.2) $P = P^{**} = \{x \in \mathbb{R}^d \mid <x^*,x> \geq 0 \quad x^* \in P^*\}.$

holds. If $A \in \mathfrak{B}$ then the support function $f_A(x^*) = \sup_{x \in A} <x^*,x>$ is finite on $P^*$. Indeed if $x_0 \in U(A)$, then $<x^*,x> \leq <x^*,x_0>$ holds for all $x \in A$.

Theorem 2. For every $A \in \mathfrak{B}$,

$$U(A) = \bigcap_{x^* \in \partial P^*} \{x \mid <x^*,x> \geq f_A(x^*)\},$$

where $\partial P^*$ denotes the boundary of $P^*$.

It is known that the dual cone $P^*$ satisfies (P1) and (P2), if $P$ is closed in $\mathbb{R}^d$. For the proof of Theorem 2, we prepare a basic lemma.

Lemma 2. Let $P \subset \mathbb{R}^d$ be a closed positive cone satisfying (P1) and (P2). Then

(1) if $0 \leq b \leq a$ and $b \neq 0$, there exists $n \in \mathbb{N}$ such that $nb \ngeq a$,
(2) if $a$ is an interior point of $P$ and $b \ngeq a$, then there exists $t > 0$ such that $a + t(a - b) \in \partial P$.

proof. Suppose that $\frac{a}{n} - b \geq 0$ for every $n = 1, 2, 3, \ldots$. Then the closedness of $P$ yields $-b \geq 0$ which contradicts (P1). Hence there exists $n \in \mathbb{N}$ such that $a - nb \ngeq 0$ and (1) follows immediately. Next we suppose that $a + t(a - b) \geq 0$ for every $t > 0$. Then $\frac{t+1}{t}a - b \geq 0$ (for all $t > 0$) and the closedness of $P$ yields $a - b \geq 0$ which contradicts the assumption. Hence we can choose $t_0 = \sup\{t > 0 \mid a + t(a - b) \in P\}$, and $a + t_0(a - b) \in \partial P$.

proof of Theorem 2. Since ‘$\subset$’ is obvious we will prove only the converse. Let $x^*$ be an arbitrary element of $P^*$. By (1) in Lemma 2, we can take $x_1^* \in \partial P^*$ such that $x_1^* \ngeq x^*$. Moreover, by (2) in Lemma 2, there exists $x_2^* \in \partial P^*$ such that $x^* = \lambda x_1^* + (1 - \lambda)x_2^*$ for some $0 < \lambda < 1$. Suppose that $x \in \bigcap_{x^* \in \partial P^*} \{x \mid <x^*,x> \geq f_A(x^*)\}$ and $y \in A$, then

$$<x^*,x - y> = \lambda <x_1^*,x - y> + (1 - \lambda) <x_2^*,x - y> \geq 0.$$
Since \( x^* \in P^* \) and \( y \in A \) are arbitrary, we can conclude by (2.2) that \( x \in U(A) \).

The following is an immediate consequence of this theorem.

**Corollary 3.** Let \( A, B \in \mathfrak{B} \) and suppose that \( f_A(x^*) = f_B(x^*) \) on \( \partial P^* \), then \([A] = [B]\).

**References**


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