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ON ENERGY DECAY ESTIMATES FOR THE WAVE EQUATION OF KIRCHHOFF TYPE

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ABSTRACT. In this paper we prove the existence and uniqueness of the solution to the mixed problem for wave equation of Kirchhoff type with nonlinear boundary damping and memory term. Moreover we discuss the uniform decay of the solution.

1. INTRODUCTION

In this paper, we are concerned with the existence, uniqueness and uniform decay of solution for nondegenerate wave equation of Kirchhoff type with nonlinear boundary damping and memory source term of the form:

$$u'' - M(\|\nabla u\|^2)\Delta u - \Delta u' = 0 \quad \text{on} \quad Q = \Omega \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on} \quad x \in \Omega, \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,\infty), \tag{1.3}$$

$$M(\|\nabla u\|^2)\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu} + u + u' + g(t)|u'|^\rho u' = g * |u|^\gamma u \quad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,\infty), \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with C^2 boundary $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and Γ_0, Γ_1 have positive measures, $M(s)$ is a C^1 class function such that $M(s) \geq m_0$ for some constant $m_0 > 0$, $g * u = \int_0^t g(t-r)u(r)dr$, $\|\nabla u\|^2 = \Sigma_{i=1}^n \int_\Omega |\frac{\partial u}{\partial x_i}|^2 dx$, $\Delta u = \Sigma_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$ and ν denotes the unit outer normal vector pointing towards Ω. Here

$$0 < \gamma, \rho \leq \frac{1}{n-2} \quad \text{if} \quad n \geq 3, \quad \text{or} \quad \gamma, \rho > 0 \quad \text{if} \quad n = 1, 2. \tag{1.5}$$

This problem has its origin in the mathematical description of small amplitude vibrations of an elastic string([1-3, 5, 7, 8, 13-16, 18 and reference therein]). There exists a

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large body of literature regarding viscoelastic problems with the memory term acting in the domain([3, 4, 6, 9]). Boundary stabilization has received considerable attention in the literature and among the numerous works in this direction, we can cite the works of Lasiecka and Tataru[10], Rao[17] and Zuazua[19].

Matsuyama [11](also see [12]) investigated the existence and asymptotic behavior of solutions of (1.1)-(1.3) with Dirichlet boundary conditions. Our work was motivated by some results of Cavalcanti et al.[3]. They have studied the existence and uniform decay of strong solutions of wave equations with nonlinear boundary damping and memory source term, that is, semilinear case. In this paper, we will study the existence of strong solutions of the problems (1.1)-(1.4). Moreover, when $\rho = \gamma$, the uniform decay of the energy

$$E(t) = \frac{1}{2}||u'(t)||^2 + \frac{1}{2}\bar{M}(||\nabla u(t)||^2) + \frac{1}{2}||u(t)||^2_{\Gamma_0} \tag{1.1}$$

is proved. Here, $\bar{M}(s) = \int_0^s M(r)dr$.

It is important to observe that as far as we concerned it has never been considered nonlinear memory terms acting in the boundary in the literature. Works of this paper may be contribute the study of wave equation of Kirchhoff type and nonlinear boundary feedback combined with a nonlinear memory source term. Our paper is organized as follows: In Section 2, we give some notations, assumptions and state the main result. In Section 3, we prove the existence of solution of the problems (1.1)-(1.4) and the uniform decay of energy is given in Section 4.

2. Assumption and main result

Throughout this paper we define

$$V := \{u \in H^1(\Omega);\ u = 0 \quad \text{on} \quad \Gamma_1\}, \quad (u, v) := \int_\Omega u(x)v(x)dx,$$

$$(u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)d\Gamma \text{ and } ||u||^p_{p,\Gamma_0} = \int_{\Gamma_0} |u(x)|^p dx.$$

For simplicity we denote $||\cdot||_{L^2(\Omega)}$ and $||\cdot||_{2,\Gamma_0}$ by $||\cdot||$ and $||\cdot||_{\Gamma_0}$.

(A_1) **Assumptions on the initial data**

Let us consider $u_0,\ u_1 \in V \cap H^{\frac{3}{2}}(\Omega)$ verifying the compatibility conditions

$$\begin{aligned}
&M(||\nabla u_0||^2)\Delta u_0 + \Delta u_1 = 0 \text{ in } \Omega,\\
&u_0 = 0 \quad \text{on} \quad \Gamma_1,\\
&M(||\nabla u_0||^2)\frac{\partial u_0}{\partial \nu} + \frac{\partial u_1}{\partial \nu} + u_0 + u_1 + g(0)|u_1|^\rho u_1 = 0 \quad \text{on} \quad \Gamma_0.
\end{aligned}$$

(A_2) **Assumptions on the kernel g of the memory**:

Let us consider the function $g \in W^{1,\infty}(0,\infty) \cap W^{1,1}(0,\infty)$ such that $g(t) \geq 0$, $\forall t \geq 0$ and

$$\begin{aligned}
&-\alpha_0 g(t) \leq g'(t) \leq -\alpha_1 g(t), \quad \forall t \geq t_0,\\
&g(0) = 0,\ |g'(t)| \leq \alpha_2 g(t), \quad \forall t \in [0, t_0]
\end{aligned}$$

for some α_0, α_1, $\alpha_2 > 0$ and $l := 1 - \int_0^\infty g(r)dr > 0$.

Now we are in position to state our main result.

Theorem 2.1. *Under the assumptions (A_1)-(A_2), suppose that γ, ρ satisfy the hypothesis (1.5) with $\rho \geq \gamma$. Then problems (1.1)-(1.4) have a unique strong solution $u : \Omega \to \mathbb{R}$ such that $u \in L^\infty(0,\infty;V)$, $u' \in L^\infty(0,\infty;V)$, $u'' \in L^2(0,\infty;L^2(\Omega))$. Moreover, if $\rho = \gamma$ and $\alpha_1 > 2(\gamma+2)$, then there exist positive constants C_1 and C_2 such that*

$$E(t) \leq C_1 E(0) exp(-C_2 t) \text{ for all } t \geq t_0.$$

3. Proof of Theorem 2.1

In this section we are going to show the existence of solution for problems (1.1)-(1.4) using Faedo-Galerkin's approximation. For this end we represent by $\{w_j\}_{j\in N}$ a basis in V which is orthonormal in $L^2(\Omega)$, by V_m the finite dimensional subspace of V generated by the first m vectors. Next we define $u_m(t) = \Sigma_{j=1}^m g_{jm}(t) w_j$, where $u_m(t)$ is the solution of the following Cauchy problem:

$$\begin{aligned}
&(u''_m, w) + M(\|\nabla u_m\|^2)(\nabla u_m, \nabla w) + (\nabla u'_m, \nabla w) + (u_m, w)_{\Gamma_0} \\
&\quad + (u'_m, w)_{\Gamma_0} + (g(t)|u'_m|^\rho u'_m, w)_{\Gamma_0} \\
&= \int_0^t g(t-r)|u_m(r)|^\gamma (u_m(r), w)_{\Gamma_0} dr, \quad w \in V_m
\end{aligned} \tag{3.1}$$

with the initial conditions,

$$\begin{aligned}
u_m(0) &= u_{0m} = \Sigma_{j=1}^m (u_0, w_j) w_j \to u_0 \quad \text{in } V \cap H^{\frac{3}{2}}(\Omega), \\
u'_m(0) &= u_{1m} = \Sigma_{j=1}^m (u_1, w_j) w_j \to u_1 \quad \text{in } V.
\end{aligned} \tag{3.2}$$

The approximate system is a system of m ordinary differential equations. It is easy to see that equation (3.1) has a local solution in $[0, T_m)$. The extension of these solutions to the whole interval $[0,\infty)$ is a consequence of the first estimate which we are going to prove below.

A Priori Estimate I.

Replacing w by $u'_m(t)$ in (3.1), assumption (A_2) yield

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\|u'_m(t)\|^2 + \frac{1}{2}\bar{M}(\|\nabla u_m(t)\|^2) + \frac{1}{2}\|u_m(t)\|^2_{\Gamma_0} \\
&\quad + \frac{1}{\gamma+2} g(t)\|u_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \int_0^t g(t-r)\|u_m(r)\|^{\gamma+2}_{\gamma+2,\Gamma_0} dr\Big) \\
&\quad + \|u'_m(t)\|^2_{\Gamma_0} + g(t)\|u'_m(t)\|^{\rho+2}_{\rho+2,\Gamma_0} + \|\nabla u'_m(t)\|^2 \\
&= \int_0^t g(t-r)|u_m(r)|^\gamma (u_m(r), u'_m(t))_{\Gamma_0} dr + \frac{1}{\gamma+2} g'(t)\|u_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\
&\quad + g(t)|u_m(t)|^\gamma (u_m(t), u'_m(t))_{\Gamma_0} + \int_0^t g'(t-r)\|u_m(r)\|^{\gamma+2}_{\gamma+2,\Gamma_0} dr
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&\le \int_0^t g(t-r)|u_m(r)|^\gamma (u_m(r), u_m'(t))_{\bar{\alpha}} dr \\
&+ \frac{\alpha_2}{\gamma+2} g(t)\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + g(t)|u_m(t)|^\gamma (u_m(t), u_m'(t))_{\Gamma_0} \\
&+ \bar{\alpha} \int_0^t g(t-r)\|u_m(r)\|_{\gamma+2,\Gamma_0}^{\gamma+2} dr,
\end{aligned} \tag{3.3}
$$

where $\bar{\alpha} = \max\{\alpha_0, \alpha_1, \alpha_2\}$.

Note that Hölder's inequality and Young's inequality give us, for any $\eta > 0$,

$$
\begin{aligned}
|u_m(r)|^\gamma (u_m(r), u_m'(t))_{\Gamma_0} &\le \int_{\Gamma_0} |u_m(r)|^{\gamma+1} |u_m'(t)| d\Gamma \\
&\le \Big(\int_{\Gamma_0} |u_m(r)|^{\gamma+2} d\Gamma\Big)^{\frac{\gamma+1}{\gamma+2}} \Big(\int_{\Gamma_0} |u_m'(t)|^{\gamma+2} d\Gamma\Big)^{\frac{1}{\gamma+2}} \\
&= \|u_m(r)\|_{\gamma+2,\Gamma_0}^{\gamma+1} \|u_m'(t)\|_{\gamma+2,\Gamma_0} \\
&\le C_1(\eta)\|u_m(r)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + \eta \|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}.
\end{aligned} \tag{3.4}
$$

Thus we have

$$
\begin{aligned}
&\int_0^t g(t-r)|u_m(r)|^\gamma (u_m(r), u_m'(t))_{\Gamma_0} dr \\
&\le \int_0^t g(t-r)\big(C_1(\eta)\|u_m(r)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + \eta \|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\big) dr \\
&= C_1(\eta) \int_0^t g(t-r)\|u_m(r)\|_{\gamma+2,\Gamma_0}^{\gamma+2} dr + \eta \|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} \int_0^t g(r) dr.
\end{aligned} \tag{3.5}
$$

Since $\rho \ge \gamma$, $L^{\rho+2}(\Gamma_0) \hookrightarrow L^{\gamma+2}(\Gamma_0)$ and therefore we can obtain

$$
\eta \|u_m'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} \int_0^t g(r) dr \le C_2(\eta) \int_0^t g(r) dr + \eta \int_0^t g(r) dr \|u_m'(t)\|_{\rho+2,\Gamma_0}^{\rho+2}. \tag{3.6}
$$

Therefore (3.5) and (3.6) yield

$$
\begin{aligned}
&\int_0^t g(t-r)|u_m(r)|^\gamma (u_m(r), u_m'(t))_{\Gamma_0} dr \\
&\le C_1(\eta) \int_0^t g(t-r)\|u_m(r)\|_{\gamma+2,\Gamma_0}^{\gamma+2} dr \\
&\quad + C_2(\eta) \int_0^t g(r) dr + \eta \int_0^t g(r) dr \|u_m'(t)\|_{\rho+2,\Gamma_0}^{\rho+2}.
\end{aligned} \tag{3.7}
$$

Similarly we have

$$
\begin{aligned}
&g(t)|u_m(t)|^\gamma (u_m(t), u_m'(t))_{\Gamma_0} \\
&\le C_3(\eta) g(t)\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + g(t)C_4(\eta) + \eta g(t)\|u_m'(t)\|_{\rho+2,\Gamma_0}^{\rho+2}.
\end{aligned} \tag{3.8}
$$

Therefore (3.3), (3.7) and (3.8) give

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\|u'_m(t)\|^2+\frac{1}{2}\bar{M}(\|\nabla u_m(t)\|^2)+\frac{1}{2}\|u_m(t)\|^2_{\Gamma_0}\\
&\qquad+\frac{1}{\gamma+2}g(t)\|u_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}+\int_0^t g(t-r)\|u_m(r)\|^{\gamma+2}_{\gamma+2,\Gamma_0}dr\Big)\\
&\qquad+\|\nabla u'_m(t)\|^2+\|u'_m(t)\|^2_{\Gamma_0}+((1-\eta)g(t)-\eta\|g\|_{L^1(0,\infty)})\|u'_m(t)\|^{\rho+2}_{\rho+2,\Gamma_0}\\
&\le (C_1(\eta)+\alpha_2)\int_0^t g(t-r)\|u_m(r)\|^{\gamma+2}_{\gamma+2,\Gamma_0}dr\\
&\qquad+(C_3(\eta)+\frac{\alpha_2}{\gamma+2})g(t)\|u_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}+C_4(\eta)g(t)+C_2(\eta)\int_0^t g(r)dr.
\end{aligned}
\tag{3.9}
$$

Note that we can choose $\eta>0$ sufficiently small such that $(1-\eta)g(t)-\eta\|g\|_{L^1(0,\infty)}>C_0g(t)$ for some constant C_0, which can be from assumption (A_2). Integrating (3.9) over $[0,t]$, choosing $\eta>0$ sufficiently small and employing Gronwall's lemma we obtain the first estimate:

$$
\begin{aligned}
&\frac{1}{2}\|u'_m(t)\|^2+\frac{1}{2}\bar{M}(\|\nabla u_m(t)\|^2)+\frac{1}{\gamma+2}g(t)\|u_m(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}\\
&\qquad+\frac{1}{2}\|u_m(t)\|^2_{\Gamma_0}+C_0\int_0^t g(s)\|u'_m(s)\|^{\rho+2}_{\rho+2,\Gamma_0}ds\\
&\qquad+\int_0^t(g(t-r)\|u_m(r)\|^{\gamma+2}_{\gamma+2,\Gamma_0}+\|\nabla u'_m(s)\|^2+\|u'_m(s)\|^2_{\Gamma_0})ds\\
&\le L_1,
\end{aligned}
\tag{3.10}
$$

where $L_1>0$ is independent of m. Since $\bar{M}(\|\nabla u_m(t)\|^2)\ge m_0\|\nabla u_m(t)\|^2$, from (3.10) we have

$$\|\nabla u_m(t)\|^2\le\frac{2L_1}{m_0}.$$

A Priori Estimate II.

Differentiating (3.1) and substituting w by $u''_m(t)$, assumption (A_2) and (3.10) yield

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\|u''_m(t)\|^2+\frac{1}{2}\|u'_m(t)\|^2_{\Gamma_0}\Big)+\|\nabla u''_m(t)\|^2+\|u''_m(t)\|^2_{\Gamma_0}\\
&\qquad+(\rho+1)g(t)(|u'_m(t)|^\rho,|u''_m(t)|^2)_{\Gamma_0}\\
&=-M(\|\nabla u_m(t)\|^2)(\nabla u'_m(t),\nabla u''_m(t))-g'(t)|u'_m(t)|^\rho(u'_m(t),u''_m(t))_{\Gamma_0}\\
&\qquad-2M'(\|\nabla u_m(t)\|^2)(\nabla u_m(t),\nabla u'_m(t))(\nabla u_m(t),\nabla u''_m(t))\\
&\qquad+\int_0^t g'(t-r)|u_m(r)|^\gamma(u_m(r),u''_m(t))_{\Gamma_0}dr
\end{aligned}
$$

$$\leq C_1\|\nabla u_m'(t)\|^2 + \frac{1}{2}\|\nabla u_m''(t)\|^2 - g'(t)|u_m'(t)|^\rho(u_m'(t), u_m''(t))_{\Gamma_0}$$
$$+ \int_0^t g'(t-r)|u_m(r)|^\gamma(u_m(r), u_m''(t))_{\Gamma_0}dr$$
$$\equiv C_1\|\nabla u_m'(t)\|^2 + \frac{1}{2}\|\nabla u_m''(t)\|^2 + I_1 + I_2,$$

where $M_1 = \sup_{0\leq s\leq \frac{2L_1}{m_0}} M(s)$, $M_2 = \sup_{0\leq s\leq \frac{2L_1}{m_0}} M'(s)$ and $C_1 = M_1^2 + 4M_2^2(\frac{2L_1}{m_0})^2$. Now, Schwarz's inequality and first estimate gives us

$$\begin{aligned} I_1 &\leq \alpha_2 g(t)\int_{\Gamma_0} |u_m'(t)|^{\frac{\rho}{2}}|u_m'(t)|^{\frac{\rho}{2}+1}|u_m''(t)|d\Gamma \\ &\leq \frac{\alpha_2^2}{4\eta}g(t)\|u_m'(t)\|_{\rho+2,\Gamma_0}^{\rho+2} + \eta g(t)(|u_m'(t)|^\rho, |u_m''(t)|^2)_{\Gamma_0}. \end{aligned} \tag{3.11}$$

Now, taking into account that $\frac{\gamma+1}{2\gamma+2} + \frac{1}{2} = 1$, using the generalized Hölder inequality and the continuity of the trace operator $\gamma_0 : H^1(\Omega) \hookrightarrow L^q(\Gamma)$ for $1 \leq q \leq \frac{2n-2}{n-2}$, we obtain

$$\begin{aligned} (|u_m(r)|^\gamma u_m(r), u_m''(t))_{\Gamma_0}dr &\leq (\int_{\Gamma_0} |u_m(r)|^{2\gamma+2}d\Gamma)^{\frac{\gamma+1}{2\gamma+2}}(\int_{\Gamma_0} |u_m''(t)|^2 d\Gamma)^{\frac{1}{2}} \\ &\leq C(\eta)\|\nabla u_m(r)\|^{2\gamma+2} + \eta\|u_m''(t)\|_{\Gamma_0}^2 \\ &\leq C(\eta)(\frac{2L_1}{m_0})^{\gamma+1} + \eta\|u_m''(t)\|_{\Gamma_0}^2. \end{aligned} \tag{3.12}$$

Thus from (3.12), we get

$$\begin{aligned} I_2 &\leq \alpha_2\int_0^t g(t-r)(C(\eta)(\frac{2L_1}{m_0})^{\gamma+1} + \eta\|u_m''(t)\|_{\Gamma_0}^2)dr \\ &\leq \alpha_2 C(\eta)(\frac{2L_1}{m_0})^{\gamma+1}\|g\|_{L^1(0,\infty)} + \eta\alpha_2\|u_m''(t)\|_{\Gamma_0}^2\|g\|_{L^1(0,\infty)}. \end{aligned} \tag{3.13}$$

Combining the estimates (3.11)-(3.13), we get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_m''(t)\|^2 + \frac{1}{2}\|u_m'(t)\|_{\Gamma_0}^2\right) + \|u_m''(t)\|_{\Gamma_0}^2 + \frac{1}{2}\|\nabla u_m''(t)\|^2 \\ &\quad + (\rho+1-\eta)g(t)(|u_m'(t)|^\rho, |u_m''(t)|^2)_{\Gamma_0} \\ &\leq \frac{m_0^2}{4\eta}g(t)\|u_m'(t)\|_{\rho+2,\Gamma_0}^{\rho+2} + C_1\|\nabla u_m'(t)\|^2 \\ &\quad + (m_2 C(T,\eta)[\frac{2L_1}{m_0}]^{\gamma+1} + \eta m_2\|u_m''(t)\|_{\Gamma_0}^2)\|g\|_{L^1(0,\infty)}. \end{aligned} \tag{3.14}$$

Integrating (3.14) over $[0,t]$, choosing $\eta > 0$ sufficiently small and employing (3.10) and Gronwall's lemma we obtain the second estimate:

$$\|u_m''(t)\|^2 + \|u_m'(t)\|_{\Gamma_0}^2 + \int_0^t (\|\nabla u_m''(s)\|^2 + \|u_m''(s)\|_{\Gamma_0}^2) ds \le L_2, \tag{3.15}$$

where $L_2 > 0$ is independent of m.

The estimates above are sufficient to pass to the limit in the linear terms of problem (3.1). Next we are going to consider the nonlinear ones.

Analysis of the nonlinear terms.

From the above estimates we have that

$$(u_m) \quad \text{is bounded in} \quad L^2(0,T; H^{\frac{1}{2}}(\Gamma_0)), \tag{3.16}$$

$$(u_m') \quad \text{is bounded in} \quad L^2(0,T; H^{\frac{1}{2}}(\Gamma_0)), \tag{3.17}$$

$$(u_m'') \quad \text{is bounded in} \quad L^2(0,T; L^2(\Gamma_0)). \tag{3.18}$$

From (3.16)-(3.18), taking into consideration that the imbedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ is continuous and compact and using Aubin compactness theorem, we can extract a subsequence (u_μ) of (u_m) such that

$$u_\mu \to u \quad \text{a.e. on} \quad \Sigma_0 \quad \text{and} \quad u_\mu' \to u' \quad \text{a.e. on} \quad \Sigma_0 \tag{3.19}$$

and therefore

$$|u_\mu|^\gamma u_\mu \to |u|^\gamma u \quad \text{and} \quad |u_\mu'|^\rho u_\mu' \to |u'|^\rho u' \quad \text{a.e. on} \quad \Sigma_0. \tag{3.20}$$

On the other hand, from the first and second estimate we obtain

$$(g * |u_\mu|^\gamma u_\mu) \quad \text{is bounded in} \quad L^2(\Sigma_0), \tag{3.21}$$

$$(g|u_\mu'|^\rho u_\mu') \quad \text{is bounded in} \quad L^2(\Sigma_0). \tag{3.22}$$

Combining (3.20)-(3.22), we deduce that

$$g * |u_\mu|^\gamma u_\mu \to g * |u|^\gamma u \quad \text{weakly in} \quad L^2(\Sigma_0),$$
$$g|u_\mu'|^\rho u_\mu' \to g|u'|^\rho u' \quad \text{weakly in} \quad L^2(\Sigma_0).$$

The last convergence is sufficient to pass to the limit in the nonlinear terms of problem (3.1). This completes the proof of the existence of solutions of the problems (1.1)-(1.4). The uniqueness is obtained in a stand way, so we omit the proof here. □

4. Uniform decay of energy

Note that the derivative of energy (1.1) is given by

$$\begin{aligned} E'(t) = & -\|\nabla u'(t)\|^2 - \|u'(t)\|_{\Gamma_0}^2 - g(t)\|u'(t)\|_{\rho+2,\Gamma_0}^{\rho+2} \\ & + \int_0^t g(t-r)|u(r)|^\gamma (u(r), u'(t))_{\Gamma_0} dr. \end{aligned} \tag{4.1}$$

Defining

$$(g\Box u)(t) := \int_0^t g(t-r)\||u(r)|^\gamma u(r) - u(t)|_{\Gamma_0}^2 dr, \tag{4.2}$$

a simple computation gives us

$$\begin{aligned} &\int_0^t g(t-r)(|u(r)|^\gamma u(r), u'(t))_{\Gamma_0} dr \\ &= -\frac{1}{2}(g\Box u)'(t) + \frac{1}{2}(g'\Box u)(t) + \frac{1}{2}\frac{d}{dt}\{\|u(t)\|_{\Gamma_0}^2 \int_0^t g(r)dr\} \\ &- \frac{1}{2}g(t)\|u(t)\|_{\Gamma_0}^2. \end{aligned} \tag{4.3}$$

Now we define the modified energy by

$$\begin{aligned} e(t) = &\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\bar{M}(\|\nabla u(t)\|^2) + \frac{1}{2}(g\Box u)(t) \\ &+ \frac{1}{2}(1 - \int_0^t g(r)dr)\|u(t)\|_{\Gamma_0}^2 + \frac{1}{\gamma+2}g(t)\|u(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}. \end{aligned} \tag{4.4}$$

Then assumption (A_2) implies

$$\begin{aligned} e'(t) &= -\|u'(t)\|_{\Gamma_0}^2 - \|\nabla u'(t)\|^2 - g(t)\|u'(t)\|_{\rho+2,\Gamma_0}^{\rho+2} \\ &\quad + \frac{1}{\gamma+2}g'(t)\|u(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + g(t)(|u(t)|^\gamma u(t), u'(t))_{\Gamma_0} \\ &\quad - \frac{1}{2}g(t)\|u(t)\|_{\Gamma_0}^2 + \frac{1}{2}(g'\Box u)(t) \\ &\le -\|u'(t)\|_{\Gamma_0}^2 - \|\nabla u'(t)\|^2 - g(t)\|u'(t)\|_{\rho+2,\Gamma_0}^{\rho+2} \\ &\quad - \frac{\alpha_1}{\gamma+2}g(t)\|u(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + g(t)(|u(t)|^\gamma u(t), u'(t))_{\Gamma_0} \\ &\quad - \frac{1}{2}g(t)\|u(t)\|_{\Gamma_0}^2 - \frac{\alpha_1}{2}(g\Box u)(t). \end{aligned} \tag{4.5}$$

Thus using Young's inequality, $\gamma = \rho$ and then choosing $\eta = 2^{-(\gamma+1)}$ and $1 - \eta > \frac{1}{2}$, we have

$$\begin{aligned} e'(t) \le &-\|u'(t)\|_{\Gamma_0}^2 - \|\nabla u'(t)\|^2 - \frac{1}{2}g(t)\|u'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} - \beta g(t)\|u(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} \\ &- \frac{1}{2}g(t)\|u(t)\|_{\Gamma_0}^2 - \frac{\alpha_1}{2}(g\Box u)(t), \end{aligned} \tag{4.6}$$

where $\beta = \frac{\alpha_1}{\gamma+2} - \eta^{-\frac{1}{\gamma+1}} > 0$.

On the other hand we note that from assumption (A_2), we obtain

$$E(t) \le l^{-1}e(t) \tag{4.7}$$

and therefore it is enough to obtain the desired exponential decay for the modified energy $e(t)$ which will be done below. For this purpose let λ be the positive number such that

$$\|v\|^2 \leq \lambda \|\nabla v\|^2, \quad \forall v \in V$$

and for every $\epsilon > 0$ let us define the perturbed modified energy by

$$e_\epsilon(t) = e(t) + \epsilon\psi(t), \quad \text{where} \quad \psi(t) = (u(t), u'(t)).$$

Proposition 4.1. *We have*

$$|e_\epsilon(t) - e(t)| \leq \epsilon(\frac{\lambda}{m_0})^{\frac{1}{2}} e(t), \quad \forall t \geq 0.$$

Proof. Applying Cauchy Schwarz's inequality

$$\begin{aligned}
|\psi(t)| &\leq \|u'(t)\|\|u(t)\| \leq (\frac{\lambda}{m_0})^{\frac{1}{2}} \|u'(t)\| m_0^{\frac{1}{2}} \|\nabla u(t)\| \\
&\leq (\frac{\lambda}{m_0})^{\frac{1}{2}} (\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\bar{M}(\|\nabla u(t)\|^2)) \\
&\leq (\frac{\lambda}{m_0})^{\frac{1}{2}} e(t).
\end{aligned}$$

Thus we have $|e_\epsilon(t) - e(t)| = \epsilon|\psi(t)| \leq \epsilon(\frac{\lambda}{m_0})^{\frac{1}{2}} e(t)$. □

Proposition 4.2. *There exist $C_1 > 0$ and ϵ_1 such that for $\epsilon \in (0, \epsilon_1]$*

$$e'_\epsilon(t) \leq -\epsilon C_1 e(t).$$

Proof. Using the problem (1.1) and the fact that $M(s)s \geq \bar{M}(s)$ for $s \geq 0$, we have

$$\begin{aligned}
\psi'(t) &= \|u'(t)\|^2 - M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 - (\nabla u'(t), \nabla u(t)) - \|u(t)\|^2_{\Gamma_0} \\
&\quad - (u'(t), u(t))_{\Gamma_0} - (g(t)|u'(t)|^\rho u'(t), u(t))_{\Gamma_0} \\
&\quad + \int_0^t g(t-r)|u(r)|^\gamma (u(r), u(t))_{\Gamma_0} dr \\
&\leq \|u'(t)\|^2 - \bar{M}(\|\nabla u(t)\|^2) - (\nabla u'(t), \nabla u(t)) - \|u(t)\|^2_{\Gamma_0} \\
&\quad - (u'(t), u(t))_{\Gamma_0} - (g(t)|u'(t)|^\rho u'(t), u(t))_{\Gamma_0} \\
&\quad + \int_0^t g(t-r)|u(r)|^\gamma (u(r), u(t))_{\Gamma_0} dr.
\end{aligned} \tag{4.8}$$

Now since

$$\begin{aligned}
&\int_0^t g(t-r)|u(r)|^\gamma (u(r), u(t))_{\Gamma_0} dr \\
&= \int_0^t g(t-r)(|u(r)|^\gamma u(r) - u(t), u(t))_{\Gamma_0} dr + \int_0^t g(t-r)\|u(t)\|^2_{\Gamma_0} dr \\
&\leq \frac{1}{2}\int_0^t g(t-r)\||u(r)|^\gamma u(r) - u(t)|^2_{\Gamma_0} dr + \frac{3}{2}\|u(t)\|^2_{\Gamma_0} \int_0^t g(r)dr \\
&= (g\Box u)(t) + \frac{3}{2}\|u(t)\|^2_{\Gamma_0} \int_0^t g(r)dr,
\end{aligned} \tag{4.9}$$

we get

$$
\begin{aligned}
\psi'(t) &\le \|u'(t)\|^2 - \bar{M}(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 - (\nabla u'(t), \nabla u(t)) - \|u(t)\|^2_{\Gamma_0} \\
&\quad - (u'(t), u(t))_{\Gamma_0} - (g(t)|u'(t)|^\rho u'(t), u(t))_{\Gamma_0} + (g\Box u)(t) \\
&\quad + \frac{3}{2}\|u(t)\|^2_{\Gamma_0} \int_0^t g(r)dr \\
&= -e(t) - \frac{1}{2}\bar{M}(\|\nabla u(t)\|^2) - (\nabla u'(t), \nabla u(t)) \\
&\quad - \frac{1}{2}\|u(t)\|^2_{\Gamma_0} + \int_0^t g(r)dr\|u(t)\|^2_{\Gamma_0} + \frac{3}{2}(g\Box u)(t) - (u'(t), u(t))_{\Gamma_0} \\
&\quad - (g(t)|u'(t)|^\rho u'(t), u(t))_{\Gamma_0} + \frac{1}{\gamma+2}g(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \frac{3}{2}\|u'(t)\|^2.
\end{aligned} \tag{4.10}
$$

Now, applying Sobolev imbedding, we have

$$
\begin{aligned}
|(u'(t), u(t))_{\Gamma_0}| &\le \|u(t)\|_{\Gamma_0}\|u'(t)\|_{\Gamma_0} \\
&\le \mu\|\nabla u(t)\|\|u'(t)\|_{\Gamma_0} \\
&\le \frac{\eta}{m_0}\bar{M}(\|\nabla u(t)\|^2) + \frac{\mu^2}{4\eta}\|u'(t)\|^2_{\Gamma_0},
\end{aligned} \tag{4.11}
$$

where μ is the positive number such that

$$\|v\|_{\Gamma_0} \le \mu\|\nabla v\|, \quad \forall v \in V.$$

Also Schwarz's inequality and Young inequality imply

$$
\begin{aligned}
|(\nabla u'(t), \nabla u(t))| &\le \|\nabla u(t)\|\|\nabla u'(t)\| \\
&\le \eta\|\nabla u(t)\|^2 + \frac{1}{4\eta}\|\nabla u'(t)\|^2 \\
&\le \frac{\eta}{m_0}\bar{M}(\|\nabla u(t)\|^2) + \frac{1}{4\eta}\|\nabla u'(t)\|^2,
\end{aligned} \tag{4.12}
$$

and

$$
\begin{aligned}
|(g(t)|u'(t)|^\rho u'(t), u(t))_{\Gamma_0}| &\le g(t)\|u'(t)\|^{\rho+1}_{\rho+2,\Gamma_0}\|u(t)\|_{\rho+2,\Gamma_0} \\
&\le \theta(\eta)g(t)\|u'(t)\|^{\rho+2}_{\rho+2,\Gamma_0} + \eta g(t)\|u(t)\|^{\rho+2}_{\rho+2,\Gamma_0}
\end{aligned} \tag{4.13}
$$

and

$$\frac{3}{2}\|u'(t)\|^2 \le \frac{3}{2}\lambda\|\nabla u'(t)\|^2. \tag{4.14}$$

Combining (4.10)-(4.14), we have

(4.15)

$$
\begin{aligned}
\psi'(t) &\le -e(t) - \frac{1}{2}(1 - \frac{4\eta}{m_0})\bar{M}(\|\nabla u(t)\|^2) - \frac{b(2\alpha+1)}{2(\alpha+1)}\|\nabla u(t)\|^{2(\alpha+1)} + 2\eta\|\nabla u(t)\|^2 \\
&\quad + \frac{\mu^2}{4\eta}\|u'(t)\|^2_{\Gamma_0} - \frac{1}{2}\|u(t)\|^2_{\Gamma_0} + \int_0^t g(r)dr\|u(t)\|^2_{\Gamma_0} + \frac{3}{2}(g\Box u)(t) \\
&\quad + (\frac{1}{4\eta} + \frac{3}{2}\lambda)\|\nabla u'(t)\|^2 + \theta(\eta)g(t)\|u'(t)\|^{\rho+2}_{\rho+2,\Gamma_0} + \eta g(t)\|u(t)\|^{\rho+2}_{\rho+2,\Gamma_0} \\
&\quad + \frac{1}{\gamma+2}g(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}.
\end{aligned}
$$

Combining (4.6), (4.14), (4.15) and assumption (A_2) and considering $\rho = \gamma$, we get

$$\begin{aligned}
e'_\epsilon(t) &= e'(t) + \epsilon\psi'(t) \\
&\le -\epsilon e(t) - (1 - \frac{\epsilon\mu^2}{4\eta})\|u'(t)\|^2_{\Gamma_0} - (\frac{1}{2} - \epsilon\theta(\eta))g(t)\|u'(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\
&\quad - (\beta - \epsilon\eta - \frac{\epsilon}{\gamma+2})g(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\
&\quad - (1 - \frac{\epsilon}{4\eta} - \frac{3\epsilon}{2}\lambda)\|\nabla u'(t)\|^2 - \frac{1}{2}(1 - \frac{4\eta}{m_0})\epsilon\bar{M}(\|\nabla u(t)\|^2) \\
&\quad - (\frac{\alpha_1}{2} - \frac{3}{2}\epsilon)(g\Box u)(t) + \epsilon\int_0^t g(r)dr\|u(t)\|^2_{\Gamma_0} - \frac{1}{2}g(t)\|u(t)\|^2_{\Gamma_0} \\
&\quad - \frac{b\epsilon(2\alpha+1)}{2(\alpha+1)}\|\nabla u(t)\|^{2(\alpha+1)} - \frac{1}{2}\epsilon\|u(t)\|^2_{\Gamma_0} \qquad (4.16) \\
&\le -C_2\epsilon e(t) - (1 - \frac{\epsilon\mu^2}{4\eta})\|u'(t)\|^2_{\Gamma_0} - (\frac{1}{2} - \epsilon\theta(\eta))g(t)\|u'(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\
&\quad - (\beta - \epsilon\eta - \frac{2\epsilon}{\gamma+2} + \frac{4\eta\epsilon}{m_0(\gamma+2)})g(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\
&\quad - (1 - \frac{\epsilon}{4\eta} - 2\epsilon\lambda + \frac{2\eta\epsilon\lambda}{m_0})\|\nabla u'(t)\|^2 - (\frac{\alpha_1}{2} - 2\epsilon + \frac{2\eta\epsilon}{m_0})(g\Box u)(t) \\
&\quad + \epsilon(\frac{1}{2} + \frac{2\eta}{m_0})\int_0^t g(r)dr\|u(t)\|^2_{\Gamma_0} - \frac{1}{2}g(t)\|u(t)\|^2_{\Gamma_0} \\
&\quad - \frac{b\epsilon}{m_0(\alpha+1)}(m_0\alpha + 2\eta)\|\nabla u(t)\|^{2(\alpha+1)} - \frac{2\eta\epsilon}{m_0}\|u(t)\|^2_{\Gamma_0},
\end{aligned}$$

where $C_2 = 2 - \frac{4\eta}{m_0}$. Defining $\epsilon_1 = \min\{\frac{4\eta}{\mu^2}, \frac{1}{2\theta(\eta)}, \frac{\beta m_0(\gamma+2)}{m_0\eta(\gamma+2)+2m_0-4\eta}, \frac{4m_0\eta}{m_0+8\eta\lambda(m_0-\eta)}, \frac{m_0\alpha_1}{4(m_0-\eta)}\}$ and sufficiently small $\eta < \frac{m_0}{4}$. Then for each $\epsilon \in (0, \epsilon_1]$, we have

$$e'_\epsilon(t) \le -\epsilon C_1 e(t) \qquad (4.17)$$

if $\|g\|_{L^1(0,\infty)}$ is sufficiently small. $\Box$

Now let $\epsilon_0 = \min\{\frac{1}{2\lambda^{\frac{1}{2}}}, \epsilon_1\}$ and let us consider $\epsilon \in (0, \epsilon_0]$. Then we conclude from Proposition 4.1, $(1 - \epsilon\lambda^{\frac{1}{2}})e(t) < e_\epsilon(t) < (1 + \epsilon\lambda^{\frac{1}{2}})e(t)$ and so

$$\frac{1}{2}e(t) < e_\epsilon(t) < \frac{3}{2}e(t). \qquad (4.18)$$

Thus we have $e'_\epsilon(t) \le -\frac{2}{3}C_1\epsilon e_\epsilon(t)$ for all $t \ge t_0$. Consequently, by virture of (4.18), we get

$$e(t) \le 3e(0)exp(-\frac{2}{3}C_1\epsilon t) \text{ for all } t \ge t_0.$$

This concludes the proof of Theorem 2.1. $\Box$

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