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NEW RESULTS ON NONLINEAR ERGODIC THEOREMS FOR NONLINEAR OPERATORS

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In this paper, we give some well-known theorems and recent new results for nonlinear ergodic theorems for semitopological semigroups of nonlinear operators either in Banach spaces or in Hilbert spaces.

1. Introduction

Let G be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each $s\in G$ the mappings $s\mapsto s\cdot t$ and $s\mapsto t\cdot s$ of G into itself are continuous. Let C be a nonempty subset of a Banach space E and let $\Im=\{T_{t}:t\in G\}$ be a semigroup on C, i.e., $T_{st}(x)=T_{s}T_{t}(x)$ for all $s,t\in G$ and $x\in C$. Recall that a semigroup $\Im$ is said to be

(a) nonexpansive if $\|T_{t}x-T_{t}y\|\leq\|x-y\|$ for $x,y\in C$ and $t\in G$.
(b) asymptotically nonexpansive [18] if there exists a function $k:G\mapsto[0,\infty)$ with $\inf_{s\in G}\sup_{t\in G}k_{ts}\leq 1$ such that $\|T_{t}x-T_{t}y\|\leq k_{t}\|x-y\|$ for $x,y\in C$ and $t\in G$.
(c) asymptotically nonexpansive type [18] if for each x in C, there is a function $r(\cdot,x):G\mapsto[0,\infty)$ with $\inf_{s\in G}\sup_{t\in G}r(ts,x)=0$ such that $\|T_{t}x-T_{t}y\|\leq\|x-y\|+r(t,x)$ for all $y\in C$ and $t\in G$.

It is easily seen that $(a)\Rightarrow(b)\Rightarrow(c)$ and that both inclusions are proper. Baillon [1] proved the first nonlinear mean ergodic theorem for a nonexpansive mapping in a Hilbert space. Simple proofs as well as extensions of this theorem to more general Banach spaces, to nonexpansive and asymptotically nonexpansive semigroups of operators and to more general semigroups of operators were given among others by Baillon-Brézis [2], Brézis-Browder [3], Bruck ([5],[6]), Hirano-Takahashi ([11],[12]), Kim-Li [16], Li-Kim ([24],[27]), Mizoguchi-Takahashi [29], Reich ([31],[32]) and Kaczor-Kuczumow-Reich [14] to which we also refer for an up-to-date and complete bibliography.

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The purpose of this paper is to give some well-known theorems and recent new results for nonlinear ergodic theorems for semitopological semigroups of nonlinear operators either in Banach spaces or in Hilbert spaces.

2. Hilbert Space Setting

The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [1].

Theorem 2.1 ([1]). *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself. If the set $\mathcal{F}(T)$ of fixed points of T is nonempty, then for each $x \in C$, the cesáro means*

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly as $n \to \infty$ to a point of $\mathcal{F}(T)$.

In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $\mathcal{F}(T)$ such that $PT = TP = P$ and $Px \in \overline{conv}\{T^n x : n = 0, 1, 2, \cdots\}$ for each $x \in C$, where $\overline{conv} A$ is the closure of the convex hull of A.

Let $m(G)$ be the Banach space of all bounded real-valued functions on G with the supremum norm and let X be a subspace of $m(G)$ containing constants. Then, an element μ of X^* (the dual space of X) is called a mean on X if $\|\mu\| = \mu(1) = 1$. We know that $\mu \in X^*$ is a mean on X if and only if

$$\inf_{s\in G} f(s) \leq \mu(f) \leq \sup_{s\in G} f(s)$$

for every $f \in X$.

Mizoguchi-Takahashi [29] first introduced the notion of submeans. A real valued function μ on X is called a submean on X if the following conditions are satisfied:

(1) $\mu(f+g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$;
(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geq 0$;
(3) for $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$;
(4) $\mu(c) = c$ for every constant function c.

Clearly every mean on X is a submean. For a submean μ on X and $f \in X$, according to time and circumstances, we use $\mu_t(f(t))$ instead of $\mu(f)$. For each $s \in G$ and $f \in m(G)$, we define elements $l_s f$ and $r_s f$ of $m(G)$ given by

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for all $t \in G$ respectively. A submean μ on X is said to be left invariant (resp. right invariant) if

$$\mu(f) = \mu(l_s f) \ (resp.\, \mu(f) = \mu(r_s f))$$

for all $f \in X$ and $s \in G$. An invariant submean is left and right invariant.

In [29], by using the notion of submean, Mizoguchi-Takahashi proved a nonlinear ergodic retraction theorem for lipschitzian semigroups. Other related results may be found in [6],[12],[13],[15],[20],[22],[32],[33],[35],[36] and [37].

Theorem 2.2 ([29]). *Let C be a closed convex subset of a Hilbert space H and X a r_s-invariant subspace of $m(G)$ containing constants which has a right invariant submean. Let $\Im = \{T_t : t \in G\}$ be a Lipschitzian semigroup on C with $\inf_s \sup_t k_{ts}^2 \leq 1$ and $\mathcal{F}(\Im) \neq \emptyset$, where k_t is the Lipschitzian constants and we denote $\mathcal{F}(\Im)$ by the set $\{x \in C : T_s(x) = x$ for all $s \in G\}$ of common fixed point of $\Im$. If for each $x, y \in C$, the function f on G defined by*

$$f(t) = \|T_t x - y\|^2 \quad \text{for all } t \in G$$

and the function g on G defined by

$$g(t) = k_t^2 \quad \text{for all } t \in G$$

belong to X, then the followings are equivalent:

(1) $\bigcap_{s \in G} \overline{conv}\{T_{ts}x : t \in G\} \bigcap \mathcal{F}(\Im) \neq \emptyset$ *for each $x \in C$.*

(2) *There is a nonexpansive retraction P of C onto $\mathcal{F}(\Im)$ such that $PT_t = T_t P = P$ for every $t \in G$ and $Px \in \overline{conv}\{T_t x : t \in G\}$ for every $x \in C$.*

Theorem 2.3 ([19]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $\Im = \{T_t : t \in G\}$ be a continuous representation of a semitopological semigroup $\Im$ as nonexpansive mappings from C into itself. If for each $x \in C$, the set $\bigcap_{s \in G} \overline{conv}\{T_{ts}x : t \in G\} \bigcap \mathcal{F}(\Im) \neq \emptyset$, then there exists a nonexpansive retraction P of C onto $\mathcal{F}(\Im)$ such that $PT_t = T_t P = P$ for every $t \in G$ and $Px \in \overline{conv}\{T_t x : t \in G\}$ for every $x \in C$.*

Now, without using the concept of submean, we shall give ergodic retraction theorem for semitopological semigroups of asymptotically nonexpansive type mappings on nonconvex domains.

Let C is a nonempty subset of a real Hilbert space H, G a semitopological semigroup, and $\Im = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C. For each $x \in C$, define $E(x)$ and $E(\Im)$ by

$$E(x) = \{z : \inf_{s \in G} \sup_{t \in G} \|T_{ts}x - z\| = \inf_{t \in G} \|T_t x - z\|\}$$

and

$$E(\Im) = \bigcap_{x \in C} E(x)$$

respectively.

We begin with the following lemma.

Lemma 2.1. *Let C be a nonempty subset of a Hilbert space H and $\Im = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C. Then $\mathcal{F}(\Im) \subset E(\Im)$.*

The following proposition plays an important role in the proof of the following two theorems.

Proposition 2.1. *Let G be a semitopological semigroup, C a nonempty subset of a Hilbert space H, and $\Im = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C. Then for every $x \in C$, the set*

$$\bigcap_{s \in G} \overline{conv}\{T_{ts}x : t \in G\} \bigcap E(x)$$

consists of at most one point.

Remark. *In the Takahashi-Zhang's result [38], it is assumed that C is a closed convex subset, G a reversible semigroup, and $\Im$ an asymptotically nonexpansive semigroup. The Proposition 2.1 shows us those key conditions are not necessary.*

Theorem 2.4 ([24]). *Let C be a nonempty subset of a Hilbert space H and let $\Im = \{T_t : t \in G\}$ be a semitopological semigroup of asymptotically nonexpansive type mappings on C such that $E(\Im) \neq \emptyset$. Then the followings are equivalent:*

(1) $\bigcap_{s \in G} \overline{conv}\{T_{ts}x : t \in G\} \bigcap E(\Im) \neq \emptyset$ *for each* $x \in C$.
(2) *There is a unique nonexpansive retraction P of C into $E(\Im)$ such that $PT_t = P$ for every $t \in G$ and $Px \in \overline{conv}\{T_t x : t \in G\}$ for every $x \in C$.*

By using Lemma 2.1 and Theorem 2.4, we have the following ergodic retraction theorem for asymptotically nonexpansive type semigroups with a nonclosed, nonconvex domain.

Theorem 2.5 ([23],[24]). *Let C be a nonempty subset of a real Hilbert space H and let $\Im = \{T_t : t \in G\}$ be a semitopological semigroup of asymptotically nonexpansive type mappings on C such that $\mathcal{F}(\Im) \neq \emptyset$. Then the followings are equivalent:*

(1) $\bigcap_{s \in G} \overline{conv}\{T_{ts}x : t \in G\} \bigcap \mathcal{F}(\Im) \neq \emptyset$ *for each* $x \in C$.

(2) *There is a unique nonexpansive retraction P of C onto $\mathcal{F}(\Im)$ such that $PT_t = T_tP = P$ for every $t \in G$ and $Px \in \overline{conv}\{T_tx : t \in G\}$ for every $x \in C$.*

Remark. *By Theorem 2.5, many key conditions, in Theorem 2.2 and Theorem 2.3, such as C is closed convex subset and $\Im$ is continuous lipschitzian semigroup, are not necessary.*

We give the corresponding results to almost-orbits of semitopological semigroup of asymptotically nonexpansive type semigroups. The notion of the almost-orbit of semigroups was first introduced by Miyadera-Kobayasi [28].

In this section, we assume that C is a nonempty bounded subset of a real Hilbert space H, G a semitopological semigroup, and $\Im = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C. We denote $F(G)$ by the set of all functions from G to C. Put

$$C_0(G) = \{f | f : G \mapsto R^+, \inf_{s\in G} \sup_{t\in G} f(ts) = 0\}.$$

For $u(\cdot) \in F(G)$. We call $u(\cdot)$ is an almost-orbit of $\Im$ if

$$\varphi(t) = \sup_{s\in G} \|u(st) - T_su(t)\| \in C_0(G).$$

Denoted $AO(\Im)$ by the set of all almost-orbits of $\Im$. For each $u \in F(G)$ and $f_u \in C_0(G)$, define $L(u, f_u)$ by

$$L(u, f_u) = \{z : \inf_{s\in G} \sup_{t\in G} \|u(tsh) - z\|^2 - \inf_{t\in G} \|u(th) - z\|^2 \leq f_u(h)\}$$

and if $u \in AO(\Im)$, we write

$$L(u) = L(u, f_u),$$

where $f_u(t) = 8d \sup_{s\in G} \varphi(st)$ and $d = \sup\{\|x - y\| : x, y \in C\}$.

We begin with the following lemma.

Lemma 2.2. *Let $u \in AO(\Im)$. Then $\mathcal{F}(\Im) \subset L(u)$.*

Proposition 2.2. *Let G be a semitopological semigroup and let C be a nonempty subset of a Hilbert space H. Then for each $u \in F(G)$ and $f_u \in C_0(G)$, the set*

$$\bigcap_{s\in G} \overline{conv}\{u(ts) : t \in G\} \bigcap L(u, f_u)$$

consists of at most one point.

Using Lemma 2.2 and Proposition 2.2, we have following proposition for almost-orbits of semitopological semigroup.

Proposition 2.3. *Let G be a semitopological semigroup, C a nonempty subset of a Hilbert space H, $\Im = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C, and $u(\cdot)$ an almost-orbit of $\Im$. Then the set*

$$\bigcap_{s\in G} \overline{conv}\{u(ts) : t \in G\} \bigcap \mathcal{F}(\Im)$$

consists of at most one point.

Remark. *In the Takahashi-Zhang's result [38], it is assumed that C is a closed convex subset, G a reversible semigroup, and $\Im$ an asymptotically nonexpansive semigroup. The above Propositions show us those key conditions are not necessary.*

By using Proposition 2.2, we shall prove ergodic retraction theorem.

Let D be a nonempty subset of $F(G)$, and for each $u(\cdot) \in D$, let $f_u \in C_0(G)$ with $\sup_{s\in G} \inf_{t\in G} f_u(ts) = 0$. Put

$$L(D) = \bigcap\{L(u, f_u) : u \in D\}.$$

Theorem 2.6 ([17]). *Let C be a nonempty subset of a Hilbert space H and let D be a nonempty subset of $F(G)$ such that $L(D) \neq \emptyset$. Then the following are equivalent:*

(*a*) $\bigcap_{s\in G} \overline{conv}\{u(ts) : t \in G\} \bigcap L(D) \neq \emptyset$ *for each* $u \in D$.

(*b*) *There is a unique mapping P of D into $L(D)$ such that*

(i) *P is nonexpansive in the sense that*

$$\|Pu - Pv\| \leq \inf_{s\in G} \sup_{t\in G} \|u(ts) - v(ts)\|.$$

(ii) $Pu = Pu_a$, *where* $u_a(t) = u(ta) \in D$ *for* $u \in D$ *and* $a \in G$.

(iii) $Pu \in \overline{conv}\{u(ts) : t \in G\}$ *for every* $u \in D$ *and* $s \in G$.

By using Lemma 2.2 and Theorem 2.6, we have the following ergodic retraction theorem for asymptotically nonexpansive type semigroups with nonconvex domain.

Theorem 2.7 ([17]). *Let C be a nonempty subset of a real Hilbert space H and let $\Im = \{T_t : t \in G\}$ be a semitopological semigroup of asymptotically nonexpansive type mappings on C such that $\mathcal{F}(\Im) \neq \emptyset$. Then the followings are equivalent:*

(a) $\bigcap_{s\in G} \overline{conv}\{T_{ts}x : t \in G\} \bigcap \mathcal{F}(\Im) \neq \emptyset$ *for each* $x \in C$.

(b) *There is a unique nonexpansive retraction P of C onto $\mathcal{F}(\Im)$ such that $PT_t = T_tP = P$ for every $t \in G$ and $Px \in \overline{conv}\{T_tx : t \in G\}$ for every $x \in C$.*

Using Theorem 2.7, we can get many well-known results. For example, the following theorem and Theorem 2.2 are immediately deduced from this theorem.

Theorem 2.8 ([19]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $\Im = \{T_t : t \in G\}$ be a continuous representation of a semitopological semigroup $\Im$ as nonexpansive mappings from C into itself. If for each $x \in C$, the set $\bigcap_{s\in G} \overline{conv}\{T_{ts}x : t \in G\} \bigcap \mathcal{F}(\Im) \neq \emptyset$, then there exists a nonexpansive retraction P of C onto $\mathcal{F}(\Im)$ such that $PT_t = T_tP = P$ for every $t \in G$ and $Px \in \overline{conv}\{T_tx : t \in G\}$ for every $x \in C$.*

3. Banach Space Setting

Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let G be a commutative semitopological semigroup with identity. The value of $x^* \in E^*$ (the dual space of E) at $x \in E$ will be denoted by $\langle x, x^*\rangle$. The duality mapping J(multivalued) from E into E^* will be defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$. We say that E is (F) if the norm of E is *Fréchet* differential, i.e. for each $x \neq 0$

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly in $y \in B_r$, where $B_r = \{z \in E : \|z\| < r\}$, $r > 0$. It is easy to see that E is (F) if and only if for any B_r and $x \in E$,

$$\lim_{t\to 0}(2t)^{-1}(\|x + ty\|^2 - \|x\|^2) = \langle y, J(x)\rangle,$$

uniformly in $y \in B_r$. E is said to satisfy the Opial's condition if $\{x_\alpha : \alpha \in A\}$ converges weakly to x, then

$$\limsup_{\alpha\in A} \|x_\alpha - x\| < \limsup_{\alpha\in A} \|x_\alpha - y\|$$

for all $y \neq x$, where A is a directed system.

Similarly results of Baillon's work [1] were also obtained in [4],[5],[7],[8], [9],[10],[11],[31],[34] and [35] in uniformly convex Banach spaces.

Let $m(G)$ be the Banach space of all bounded real valued functions on G with the supremum norm. Then, for each $s \in G$ and $f \in m(G)$, we can define r_sf in $m(G)$ by $(r_sf)(t) = f(t + s)$. Let D be a subspace of $m(G)$ and let μ be an element of D^*, where D^* is the dual space of D. Then, we denote by $\mu(f)$ the value of μ at the element f of D. According to the circumstance, we write by $\mu_t(f(t))$ or $\int f(t)d\mu(t)$ for the value $\mu(f)$. When D contains constants, a linear functional μ on D is called a mean on D if $\|\mu\| = \mu(1) = 1$. Further, let D be invariant under every $r_s, s \in G$. Then a mean μ on D is called invariant if $\mu(r_s f) = \mu(f)$ for all $s \in G$ and $f \in D$. For $s \in G$, we can define a point evaluation δ_s by $\delta_s = f(s)$ for every $f \in m(G)$. A convex combination of point evaluations is called a finite mean on G.

Let $\Im = \{T(t) : t \in G\}$ be a family of mappings from C into itself. $\Im$ is said to be a commutative semigroup of asymptotically nonexpansive type mappings on C if the following conditions are satisfied:

(a) $T(t+s)x = T(t)T(s)x$ for all $t, s \in G$ and $x \in C$:
(b) for each $t \in G$, there exists $\alpha(t) \geq 0$ such that

$$\|T(t)x - T(t)y\| \leq \|x - y\| + \alpha(t) \quad \text{for all} \quad x, y \in C$$

with

$$\lim_{t \in G} \alpha(t) = 0, \tag{3.1}$$

where $\lim_{t \in G} \alpha(t)$ denotes the limit of a net $\alpha(\cdot)$ on the directed system $(G, \leq)$ and the binary relation $\leq$ on G is defined by $a \leq b$ if and only if there is $c \in G$ with $a + c = b$.

As in ([28], [30]), a function $u(\cdot) : G \mapsto C$ is said to be an almost-orbit of $\Im = \{T(t) : t \in G\}$ if

$$\lim_{t \in G} [\sup_{h \in G} \|u(h+t) - T(h)u(t)\|] = 0.$$

Throughout the rest of this paper, $\Im = \{T(t) : t \in G\}$ is a commutative semigroup of asymptotically nonexpansive type mappings on C such that each $T(t)$ is continuous, $u(\cdot)$ is an almost-orbit of $\Im$, and D is a subspace of $m(G)$ containing constant functions and invariant under r_s for every $s \in G$. Furthermore suppose for each $x^* \in E^*$, the function $h_{x^*} : t \mapsto \langle u(t), x^* \rangle$ is in D. Since E is reflexive, for any $\mu \in D^*$ there exists a unique element u_μ in E such that

$$\langle u_\mu, x^* \rangle = \int \langle u(t), x^* \rangle d\mu(t)$$

for all $x^* \in E^*$. We denote u_μ by $\mu(t)\langle u(t) \rangle$ or $\int u(t)d\mu(t)$. If μ is a mean on D, then $\int u(t)d\mu(t)$ is contained in $\overline{co}\{u(t) : t \in G\}$. Also, if μ is a finite mean on G, say

$$\mu = \sum_{i=1}^{n} a_i \delta_{t_i} (t_i \in G, a_i \geq 0, i = 1, 2, \cdots, n, \sum_{i=1}^{n} a_i = 1),$$

then

$$\mu(t)\langle u(t)\rangle = \sum_{i=1}^{n} a_i u(t_i).$$

First of all, we need several lemmas which play a crucial role in the proof of our main theorems in this section.

Lemma 3.1 **([21]).** *Let $\{v_1(t) : t \in G\}$ and $\{v_2(t) : t \in G\}$ be almost-orbits of $\Im$. Then $\lim_{t\in G} \|v_1(t) - v_2(t)\|$ exists.*

To simplify, in the following, for each $\varepsilon \in (0, 1]$, we define

$$a(\varepsilon) = \frac{\varepsilon^2}{10R}\delta(\frac{\varepsilon}{R})$$

and

$$G_\varepsilon = \{h_\varepsilon \in G : \alpha(h + h_\varepsilon) < a(\varepsilon) \; for \; each \; h \in G\},$$

where δ is the modulus of convexity of the norm, $M = \sup\{\|x\| : x \in C\}$, and $R = 8M + 1$. Note that from (3.1), G_ε is nonempty for each $\epsilon > 0$ and for each $h \in G, h + h_\varepsilon \in G_\varepsilon$ whenever $h_\varepsilon \in G_\varepsilon$.

Lemma 3.2. *Let λ be a finite mean on G and let $\varepsilon_i \in (0, 1] (i = 1, 2)$. Then there exists $t_{\varepsilon_2} \in G$, where t_{ε_2} is independent of ε_1, such that*

$$\|T(h)\lambda(s)\langle u(t + s)\rangle - \lambda(s)\langle u(t + s + h)\rangle\| < \varepsilon_1 + \varepsilon_2$$

for all $t \geq t_{\varepsilon_2}$ and $h \in G_{\varepsilon_1}$.

Since G is a commutative semigroup, there exists a net $\{\lambda_\alpha : \alpha \in A\}$ of finite means on G such that

$$\lim_{\alpha\in A} \|\lambda_\alpha - r_s^*\lambda_\alpha\| = 0 \tag{3.2}$$

for every $s \in G$, where A is a directed set and r_s^* is the conjugate operator of r_s.

Let $\mathbb{N}$ be the set of positive integers and let $I = A \times \mathbb{N}$ be the product net. For $\beta = (\alpha, n) \in I$, put $P_1\beta = \alpha$, $P_2\beta = n$, $\lambda_\beta = \lambda_\alpha$, and $\varepsilon_\beta = \frac{1}{P_2\beta} = \frac{1}{n}$. Then, from (3.2), we have

$$\lim_{\beta\in I} \|\lambda_\beta - r_s^*\lambda_\beta\| = 0$$

for all $s \in G$ and

$$\lim_{\beta\in I} \varepsilon_\beta = 0.$$

By Lemma 3.2, for each $\beta \in I$, we can choose $s_\beta \in G$ such that

$$s_\beta \in G_{\varepsilon_\beta}, \tag{3.3}$$

and for each $\varepsilon > 0$,

$$\|T(h)\lambda_\beta(t)\langle u(t+s)\rangle - \lambda_\beta(t)\langle u(t+s+h)\rangle\| < \varepsilon + \varepsilon_\beta \tag{3.4}$$

for all $h \in G_\varepsilon$ and $s \geq s_\beta$.

Let

$$\Lambda = \{\{t_\alpha\}_{\alpha\in I} : t_\alpha \geq s_\alpha \quad \text{for all} \quad \alpha \in I\}.$$

Combining (3.3) and (3.4), we have, for every $\{t_\alpha\} \in \Lambda$ and $h \geq s_\beta$, that

$$\|T(h)\lambda_\alpha(t)\langle u(t_\alpha+s)\rangle - \lambda_\alpha(t)\langle u(t_\alpha+s+h)\rangle\| < \varepsilon_\beta + \varepsilon_\alpha.$$

Lemma 3.3. *Let $\{t_\alpha\} \in \Lambda$ and $f \in \mathcal{F}(\Im)$. Then $\lim_{\alpha\in I}\|\lambda_\alpha(t)\langle u(t+t_\alpha)\rangle - f\|$ exists.*

Lemma 3.4. *Suppose that E satisfies Opial's condition. Then we have the following.*

(a) *For each $\{t_\alpha\} \in \Lambda$, there exists $z \in \mathcal{F}(\Im)$ such that*

$$w - \lim_{\alpha\in I} \lambda_\alpha(t)\langle u(t+t_\alpha)\rangle = z.$$

(b) *$\mathcal{F}(\Im)$ is a nonempty closed convex subset of E.*

Now we give an ergodic theorem for almost-orbits of an asymptotically nonexpansive type semigroup in uniformly convex Banach spaces with Opial's condition.

As in [9], a net $\{\mu_\alpha : \alpha \in A\}$ of continuous linear functionals on D is called strongly regular if it satisfies the following conditions :

(a) $\sup_{\alpha\in A}\|\mu_\alpha\| < +\infty$;
(b) $\lim_{\alpha\in A}\mu_\alpha(1) = 1$;
(c) $\lim_{\alpha\in A}\|\mu_\alpha - r_s^*\mu_\alpha\| = 0$ for every $s \in G$, where A is a directed set.

Theorem 3.1 ([26]). *Let E be a uniformly convex Banach space which satisfies Opial's condition and let $\{\mu_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on D. Then, $\int u(t+h)d\mu_\alpha(t)$ converges weakly to an element p in $\mathcal{F}(\Im)$ uniformly in $h \in G$ and $u_\mu = p$ for all invariant mean μ on D.*

Next, we give the existence of a nonexpansive retraction for almost-orbits of an asymptotically nonexpansive type semigroup.

Let $AO(\Im)$ be the set of all almost-orbits of $\Im$. Then, for each $h \in G$ and $u \in AO(\Im)$, the function $v : G \mapsto C$ defined by $v(t) = T(h)u(t)$ is also an almost-orbit of $\Im$. In fact, if we set $\varphi(t) = \sup_{s\in G}\|u(t+s) - T(s)u(t)\|$, the result follows from $\|v(s+t) - T(s)v(t)\| = \|T(h)u(s+t) - T(s)T(h)u(t)\| \leq \|T(h)u(s+t) - u(h+s+t)\| + \|u(h+s+t) - T(s+h)u(t)\| \leq \varphi(s+t) + \varphi(t)$.

Theorem 3.2 ([26]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $\Im = \{T(t) : t \in G\}$ be an asymptotically nonexpansive type semigroup actting on C such that each $T(t)$ is continuous. Suppose that E satisfies Opial's condition. Then there exists a retraction P from the set $AO(\Im)$ onto $\mathcal{F}(\Im)$ such that*

(1) *P is nonexpansive in the sense that*

$$\|Pu - Pv\| \leq \lim_{t \in G} \|u(t) - v(t)\| \quad for \;\; u, v \in AO(\Im),$$

(2) *$PT(h)u = T(h)Pu = Pu$ for $u \in AO(\Im)$ and $h \in G$, and*
(3) *$Pu \in \bigcap_{s \in G} \overline{co}\{u(t) : t \geq s\}$ for $u \in AO(\Im)$.*

Also now, we give the ergodic theorems for almost-orbits of an asymptotically nonexpansive type semigroup in uniformly convex Banach spaces with *Fréchet* differentiable norm.

For each $\varepsilon > 0$ and $h \in G$, we set

$$\mathcal{F}_\varepsilon(T(h)) = \{x \in C : \|T(h)x - x\| \leq \varepsilon\}.$$

Lemma 3.5. *For each $0 < \epsilon < 1$, there exist $\epsilon_0 > 0$ and $h_0 \in G$ such that*

$$\overline{conv}\mathcal{F}_{\epsilon_0}(T(h)) \subset \mathcal{F}_\epsilon(T(h))$$

for each $h \geq h_0$.

Lemma 3.6. *Let $\{x_\alpha : \alpha \in A\}$ be a net of C. Then the conditions $w-\lim_{\alpha\in A} x_\alpha = x$ and $\limsup_{h\in G}(\limsup_{\alpha\in A} \|T(h)x_\alpha - x_\alpha\|) = 0$ imply that $x \in \mathcal{F}(\Im)$.*

The following proposition was proved in [23].

Proposition 3.1. *Let E be (F). Then $\cap_{s\in G}\overline{conv}\{u(t) : t \geq s\} \cap \mathcal{F}(\Im)$ is at most a singleton.*

Lemma 3.7. *Let $\{t_\alpha\} \in \Lambda$ and let W be the set of all weak limit points of $\{\lambda_\alpha(t)\langle u(t + t_\alpha)\rangle\}$. Then $W \subset \mathcal{F}(\Im)$. Moreover, if X is (F), then*

$$w - \lim_{\alpha \in A} \lambda_\alpha(t)\langle u(t + h + t_\alpha)\rangle = p \in \cap_{s\in G}\overline{conv}\{u(t) : t \geq s\} \cap \mathcal{F}(\Im)$$

uniformly in $h \in G$.

Combining Proposition 3.1, Lemma 3.7 and the proof of Theorem 2 in [9], we have the following main result.

Theorem 3.3 **([25].** *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E which is (F), $\Im = \{T(t) : t \in G\}$ a commutative semigroup of asymptotically nonexpansive type mappings on C such that each $T(t)$ is continuous, and $u(\cdot)$ an almost-orbit of $\Im$. Let $\{\mu_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on D. Then,*

$$w - lim_{\alpha \in A} \int u(t+h) d\mu_\alpha(t) = p \in \mathcal{F}(\Im) \text{ uniformly in } h \in G.$$

Furthermore, $u_\mu = u_\nu$ for invariant means μ and ν on D.

Remark. *Theorems 3.3 is extension of Theorem 3, Theorem 4 in [9] and Theorem 1 in [30] to the case of commutative semigroups of non-lipschitzian mappings.*

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