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<td>COXETER GROUPの境界とVIRTUAL COHOMOLOGICAL DIMENSIONについて（集合論的・幾何学的トポロジーとその応用の研究）</td>
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The purpose of this note is to introduce our recent paper [Ho-Y] about Coxeter groups and their boundaries. Let $V$ be a finite set and $m: V \times V \to \mathbb{N} \cup \{\infty\}$ a function satisfying the following conditions:

1. $m(v, w) = m(w, v)$ for all $v, w \in V$,
2. $m(v, v) = 1$ for all $v \in V$, and
3. $m(v, w) \geq 2$ for all $v \neq w \in V$.

A Coxeter group is a group $\Gamma$ having the presentation

$$\langle V \mid (vw)^{m(w)}v = 1 \text{ for } v, w \in V \rangle,$$

where if $m(v, w) = \infty$, then the corresponding relation is omitted, and the pair $(\Gamma, V)$ is called a Coxeter system. If $m(v, w) = 2$ or $\infty$ for all $v \neq w \in V$, then $(\Gamma, V)$ is said to be right-angled. For a Coxeter system $(\Gamma, V)$ and a subset $W \subset V$, $\Gamma_W$ is defined as the subgroup of $\Gamma$ generated by $W$. The pair $(\Gamma_W, W)$ is also a Coxeter system. $\Gamma_W$ is called a parabolic subgroup.

For a Coxeter system $(\Gamma, V)$, the simplicial complex $K(\Gamma, V)$ is defined by the following conditions:

1. the vertex set of $K(\Gamma, V)$ is $V$,
2. for $W = \{v_0, \ldots, v_k\} \subset V$, $\{v_0, \ldots, v_k\}$ spans a $k$-simplex of $K(\Gamma, V)$ if and only if $\Gamma_W$ is finite.

A simplicial complex $K$ is called a flag complex if any finite set of vertices, which are pairwise joined by edges, spans a simplex of $K$. For example, the barycentric subdivision of a simplicial complex is a flag complex.

For any finite flag complex $K$, there exists a right-angled Coxeter system $(\Gamma, V)$ with $K(\Gamma, V) = K$. Namely, let $V$ be the vertex set of $K$ and define $m: V \times V \to \mathbb{N} \cup \{\infty\}$ by

$$m(v, w) = \begin{cases} 
1 & \text{if } v = w, \\
2 & \text{if } \{v, w\} \text{ spans an edge in } K, \\
\infty & \text{otherwise.}
\end{cases}$$

The associated right-angled Coxeter system $(\Gamma, V)$ satisfies $K(\Gamma, V) = K$. Conversely, if $(\Gamma, V)$ is a right-angled Coxeter system, then $K(\Gamma, V)$ is a finite flag complex ([D2, Corollary 9.4]).
For a group $\Gamma$ and a ring $R$ with identity, the cohomological dimension of $\Gamma$ over $R$ is defined as

$$\text{cd}_R \Gamma = \sup \{ i \mid H^i(\Gamma; M) \neq 0 \text{ for some } R\Gamma\text{-module } M \}.$$

If $R = \mathbb{Z}$ then $\text{cd}_R \Gamma$ is simply called the cohomological dimension of $\Gamma$, and denoted $\text{cd} \Gamma$. It is obvious that $\text{cd}_R \Gamma \leq \text{cd} \Gamma$ for a ring $R$ with identity. It is known that $\text{cd} \Gamma = \infty$ if $\Gamma$ is not torsion-free ([Br, Corollary VIII.2.5]). A group $\Gamma$ is said to be virtually torsion-free if $\Gamma$ has a torsion-free subgroup of finite index. For a virtually torsion-free group $\Gamma$ the virtual cohomological dimension of $\Gamma$ over a ring $R$ is defined as $\text{cd}_R \Gamma'$, where $\Gamma'$ is a torsion-free subgroup of $\Gamma$ of finite index, and denoted $\text{vcd}_R \Gamma$. It is a well-defined invariant by Serre's Theorem: if $G$ is a torsion-free group and $G'$ is a subgroup of finite index, then $\text{cd}_R G' = \text{cd}_R G$ ([Br, Theorem VIII.3.1]). If $R = \mathbb{Z}$ then $\text{vcd}_R \Gamma$ is simply called the virtual cohomological dimension of $\Gamma$, and denoted $\text{vcd} \Gamma$. It is known that every Coxeter group is virtually torsion-free and the virtual cohomological dimension of each Coxeter group is finite (cf. [D1, Corollary 5.2, Proposition 14.1]).

For a simplicial complex $K$ and a simplex $\sigma$ of $K$, the closed star $\text{St}(\sigma, K)$ of $\sigma$ in $K$ is the union of all simplices of $K$ having $\sigma$ as a face, and the link $\text{Lk}(\sigma, K)$ of $\sigma$ in $K$ is the union of all simplices of $K$ lying in $\text{St}(\sigma, K)$ that are disjoint from $\sigma$.

In [Dr2], Dranishnikov gave the following formula.

**Theorem 1 (Dranishnikov [Dr2]).** Let $(\Gamma, V)$ be a Coxeter system and $R$ a principal ideal domain. Then there exists the formula

$$\text{vcd}_R \Gamma = \text{lcd}_R CK = \max \{ \text{lcd}_R K, \text{cd}_R K + 1 \},$$

where $K = K(\Gamma, V)$ and $CK$ is the simplicial cone of $K$.

Here, for a finite simplicial complex $K$ and an abelian group $G$, the local cohomological dimension of $K$ over $G$ is defined as

$$\text{lcd}_G K = \max_{\sigma \in K} \{ i \mid H^i(\text{St}(\sigma, K), \text{Lk}(\sigma, K); G) \neq 0 \},$$

and the global cohomological dimension of $K$ over $G$ is

$$\text{cd}_G K = \max \{ i \mid \tilde{H}^i(K; G) \neq 0 \}.$$

When $\tilde{H}^i(K; G) = 0$ for each $i$, then we consider $\text{cd}_G K = -1$. We note that $H^i(\text{St}(\sigma, K), \text{Lk}(\sigma, K); G)$ is isomorphic to $\tilde{H}^{i-1}(\text{Lk}(\sigma, K); G)$. Hence, we have

$$\text{lcd}_G K = \max_{\sigma \in K} \{ \text{cd}_G \text{Lk}(\sigma, K) + 1 \}.$$

**Remark.** We recall Dranishnikov's remark in [Dr3]. The definition of the local cohomological dimension in [Dr2] is given by the terminology of the normal star and link. Since $\text{Lk}(\sigma, K)$ is homeomorphic to the normal link of $\sigma$ in $K$, their definitions are equivalent by the formula above.

Dranishnikov also proved the following theorem as an application of Theorem 1.
Theorem 2 (Dranishnikov [Dr2]). A Coxeter group $\Gamma$ has the following properties:

(a) $\text{vcd}_Q \Gamma \leq \text{vcd}_R \Gamma$ for any principal ideal domain $R$.
(b) $\text{vcd}_{Z_p} \Gamma = \text{vcd}_Q \Gamma$ for almost all primes $p$.
(c) There exists a prime $p$ such that $\text{vcd}_{Z_p} \Gamma = \text{vcd} \Gamma$.
(d) $\text{vcd} \Gamma \times \Gamma = 2 \text{vcd} \Gamma$.

We extend this theorem to one over principal ideal domain coefficients.

Theorem A. Let $\Gamma$ be a Coxeter group and $R$ a principal ideal domain. Then $\Gamma$ has the following properties:

(a) $\text{vcd}_Q \Gamma \leq \text{vcd}_{R/I} \Gamma \leq \text{vcd}_R \Gamma \leq \text{vcd} \Gamma$ for any prime ideal $I$ in $R$.
(b) $\text{vcd}_{R/I} \Gamma = \text{vcd}_Q \Gamma$ for almost all prime ideals $I$ in $R$, if $R$ is not a field.
(c) There exists a non-trivial prime ideal $I$ in $R$ such that $\text{vcd}_{R/I} \Gamma = \text{vcd}_R \Gamma$, if $R$ is not a field.
(d) $\text{vcd}_R \Gamma \times \Gamma = 2 \text{vcd}_R \Gamma$.

Let $(\Gamma, V)$ be a Coxeter system and $K = K(\Gamma, V)$. Consider the product space $\Gamma \times |CK|$ of $\Gamma$ with the discrete topology and the underlying space $|CK|$ of the cone of $K$. Define an equivalence relation $\sim$ on the space as follows: for $(\gamma_1, x_1), (\gamma_2, x_2) \in \Gamma \times |CK|$, $(\gamma_1, x_1) \sim (\gamma_2, x_2)$ if and only if $x_1 = x_2$ and $\gamma_1^{-1}\gamma_2 \in \Gamma_{V(x_1)}$, where $V(x) = \{v \in V| x \in \text{St}(v, \beta^1 K)\}$. Here we consider that $|K|$ is naturally embedded in $|CK|$ as the base of the cone and $\beta^1 K$ denotes the barycentric subdivision of $K$. The natural left $\Gamma$-action on $\Gamma \times |CK|$ is compatible with the equivalence relation; hence, it passes to a left action on the quotient space $\Gamma \times |CK|/\sim$. Denote this quotient space by $A(\Gamma, V)$. The space $A(\Gamma, V)$ is contractible and $\Gamma$ acts cocompactly and properly discontinuously on the space ([D1, Theorem 13.5]).

We can also give the space $A(\Gamma, V)$ a structure of a piecewise Euclidean cell complex with the vertex set $\Gamma \times \{v_0\}$ ([D2, §9]). $\Sigma(\Gamma, V)$ denotes this piecewise Euclidean cell complex. Refer to [D2, Definition 2.2] for the definition of a piecewise Euclidean cell complex. In particular, if $(\Gamma, V)$ is right-angled, then each cell of $\Sigma(\Gamma, V)$ is a cube, hence, $\Sigma(\Gamma, V)$ is a cubical complex. More precisely, for a right-angled Coxeter system $(\Gamma, V)$, we can define the cubical complex $\Sigma(\Gamma, V)$ by the following conditions:

1. the vertex set of $\Sigma(\Gamma, V)$ is $\Gamma$,
2. for $\gamma, \gamma' \in \Gamma$, $\{\gamma, \gamma'\}$ spans an edge in $\Sigma(\Gamma, V)$ if and only if the length $l_V(\gamma^{-1}\gamma') = 1$, and
3. for $\gamma \in \Gamma$ and $v_0, \ldots, v_k \in V$, the edges $|\gamma, \gamma v_0|, \ldots, |\gamma, \gamma v_k|$ form a $(k+1)$-cube in $\Sigma(\Gamma, V)$ if and only if $\{v_0, \ldots, v_k\}$ spans a $k$-simplex in $K(\Gamma, V)$.

We note that $l_V(\gamma^{-1}\gamma') = 1$ if and only if $\gamma$ and $\gamma'$ are in the same $S$-orbit. Hence, $\Sigma(\Gamma, V)$ is a CAT(0) geodesic space by a piecewise Euclidean metric (cf. [D2, Theorem 7.8]). We define the boundary $\partial \Gamma$ as the set of
geodesic rays in $\Sigma(\Gamma, V)$ emanating from the unit element $e \in \Gamma \subset \Sigma(\Gamma, V)$ with the topology of the uniform convergence on compact sets, i.e., $\partial \Gamma$ is the visual sphere of $\Sigma(\Gamma, V)$ at the point $e \in \Sigma(\Gamma, V)$. In general, for all points $x, y$ in a CAT(0) space $X$, the visual spheres of $X$ at points $x$ and $y$ are homeomorphic (cf. [Dr1, Assertion 1]). This boundary is known to be a finite-dimensional compactum (i.e., metrizable compact space). Details of the boundaries of CAT(0) spaces can be found in [D2] and [D-J].

It is still unknown whether the following conjecture holds.

**Rigidity Conjecture (Dranishnikov [Dr4]).** Isomorphic Coxeter groups have homeomorphic boundaries.

We note that there exists a Coxeter group $\Gamma$ with different Coxeter systems $(\Gamma, V_1)$ and $(\Gamma, V_2)$.

Let $X$ be a compact metric space and $G$ an abelian group. The cohomological dimension of $X$ over $G$ is defined as

$$c \text{-dim}_G X = \sup \{ i \mid \check{H}^i (X, A; G) \neq 0 \text{ for some closed set } A \subset X \},$$

where $\check{H}^i (X, A; G)$ is the Čech cohomology of $(X, A)$ over $G$.

In [B-M], Bestvina-Mess proved the following theorem for hyperbolic groups. An analogous theorem for Coxeter groups is proved by the same argument (cf. [Dr1]).

**Theorem 3 (Bestvina-Mess [B-M]).** Let $\Gamma$ be a Coxeter group and $R$ a ring with identity. Then there exists the formula

$$c \text{-dim}_R \partial \Gamma = \operatorname{vcd}_R \Gamma - 1.$$  

We have a dimension theorectic theorem in the study of Coxeter groups.

**Theorem B.** Let $(\Gamma, V)$ be a right-angled Coxeter system with $\operatorname{vcd}_R \Gamma = n$, where $R$ is a principal ideal domain. Then there exists a sequence $W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset V$ such that $\operatorname{vcd}_R \Gamma_{W_i} = i$ for $i = 0, \ldots , n - 1$. In particular, we can obtain a sequence of simplexes $\tau_0 \succ \tau_1 \succ \cdots \succ \tau_{n-1}$ such that $W_i$ is the vertex set of $\operatorname{Lk}(\tau_i, K(\Gamma, V))$ and $K(\Gamma_{W_i}, W_i) = \operatorname{Lk}(\tau_i, K(\Gamma, V))$.

We note that Theorem B is not always true for general Coxeter groups.

**Example.** We consider the Coxeter system $(\Gamma, V)$ defined by $V = \{v_1, v_2, v_3\}$ and

$$m(v_i, v_j) = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i \neq j. \end{cases}$$

Then $\Gamma$ is not right-angled, and $K(\Gamma, V)$ is not a flag complex. Indeed, $\Gamma_{\{v_i, v_j\}}$ is finite for each $i, j \in \{1, 2, 3\}$, but $\Gamma$ is infinite (cf. [Bo, p.98, Proposition 8]). Since $\cd K(\Gamma, V) = 1$ and $\lcd K(\Gamma, V) = 1$, we have $\operatorname{vcd} \Gamma = 2$ by Theorem 1. For any proper subset $W \subset V$, $\operatorname{vcd} \Gamma_W = 0$, because $\Gamma_W$ is a finite group. Hence there does not exist a subset $W \subset V$ such that $\operatorname{vcd} \Gamma_W = 1$. □

By Theorem 3, we can obtain the following corollary.
Corollary B'. For a right-angled Coxeter system $(\Gamma, V)$ with $c\dim_R \partial \Gamma = n$, where $R$ is a principal ideal domain, there exists a sequence $\partial \Gamma_{W_0} \subset \partial \Gamma_{W_1} \subset \cdots \subset \partial \Gamma_{W_{n-1}}$ of the boundaries of parabolic subgroups of $(\Gamma, V)$ such that $c\dim_R \partial \Gamma_{W_i} = i$ for each $i = 0, 1, \ldots, n - 1$.

In general, for a finite dimensional compactum $X$, the equality $c\dim_Z X = \dim X$ holds ([K, §2, Remark 4]). Since the boundaries of Coxeter groups are always finite dimensional, we obtain the following corollary.

Corollary B". For a right-angled Coxeter system $(\Gamma, V)$ with $\dim \partial \Gamma = n$, there exists a sequence $\partial \Gamma_{W_0} \subset \partial \Gamma_{W_1} \subset \cdots \subset \partial \Gamma_{W_{n-1}}$ of the boundaries of parabolic subgroups of $(\Gamma, V)$ such that $\dim \partial \Gamma_{W_i} = i$ for each $i = 0, 1, \ldots, n - 1$.

Finally, we see a relation between a subgroup of a Coxeter group which is of finite index and their boundaries.

If $X$ and $Y$ are topological spaces, let us define $X \star Y$ to be the quotient space of $X \times Y \times [0, 1]$ obtained by identifying each set $x \times Y \times 0$ to a point and each set $X \times y \times 1$ to a point.

Theorem C. Let $(\Gamma, V)$ be a right-angled Coxeter system and $W$ a subset of $V$. Then the following conditions are equivalent:

1. The parabolic subgroup $\Gamma_W \subset \Gamma$ is of finite index.
2. $\{v, v'\}$ spans an edge of $K(\Gamma, V)$ for any $v \in V \setminus W$ and $v' \in V$.
3. $\Gamma = \Gamma_W \times \Gamma_{V \setminus W}$ and $\Gamma_{V \setminus W} \approx \mathbb{Z}_2^{V \setminus W}$.
4. $\partial \Gamma = \partial \Gamma_W$.

REFERENCES


**INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA, 305-8571, JAPAN**

*E-mail address: thosaka@math.tsukuba.ac.jp*

**DEPARTMENT OF MATHEMATICS, INTERDISCIPLINARY FACULTY OF SCIENCE AND ENGINEERING, SHIMANE UNIVERSITY, MATSUE, 690-8504, JAPAN**

*E-mail address: yokoi@math.shimane-u.ac.jp*