Related Topics to Transferable Utility Games
— An Infinite Market Game —

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1 Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space throughout this paper. A game $v$ is a nonnegative real valued function, defined on the $\sigma$-field $\mathcal{F}$, which maps the empty set to zero. An outcome of a game $v$ is a finitely additive real valued function $\alpha$ on $\mathcal{F}$ such that $\alpha(\Omega) = v(\Omega)$. For an outcome $\alpha$ of $v$, an integrable function $f$ satisfying $\int_S f \, d\mu = \alpha(S)$ for all $S \in \mathcal{F}$ is said to be an outcome density of $\alpha$ with respect to $\mu$. An outcome indicates outcomes to each coalition while an outcome density designates outcomes to every player. The core of $v$ is the set of outcomes $\alpha$ satisfying $\alpha(S) \geq v(S)$ for all $S \in \mathcal{F}$.

To every game $v$ we associate an extended real number $|v|$ defined by

$$|v| = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_{\Omega} \right\},$$

where $n = 1, 2, \ldots, S_i \in \mathcal{F}$, $\lambda_i$ is a real number. The notation $\chi_A$ denotes the characteristic function of a subset $A$ of $\Omega$. For a game $v$ with $|v| < \infty$, we define two games $\overline{v}$ and $\hat{v}$ by

$$\overline{v}(S) = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_{S} \right\}, \quad S \in \mathcal{F},$$

$$\hat{v}(S) = \min \{ \alpha(S) : \alpha \text{ is additive, } \alpha \geq v, \, \alpha(\Omega) = |v| \}, \quad S \in \mathcal{F},$$

where $n = 1, 2, \ldots, S_i \in \mathcal{F}$, $\lambda_i$ is a real number.
A game $v$ is said to be balanced if $v(\Omega) = |v|$, totally balanced if $v = \overline{v}$ and exact if $v = \hat{v}$, respectively. It is proved in [3] that the core of a game is nonempty if and only if it is balanced, every exact game is totally balanced, and every totally balanced game is balanced.

A game $v$ is said to be monotone if $S \subset T$ implies $v(S) \leq v(T)$ for any $S$ and $T$ in $\mathcal{F}$. A game $v$ is said to be continuous at $\Omega$ if it follows that $\lim_{n \to \infty} v(S_n) = v(\Omega)$ for any nondecreasing sequence $\{S_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} S_n = \Omega$.

### 2 Market Games

We denote utilities of players by a Carathéodory type function $u$ defined on $\Omega \times R^l_+ \to R^l_+$, where $R^l_+$ denotes the nonnegative orthant of the $l$-dimensional Euclidean space $R^l$, and $R^l_+$ is the set of nonnegative real numbers. The nonnegative number $u(\omega, x)$ designates the density of the utility of a player $\omega$ getting goods $x$. We always use the ordinary coordinatewise order when having concern with an order in $R^l_+$. We suppose that the function $u : \Omega \times R^l_+ \to R^l_+$ satisfies the conditions:

1. The function $\omega \mapsto u(\omega, x)$ is measurable for all $x \in R^l_+$;
2. The function $x \mapsto u(\omega, x)$ is continuous, concave, nondecreasing, and $u(\omega, 0) = 0$, for almost all $\omega$ in $\Omega$;
3. $\sigma \equiv \sup\{u(\omega, x) : (\omega, x) \in \Omega \times B_+\} < \infty$, where $B_+ = \{x \in R^l_+ : \|x\| \leq 1\}$, and $\|x\|$ denotes the Euclidean norm of $x \in R^l_+$.

For any set $S$ in $\mathcal{F}$, the set of integrable functions on $S$ to $R^l_+$ is denoted by $L_1(S, R^l_+)$. We take an element $e$ of $L_1(\Omega, R^l_+)$ as the density of initial endowments for the players. For any $S$ in $\mathcal{F}$, define

$$v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega)) d\mu(w) : x \in L_1(S, R^l_+), \int_S x \, d\mu = \int_S e \, d\mu \right\}.$$

The set function $v$ defined above is called a market game derived from the market $(\Omega, \mathcal{F}, \mu, u, e)$.

It is well known that the function $\omega \mapsto u(\omega, x(\omega))$ is measurable for any $x \in L_1(S, R^l_+)$. Moreover we need to show that the function $\omega \mapsto u(\omega, x(\omega))$ is integrable in order to define $v(S)$ as a real number.
Lemma 1 If $x \in L_1(S, R_+^l)$, then $u(\cdot, x(\cdot)) \in L_1(S, R_+)$ for any $S \in \mathcal{F}$ and the map $x \mapsto u(\cdot, x(\cdot))$ is continuous with respect to the norm topologies of $L_1(S, R_+^l)$ and $L_1(S, R_+)$.  

Proof Let $x \in L_1(S, R_+^l)$. Since $u(\omega, \cdot)$ is concave, for any $x \in R_+^l$ with $\|x\| > 1$, we have the inequality
\[
\frac{u(\omega, x) - u(\omega, x/\|x\|)}{\|x - x/\|x\|\|} \leq \frac{u(\omega, x/\|x\|) - u(\omega, 0)}{\|x/\|x\| - 0}\]
and hence we have $u(\omega, x) \leq \|x\|\sigma$ for any $\omega \in \Omega$ and $x \in R_+^l$ with $\|x\| > 1$. It is obvious from the definition of $\sigma$ that $u(\omega, x) \leq \sigma$ for any $\omega \in \Omega$ and $x \in R_+^l$ with $\|x\| \leq 1$. Thus we have $u(\omega, x) \leq \sigma(1 + \|x\|)$ for any $(\omega, x) \in \Omega \times R_+^l$ and this leads to the inequalities
\[
\int_S u(\omega, x(\omega)) d\mu(\omega) \leq \sigma \int_S (1 + \|x(\omega)\|) d\mu(\omega) = \sigma \left( \mu(S) + \int_S \|x(\omega)\| d\mu(\omega) \right) < \infty.
\]
Thus it follows that $u(\cdot, x(\cdot)) \in L_1(S, R_+)$. The second part of the assertion is verified in Theorem 2.1 of [2]. Although Theorem 2.1 of [2] is proved under the hypotheses that $S$ is a measurable set in $R^d$ and the second argument $x$ of the function $u$ runs over $R$, the proof of Theorem 2.1 of [2] is valid even in our setting. Thus the map $x \mapsto u(\cdot, x(\cdot))$ is norm continuous. Q.E.D.

Lemma 2 A market game $v$ is actually a game and is monotone.  

Proof It is obvious $v(\emptyset) = 0$. The finiteness of $v(S)$ follows since the inequalities
\[
\int_S u(\omega, x(\omega)) d\mu(\omega) \leq \sigma \int_S (1 + \|x\|) d\mu
\]
\[
\leq \sigma \left( \mu(S) + \sum_{i=1}^l \int_S x^i d\mu \right) = \sigma \left( \mu(S) + \sum_{i=1}^l \int_S e^i d\mu \right)
\]
hold if
\[
\int_S x d\mu = \int_S e d\mu,
\]
where $x^i$ and $e^i$ are the $i$-th coordinate functions of $x$ and $e$, respectively. Moreover $v$ is monotone because $u$ has nonnegative values. Q.E.D.
We start with a lemma on concave functions.

**Lemma 3** If $f : R_+^n \rightarrow R$ is concave and $f(0) = 0$, then for any $x_1, \ldots, x_n \in R_+^n$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i \leq 1$, it follows that

$$\sum_{i=1}^{n} \lambda_i f(x_i) \leq f(\sum_{i=1}^{n} \lambda_i x_i).$$

**Proof** We can assume that $\lambda = \sum_{i=1}^{n} \lambda_i > 0$ without loss of generality. It follows that

$$\sum_{i=1}^{n} \lambda_i f(x_i) = \lambda \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} f(x_i)$$

$$\leq \lambda f(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} x_i)$$

$$= (1 - \lambda)f(0) + \lambda f(\frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i x_i)$$

$$\leq f(\sum_{i=1}^{n} \lambda_i x_i).$$

Q.E.D.

Let $S$ be a measurable set. For any $x \in L_1(S, R_+^n)$, define $\overline{x} \in L_1(\Omega, R_+)$ by

$$\overline{x}(\omega) = \begin{cases} x(\omega), & \text{if } \omega \in S; \\ 0, & \text{if } \omega \in S^c. \end{cases}$$

**Proposition 1** A market game $v$ is totally balanced.

**Proof** Take any $S \in \mathcal{F}$ and $S_i \in \mathcal{F}$ and $\lambda_i > 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_S$. We can assume that $\mu(S) > 0$ without loss of generality.

Let $\epsilon$ be an arbitrary positive number. Take $x_i \in L_1(S_i, R_+^n)$ such that

$$\int_{S_i} x_i d\mu = \int_{S_i} e d\mu \quad \text{and} \quad v(S_i) - \frac{\epsilon}{n} < \int_{S_i} u(\omega, x_i(\omega)) d\mu(\omega),$$

3 Cores of Market Games
and define \( y \in L_1(S, \mathcal{R}_+^1) \) by
\[
y = \sum_{i=1}^{n} \lambda_i \overline{x}_i.
\]

Then we have the following:
\[
\int_S y \, d\mu = \sum_{i=1}^{n} \lambda_i \int_S \overline{x}_i \, d\mu
= \sum_{i=1}^{n} \lambda_i \int_{S_i} e \, d\mu
= \int_S e \sum_{i=1}^{n} \lambda_i \chi_{S_i} \, d\mu
\leq \int_S e \, d\mu.
\]

Define \( y' \in L_1(S, \mathcal{R}_+^1) \) by
\[
y' = y + \frac{1}{\mu(S)} \left( \int_S e \, d\mu - \int_S y \, d\mu \right).
\]

Then it is easily seen that \( \int_S y' \, d\mu = \int_S e \, d\mu \).

On the other hand, let \( A \) be the family of all nonempty subsets \( A \) of \( \{1, \ldots, n\} \) such that \( T_A \equiv \cap_{i \in A} S_i \cap \cap_{j \in A^c} (S \setminus S_j) \neq \emptyset \). Then it is easily seen that \( S_i = \bigcup_{A \ni i} T_A \) for \( i = 1, \ldots, n \) and \( \{T_A : A \in \mathcal{A}\} \) is a partition of \( \bigcup_{i=1}^{n} S_i \), and \( \sum_{i \in A} \lambda_i \leq 1 \) for all \( A \in \mathcal{A} \). For any \( i \) and \( A \) with \( i \in A \in \mathcal{A} \), define \( x_i^A = x_i|_{T_A} \), the restriction of \( x_i \) to \( T_A \). Then we have
\[
\overline{x}_i = \sum_{A \ni i} \overline{x}_i^A \quad \text{and} \quad y = \sum_{A \in \mathcal{A}} \sum_{i \in A} \lambda_i \overline{x}_i^A.
\]
Thus we have
\[
\sum_{i=1}^{n} \lambda_i v(S_i) - \epsilon < \sum_{i=1}^{n} \lambda_i \int_{S_i} u(\omega, x_i(\omega)) d\mu(\omega)
\]
\[
= \sum_{i=1}^{n} \sum_{A \ni i} \lambda_i \int_{T_A} u(\omega, x_i^A(\omega)) d\mu(\omega)
\]
\[
= \sum_{A \in A} \int_{T_A} \sum_{i \in A} \lambda_i u(\omega, x_i^A(\omega)) d\mu(\omega)
\]
\[
\leq \sum_{A \in A} \int_{T_A} u(\omega, \sum_{i \in A} \lambda_i x_i^A(\omega)) d\mu(\omega) \quad \text{by Lemma 3}
\]
\[
= \int_{S} u(\omega, \sum_{A \in A} \sum_{i \in A} \lambda_i x_i^A(\omega)) d\mu(\omega) \quad \text{by } u(\omega, 0) = 0
\]
\[
= \int_{S} u(\omega, y(\omega)) d\mu(\omega)
\]
\[
\leq \int_{S} u(\omega, y'(\omega)) d\mu(\omega) \quad \text{by monotonicity of } u(\omega, \cdot)
\]
\[
\leq v(S).
\]

Therefore, we have
\[
\sum_{i=1}^{n} \lambda_i v(S_i) \leq v(S).
\]

Thus \( \bar{v}(S) \leq v(S) \) and the reverse inequality is obvious. Hence we have \( \bar{v} = v \). Q.E.D.

**Proposition 2** A market game \( v \) is continuous at \( \Omega \).

**Proof** Let \( \{S_n\} \) be a nondecreasing sequence of measurable sets with \( \Omega = \bigcup_{n=1}^{\infty} S_n \) and \( \epsilon \) an arbitrary positive number. Then, there is \( x \in L_1(S, R_+) \) such that

\[
v(\Omega) - \epsilon < \int_{\Omega} u(\omega, x(\omega)) d\mu(\omega) \quad \text{and} \quad \int_{\Omega} x d\mu = \int_{\Omega} e d\mu.
\]
Let $x_n$ be the restriction $x|_{S_n}$ and define a sequence $\{y_n\}$ of functions in $L_1(S_n, R_+)$ by

$$y_n^i = \begin{cases} 
\frac{\int_{S_n} e^i d\mu}{\int_{S_n} x_n^i d\mu} x_n^i, & \text{if } \int_{S_n} x_n^i d\mu > \int_{S_n} e^i d\mu; \\
\frac{1}{\mu(S_n)} \left( \int_{S_n} e^i d\mu - \int_{S_n} x_n^i d\mu \right), & \text{if } \int_{S_n} x_n^i d\mu \leq \int_{S_n} e^i d\mu,
\end{cases}$$

for $i = 1, \ldots, l$. It is obvious that

$$\int_{S_n} y_n d\mu = \int_{S_n} e^i d\mu.$$  

On the other hand, since

$$\lim_{n \rightarrow \infty} \int_{S_n} |y_n^i - x_n^i| d\mu = \lim_{n \rightarrow \infty} \left| \int_{S_n} e^i d\mu - \int_{S_n} x_n^i d\mu \right| = 0,$$

for $i = 1, \ldots, l$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} \|\overline{y}_n - x\| d\mu = \lim_{n \rightarrow \infty} \int_{S_n} \|y_n - x\| d\mu + \lim_{n \rightarrow \infty} \int_{S_n} \|x\| d\mu = 0,$$

and hence $\overline{y}_n$ converges to $x$ with respect to the norm topology of $L_1(\Omega, R_+)$. Therefore, by Lemma 1, it follows that

$$\lim_{n \rightarrow \infty} \int_{S_n} u(\omega, y_n(\omega)) d\mu(\omega) = \lim_{n \rightarrow \infty} \int_{\Omega} u(\omega, \overline{y}_n(\omega)) d\mu(\omega) = \int_{\Omega} u(\omega, x(\omega)) d\mu(\omega)$$

and hence, for sufficiently large $n$,

$$v(\Omega) - \epsilon < \int_{S_n} u(\omega, y_n(\omega)) d\mu(\omega) \leq v(S_n).$$

Thus we have $\lim_{n \rightarrow \infty} v(S_n) = v(\Omega)$. Q.E.D.

Now we have reached our main theorem combining Proposition 1 and Proposition 2.

**Theorem 1** A market game $v$ has a nonempty core, and every element $\alpha$ of the core is countably additive and has a unique outcome density $f$ in $L_1(\Omega, R_+)$ with respect to $\mu$, and hence it follows that

$$\alpha(S) = \int_S f d\mu, \quad S \in \mathcal{F}.$$
Proof The core is nonempty by Proposition 1. Since $v$ is continuous at $\Omega$ by Proposition 2, any element $\alpha$ of the core is continuous at $\Omega$, which implies that $\alpha$ is countably additive. To prove existence of an outcome density for $\alpha$, it is sufficient to show that $\alpha$ is absolutely continuous with respect to $\mu$ by virtue of the Radon-Nikodym theorem. If $\mu(S) = 0$, then $v(S^c) = v(\Omega)$ by the definition of the market game $v$, and hence we have $\alpha(S^c) \geq v(S^c) = v(\Omega) = \alpha(\Omega)$, that is, $\alpha(S) = 0$. Q.E.D.

Remark 1 Similar to the assertion of Theorem 1, an exact game which is continuous at $\Omega$ has a nonempty core and every element of the core is countably additive. Moreover, there is a measure $\lambda$ on $\mathcal{F}$ such that every element of the core is absolutely continuous with respect to $\lambda$ according to [3]. The following example shows that there is a market game which is not exact and Theorem 1 is independent of the results of [3].

Example 1 [[1], pp. 192] Let $l = 1$, $\Omega = [0,1]$ and $\mu$ be the Lebesgue measure. Define $u: [0,1] \times R_+ \rightarrow R_+$ by

$$u(\omega, x) = \sqrt{x + \omega} - \sqrt{\omega} \quad \text{and} \quad e(\omega) = \frac{1}{32} \quad \text{for all } \omega \in [0,1].$$

According to [1], the supremum is attained for every measurable sets in the definition of the market game, and the core has only one element $\alpha$ and the outcome density $f$ of $\alpha$ is given by

$$f(\omega) = \begin{cases} \left(\frac{\omega}{2} - \sqrt{\omega}\right)^2 + \frac{1}{32}, & \text{if } \omega \in \left[0, \frac{1}{4}\right]; \\ \frac{1}{32}, & \text{if } \omega \in \left[\frac{1}{4}, 1\right]. \end{cases}$$

Thus it follows $\alpha([\frac{1}{2}, 1]) = \frac{1}{64}$, and hence $\hat{v}([\frac{1}{2}, 1]) = \frac{1}{64}$. On the other hand, we have

$$\sqrt{x + \omega} - \sqrt{\omega} \leq \sqrt{x + \frac{1}{2}} - \sqrt{\frac{1}{2}} \leq \sqrt{\frac{1}{2}}x$$

for $1/2 \leq \omega \leq 1$ and $x \geq 0$. Thus, if $x \in L_1([0,1], R_+)$ satisfies

$$\int_{\frac{1}{2}}^{1} x \, d\mu = \int_{\frac{1}{2}}^{1} e \, d\mu = \frac{1}{64},$$

then

$$\int_{\frac{1}{2}}^{1} u(\omega, x(\omega)) \, d\mu(\omega) \leq \int_{\frac{1}{2}}^{1} \sqrt{\frac{1}{2}}x \, d\mu = \frac{1}{64\sqrt{2}} < \frac{1}{64}.$$

Therefore we have $v([\frac{1}{2}, 1]) < \hat{v}([\frac{1}{2}, 1])$ and $v$ is not exact.
References


