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<thead>
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<th>Title</th>
<th>An Elementary Construction of a Cantor Set with Arbitrary Hausdorff Dimension (Research of Set-Theoretic and Geometric Topology and Their Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Fukaishi, Hiroo; Yamaji, Hironobu</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1188: 86-95</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64702">http://hdl.handle.net/2433/64702</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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An Elementary Construction of a Cantor Set with Arbitrary Hausdorff Dimension

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Abstract

Recently H. Kurata gave an evaluation formula of the Hausdorff dimension of the boundary of a tree with a distance function as follows.

**Theorem 1** (Kurata).

\[
\sup_{x \in X} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right) \leq \dim_H(\Omega, \ell) \leq \sup_{\xi \in \Omega} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right).
\]

In the paper we shall investigate the usefulness of Kurata’s formula and obtain the following results.

**Theorem 2.** There exists a Cantor set for which both sides of Kurata's formula do not coincide.

**Theorem 3.** For each \( \gamma, 0 \leq \gamma \leq \infty \), there exists a Cantor set \( E \) with Hausdorff dimension \( \gamma \).

§1 Introduction

Recently H. Kurata gave an evaluation formula of the Hausdorff dimension of the boundary of a tree and calculated the Hausdorff dimension of certain sets of \( \mathbb{R}^n \) by using it.
**Theorem 1** (Kurata's formula [7]). Let $\Omega$ be the boundary of a tree $(X, A, o)$ with a distance function $\ell$. Then

$$\sup_{x \in X} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right) \leq \dim_H(\Omega, \ell) \leq \sup_{\xi \in \Omega} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right).$$

In the present paper we shall show the following:

**Theorem 2.** There exists a Cantor set for which the both sides of Kurata's formula do not coincide.

**Theorem 3.** For each $\gamma$, $0 \leq \gamma \leq \infty$, there exists a Cantor set $E$ with Hausdorff dimension $\gamma$.

Our Cantor sets satisfying the condition in each of Theorems 2 and 3 are not self-similar, in general. So we cannot apply the formula $c_1^D + c_2^D = 1$ of the Hausdorff dimension $D$, where each $c_i$ denotes the ratio of similarity. We use the Kurata's formula to calculate the Hausdorff dimension of our Cantor sets.

Theorem 3 is known, for example [6], but our construction of required Cantor sets is elementary and geometrical. The ratios of contraction vary in each inductive step in the construction.

Let us recall a tree and the Hausdorff dimension of its boundary with a distance function.

**Definition** (Kurata [7]). Let $(X, A, o)$ be a tree, i.e. simply connected and locally finite graph. The set $X$ is an infinite set of points and the collection $A$ is a set of arcs. The point $o \in X$ is called the root point. For $x, y \in X$ with $x \neq y$ let $\rho(x, y)$ be the least number of arcs which join $x$ and $y$, and $\rho(x, x) = 0$. Then $\rho$ is a metric on $X$. We assume that $\#\{y \in X : \rho(x, y) = 1\} \geq 2$ for each $x \in X$. We set $X_n = \{x \in X : \rho(o, x) = n\}$ for $n = 0, 1, 2, \ldots$.

Let $\Omega$ be the set of all paths from $o$. A path is a sequence of points $(x_0, x_1, x_2, \cdots)$ such that $x_0 = o$, and $\rho(x_n, x_{n+1}) = 1$ for any $x_n \in X_n$, $n = 0, 1, 2, \cdots$. For $\xi = (x_n)_n$, $\eta = (y_n)_n \in \Omega$ we define

$$[\xi] = \{x_0, x_1, x_2, \cdots\} \quad \text{where} \quad x_0 = o,$$
and

\[ P(\xi, \eta) = x_n \quad \text{if} \quad x_0 = y_0, \ x_1 = y_1, \ldots, x_n = y_n, \ x_{n+1} \neq y_{n+1}. \]

Now \( P(\xi, \xi) \) is not defined. The space \( \Omega \) is called the \textit{boundary} of a tree \((X, A, o)\).

Let \( \ell \) be a positive function from \( X \) to \( \mathbb{R}^1 \) with the following properties:

For any path \( \xi = (x_n)_n \),

\begin{enumerate}
  \item[(L1)] \( \ell(x_n) \) is strictly decreasing in \( n \),
  \item[(L2)] \( \lim_{n \to \infty} \ell(x_n) = 0 \).
\end{enumerate}

For \( \xi = (x_n)_n, \eta = (y_n)_n \in \Omega \) define

\[ d(\xi, \eta) = \begin{cases} 
\ell(P(\xi, \eta)) & \text{if} \ \xi \neq \eta, \\
0 & \text{if} \ \xi = \eta.
\end{cases} \]

Then \( d \) is a metric on \( \Omega \), and \( \Omega \) is a compact space. For \( x \in X \) let \( B(x) = \{ \xi \in \Omega : x \in [\xi] \} \). If we take \( \eta \in \Omega \) with \( x \in [\eta] \), we have that \( B(x) = \{ \xi \in \Omega : d(\xi, \eta) \leq \ell(x) \} \). The set \( B(x) \) is both open and closed in \( \Omega \).

For \( K \subset \Omega \) and \( \alpha > 0 \) we define

\[ \Lambda_\alpha^r(K, \ell) = \inf \left\{ \sum_j (\ell(z_j))^\alpha : K \subset \bigcup_j B(z_j), \ \ell(z_j) < r \right\} \quad \text{for} \ r > 0, \]

and

\[ \Lambda_\alpha(K, \ell) = \lim_{r \to +0} \Lambda_\alpha^r(K, \ell) = \sup_{r > 0} \Lambda_\alpha^r(K, \ell). \]

We have that \( 0 \leq \Lambda_\alpha(K, \ell) \leq \infty \). The value \( \Lambda_\alpha(K, \ell) \) is called the \( \alpha \)-dimensional \textit{Hausdorff measure} of \((K, \ell)\). Define the \textit{Hausdorff dimension} of \( K \) with a distance function \( \ell \) as

\[ \dim_H(K, \ell) = \inf \{ \alpha : \Lambda_\alpha(K, \ell) = 0 \} = \sup \{ \alpha : \Lambda_\alpha(K, \ell) = \infty \}. \]

Note that \( 0 \leq \dim_H(K, \ell) \leq \infty \).

Now we define a function \( \varphi(x) \) as follows. Let \( \varphi(o) = 1 \). For \( x \in X_n, \ n > 1 \), we take \( y \in X_{n-1} \) such that \( \rho(x, y) = 1 \) and let

\[ \varphi(x) = \frac{\varphi(y)}{\# \{ z \in X_n : \rho(y, z) = 1 \}}. \]
§2 A construction of a Cantor set with variable ratios of contraction in each inductive step

In this section we construct a Cantor set $E$ with variable ratios of contraction in each inductive step.

For any number $n \geq 1$, let $\{c^{(n)}_j\}_{j=0,1,2,\cdots,2^n-1}$ be a sequence of real numbers with the properties:

(C1) $0 < c^{(n)}_j < 1$ for each $n \geq 1$,
(C2) $\lim_{n \to \infty} a^{(1)} a^{(2)} \cdots a^{(n)} = 0$ where $a^{(n)} = \max \{c^{(n)}_j : j = 0, 1, 2, \cdots, 2^n - 1\}$ for $n \geq 1$.

Let $E_0$ be a bounded closed interval in $\mathbb{R}^1$. Denote the diameter of a set $E \subset \mathbb{R}^1$ by $|E|$. Note that a natural number $j$ can be written by $i_1 i_2 \cdots i_n$ as a number of $n$ figures in a binary notation. For example,

Case $n = 2$: $0=00$, $1=01$, $2=10$, $3=11$, in a binary notation;
Case $n = 3$: $0=000$, $1=001$, $2=010$, $3=011$, in a binary notation.

Put $c_{i_1 i_2 \cdots i_n} = c^{(n)}_j$ if $j = i_1 i_2 \cdots i_n$ in a binary notation. Define a family $\{M_{i_1 i_2 \cdots i_n}\}_{i_1 i_2 \cdots i_n}$ of subintervals of $E_0$ indexed by a finite sequence of figures $0, 1$ as follows by induction:

(i) For $n = 1$, let $M_0$ and $M_1$ be two closed subintervals of $E_0$ such that

$$E_0 \setminus \text{(a middle open interval)} = M_0 \cup M_1,$$
where $\min M_0 = \min E_0$, $\max M_1 = \max E_0$ and $|M_{i_1}| = |E_0| c_{i_1}$
for $i_1 = 0, 1$.

(ii) If $M_{i_1 i_2 \cdots i_n}$ is defined, let $M_{i_1 i_2 \cdots i_n 0}$ and $M_{i_1 i_2 \cdots i_n 1}$ be two closed subintervals
of $M_{i_1 i_2 \cdots i_n}$ such that

$$M_{i_1 i_2 \cdots i_n} \setminus \text{(a middle open subinterval)} = M_{i_1 i_2 \cdots i_n 0} \cup M_{i_1 i_2 \cdots i_n 1},$$
where $\min M_{i_1 i_2 \cdots i_n 0} = \min M_{i_1 i_2 \cdots i_n}$, $\max M_{i_1 i_2 \cdots i_n 1} = \max M_{i_1 i_2 \cdots i_n}$ and

$$|M_{i_1 i_2 \cdots i_n j}| = |M_{i_1 i_2 \cdots i_n}| c_{i_1 i_2 \cdots i_n i_{n+1}}$$
for $j = i_1 i_2 \cdots i_{n+1}$ in a binary notation.

Then the family $\{M_{i_1 i_2 \cdots i_n}\}_{i_1 i_2 \cdots i_n}$ satisfies the following:

(M1) For any infinite sequence $i_1 i_2 \cdots i_n \cdots$ in $\{0, 1\}$,
\[ M_{i_1} \supset M_{i_1i_2} \supset \cdots \supset M_{i_1i_2\cdots i_n} \supset M_{i_1i_2\cdots i_ni_{n+1}} \supset \cdots. \]

(M2) If \( i_1i_2\cdots i_n \neq k_1k_2\cdots k_n \), then \( M_{i_1i_2\cdots i_n} \cap M_{k_1k_2\cdots k_n} = \emptyset \).

(M3) \( |M_{i_1i_2\cdots i_n}| = |E_0|c_{i_1}c_{i_1i_2}\cdots c_{i_1i_2\cdots i_n} \).

(M4) For any infinite sequence \( i_1i_2\cdots i_n \) in \( \{0,1\} \),

\[ \lim_{n \to \infty} |M_{i_1i_2\cdots i_n}| = 0. \]

Hence, \( \bigcap_{n=1}^{\infty} M_{i_1i_2\cdots i_n} = \) one point.

Let

\[ E_n = \bigcup_{n=1}^{\infty} \{ M_{i_1i_2\cdots i_n} : i_1i_2\cdots i_n \text{ is a sequence in } \{0, 1\} \text{ with length } n \} \quad \text{for } n \geq 1. \]

Then the set \( E = \bigcap_{n=1}^{\infty} E_n \) is a Cantor set in \( \mathbb{R}^1 \).

**Remark.** The 1/3-Cantor set is a set \( E \) with

\[ e_j^{(n)} = \frac{1}{3} \quad \text{for } n \geq 1 \text{ and } j = 0, 1, \ldots, 2^n - 1. \]

Next we define a tree \((X, A, o)\) corresponding to the Cantor set \( E \) as follows:

(T1) \( X = X_0 \cup \bigcup_{n=1}^{\infty} X_n \), where \( X_0 = \{o\}, X_1 = \{0,1\}, \cdots \), and

\[ X_n = \{ i_1i_2\cdots i_n : \text{a sequence in } \{0,1\} \text{ with length } n \} \quad \text{for } n \geq 1. \]

(T2) \( A = \{ [0,o], [0,1] \} \cup \bigcup_{n=1}^{\infty} \{ [x_n, y_{n+1}] : x_n \in X_n, y_{n+1} \in X_{n+1}, x_n = i_1i_2\cdots i_n, y_{n+1} = i_1i_2\cdots i_ni_{n+1} \}, \)

where \([x,y]\) means the arc joining \( x \) and \( y \) in \( X \).

Then \( \varphi(x_n) = \frac{1}{2^n} \) for \( x_n \in X_n \).

Define \( \ell(x_n) = c_{i_1}c_{i_1i_2}\cdots c_{i_1i_2\cdots i_n} \) for \( x_n = i_1i_2\cdots i_n \).

Then the function \( \ell \) satisfies the requirements in the definition of the boundary of a tree.
We have a bijection $g : \Omega \rightarrow E$ defined by

$$g(\xi) = s \quad \text{where} \quad \{s\} = \bigcap_{n=1}^{\infty} M_{i_1i_2 \cdots i_n}$$

for $\xi = (o, y_1, y_2, \cdots, y_n, \cdots)$ with $y_n = i_1i_2 \cdots i_n$, $n \geq 1$.

Then $\dim_{H}(\Omega, \ell) = \dim_{H}E$.

§3 Proofs

Example 1 in the following shows Theorem 2.

Example 1. For each $n$, define

$$c_{j}^{(n)} = \begin{cases} \frac{1}{3} : & j = 0, 2, \cdots, 2^n - 2, \\ \frac{1}{9} : & j = 1, 3, \cdots, 2^n - 1, \end{cases}$$

and

$$\ell(y_n) = c_{i_1}c_{i_2} \cdots c_{i_1i_2 \cdots i_n} \quad \text{for} \quad y_n = i_1i_2 \cdots i_n$$

$$= c_{j_1}^{(1)}c_{j_2}^{(2)} \cdots c_{j_n}^{(n)},$$

where $j_r = 2^{r-1}i_1 + 2^{r-2}i_2 + \cdots + 2i_{r-1} + i_r$, $r = 1, 2, \cdots, n$.

Then, the resulting Cantor set $E$ gives an example of Theorem 2 (see Fig. 1).

(1) The right side of Kurata's formula $= \frac{\log 2}{\log 3}$.

In fact, take a path $\xi = (o, y_1, y_2, \cdots, y_n, \cdots) \in \Omega$ with $y_n = 00 \cdots 0$ for any $n$.

We have that

$$\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{\log 3} \quad \text{for any} \quad n.$$

Hence,

$$\liminf_{n \rightarrow \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{\log 3}.$$

(2) The left side of Kurata's formula $= \frac{\log 2}{2 \log 3}$. 
In fact, take any $x \in X$ with $x = i_1 i_2 \cdots i_n$. Let $y_n$ be any point in $X$ such that $B(y_n) \subset B(x)$. For any $n > m$, set $y_n = i_1 i_2 \cdots i_m i_{m+1} \cdots i_n$ and $i_{m+1} = \cdots = i_n = 1$. Then, for any $n > m$,

$$\ell(y_n) = c_{j_1}^{(1)} c_{j_2}^{(2)} \cdots c_{j_m}^{(m)} \left(\frac{1}{3}\right)^{n-m}$$

and

$$\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{2(1 - \frac{m}{n}) \log 3 - \frac{1}{n} \log c_{j_1}^{(1)} c_{j_2}^{(2)} \cdots c_{j_n}^{(n)}}.$$ 

Hence,

$$\liminf_{B(y_n) \subset B(x)} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{2 \log 3}.$$

Fig. 1

Theorem 3 is established by Examples 2 - 6 in the following.

**Example 2. Case:** $\gamma = 0$. For each $n$, define

$$c_j^{(n)} = \left(\frac{1}{3}\right)^n$$

for $j = 0, 1, \cdots, 2^n - 1$.

Then, the resulting Cantor set $E$ has Hausdorff dimension 0.

In fact, take any $y_n \in \Omega$ with $y_n = i_1 i_2 \cdots i_n$. Then

$$\ell(y_n) = \left(\frac{1}{3}\right)^1 \left(\frac{1}{3}\right)^2 \cdots \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^{\frac{1}{2}n(n+1)}.$$
The function $\ell$ satisfies the conditions (L1) - (L2).

Since
$$\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{2 \log 2}{(n + 1) \log 3} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$
we have that $\dim_H E = 0$ from Theorem 1. $\square$

Example 3. Case: $\gamma = 1$. For each $n$, define
$$c_j^{(n)} = \frac{1}{4} \frac{2^n + 1}{2^{n-1} + 1} \quad \text{for} \quad j = 0, 1, \ldots, 2^n - 1.$$ 
Then, the resulting Cantor set $E$ has Hausdorff dimension 1.

In fact, take any $y_n \in \Omega$ with $y_n = i_1 i_2 \cdots i_n$. Then
$$\ell(y_n) = \left(\frac{1}{4}\right)^n \frac{2^n + 1}{2} = \frac{2^n + 1}{2^{2n+1}}.$$ 
The $\ell$ satisfies the conditions (L1) - (L2).

Since
$$\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{1}{(2 + \frac{1}{n}) - \frac{\log(2^n+1)}{\log 2^n}} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty,$$
we have that $\dim_H E = 1$ from Theorem 1. $\square$

Example 4. Case: $0 < \gamma < 1$. For each $n$, define
$$c_j^{(n)} = \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} \quad \text{for} \quad j = 0, 1, \ldots, 2^n - 1.$$ 
Then, the resulting Cantor set $E$ has Hausdorff dimension $\gamma$.

In fact, the both sides of Kurata's formula are equal to $\gamma$. $\square$

Example 5. Case: $1 < \gamma < \infty$. For some integer $N \geq 2$ with $\gamma \leq N$, we can obtain
a Cantor set $E$ in $\mathbb{R}^N$ with $\dim_H E = \gamma$ by appropriate modifications to that of § 2.
We explain how to construct such a Cantor set $E$ in $\mathbb{R}^2$ for $N = \gamma = 2$.

Let $E_0$ be a closed regular square in $\mathbb{R}^2$. For each $n$, define
$$c_j^{(n)} = \frac{1}{4} \frac{2^n + 1}{2^{n-1} + 1} \quad \text{for} \quad j = 0, 1, \ldots, 2^n - 1.$$
and \[ c_{i_1i_2\cdots i_n} = c_j^{(n)} \text{ for } j = i_1i_2\cdots i_n \text{ in a 4-ary notation.} \]

Define a family \( \{M_{i_1i_2\cdots i_n}\}_{i_1i_2\cdots i_n} \) of closed subsquares of \( E_0 \) indexed by a finite sequence of figures 0, 1, 2, 3 with the properties (M1) - (M4). Analogously in §2 we have a Cantor set \( E \subset \mathbb{R}^2 \) with \( \dim_H E = 2 \) (Fig. 2). \( \square \)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example6.png}
\caption{Fig. 2}
\end{figure}

**Example 6.** Case : \( \gamma = \infty \). We construct a Cantor set \( E \) in the Hilbert cube \( Q \) with \( \dim_H E = \infty \). The Hilbert cube means a space \[ Q = \{(t_i) : 0 \leq \frac{1}{t_i} \leq \frac{1}{i} \text{ for } i = 1, 2, 3, \cdots \} \]

with the metric \[ d(s, t) = \sqrt{\sum_{n=1}^{\infty} (s_i - t_i)^2} \text{ for } s = (s_i), \ t = (t_i). \]

Define a set \( E \subset Q \) as follows : \[ E = \bigcup_{n=1}^{\infty} A_n \cup \{a_0\}, \]

where \( a_0 = (0, 0, 0, \cdots) \), and for any \( n \), \( A_n \) is a Cantor set such that

(A1) \[ A_n \subset \left[ \frac{1}{n+1}, \frac{1}{n} \right]^n \times \{0\} \times \{0\} \times \cdots, \]
(A2) \( \dim_H A_n = n \),
(A3) \( A_m \cap A_n = \emptyset \) if \( m \neq n \).

Since \( E \) is a totally disconnected compact metric space with no isolated points, it is a Cantor set. We have that

\[
\dim_H E = \sup_n \dim_H A_n = \infty. \quad \square
\]

References