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<th>An Elementary Construction of a Cantor Set with Arbitrary Hausdorff Dimension (Research of Set-Theoretic and Geometric Topology and Their Applications)</th>
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<tr>
<td>Author(s)</td>
<td>Fukaishi, Hiroo; Yamaji, Hironobu</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1188: 86-95</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64702">http://hdl.handle.net/2433/64702</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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An Elementary Construction of a Cantor Set with Arbitrary Hausdorff Dimension

Hiroo FUKAISHI
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Abstract

Recently H. Kurata gave an evaluation formula of the Hausdorff dimension of the boundary of a tree with a distance function as follows.

**Theorem 1** (Kurata).

$$
\sup_{x \in X} \left( \lim \inf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right) \leq \dim_H(\Omega, \ell) \leq \sup_{\xi \in \Omega} \left( \lim \inf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right).
$$

In the paper we shall investigate the usefulness of Kurata's formula and obtain the following results.

**Theorem 2.** There exists a Cantor set for which both sides of Kurata's formula do not coincide.

**Theorem 3.** For each $\gamma$, $0 \leq \gamma \leq \infty$, there exists a Cantor set $E$ with Hausdorff dimension $\gamma$.

§1 Introduction

Recently H. Kurata gave an evaluation formula of the Hausdorff dimension of the boundary of a tree and calculated the Hausdorff dimension of certain sets of $\mathbb{R}^n$ by using it.
**Theorem 1** (Kurata's formula [7]). Let $\Omega$ be the boundary of a tree $(X, A, o)$ with a distance function $\ell$. Then

$$\sup_{x \in X} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right) \leq \dim_H(\Omega, \ell) \leq \sup_{\xi \in \Omega} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right).$$

In the present paper we shall show the following:

**Theorem 2.** There exists a Cantor set for which the both sides of Kurata's formula do not coincide.

**Theorem 3.** For each $\gamma$, $0 \leq \gamma \leq \infty$, there exists a Cantor set $E$ with Hausdorff dimension $\gamma$.

Our Cantor sets satisfying the condition in each of Theorems 2 and 3 are not self-similar, in general. So we cannot apply the formula $c_1^D + c_2^D = 1$ of the Hausdorff dimension $D$, where each $c_i$ denotes the ratio of similarity. We use the Kurata's formula to calculate the Hausdorff dimension of our Cantor sets.

Theorem 3 is known, for example [6], but our construction of required Cantor sets is elementary and geometrical. The ratios of contraction vary in each inductive step in the construction.

Let us recall a tree and the Hausdorff dimension of its boundary with a distance function.

**Definition** (Kurata [7]). Let $(X, A, o)$ be a tree, i.e. simply connected and locally finite graph. The set $X$ is an infinite set of points and the collection $A$ is a set of arcs. The point $o \in X$ is called the root point. For $x, y \in X$ with $x \neq y$ let $\rho(x, y)$ be the least number of arcs which join $x$ and $y$, and $\rho(x, x) = 0$. Then $\rho$ is a metric on $X$. We assume that $\#\{y \in X : \rho(x, y) = 1\} \geq 2$ for each $x \in X$. We set $X_n = \{x \in X : \rho(o, x) = n\}$ for $n = 0, 1, 2, \cdots$.

Let $\Omega$ be the set of all paths from $o$. A path is a sequence of points $(x_0, x_1, x_2, \cdots)$ such that $x_0 = o$, and $\rho(x_n, x_{n+1}) = 1$ for any $x_n \in X_n$, $n = 0, 1, 2, \cdots$. For $\xi = (x_n)_n, \eta = (y_n)_n \in \Omega$ we define $[\xi] = \{x_0, x_1, x_2, \cdots\}$ where $x_0 = o$. 
and
\[ P(\xi, \eta) = x_n \quad \text{if} \quad x_0 = y_0, x_1 = y_1, \ldots, x_n = y_n, x_{n+1} \neq y_{n+1}. \]
Now \( P(\xi, \xi) \) is not defined. The space \( \Omega \) is called the boundary of a tree \( (X, \mathcal{A}, o) \).

Let \( \ell \) be a positive function from \( X \) to \( \mathbb{R}^1 \) with the following properties:

For any path \( \xi = (x_n)_n \),

(L1) \( \ell(x_n) \) is strictly decreasing in \( n \),

(L2) \( \lim_{n \to \infty} \ell(x_n) = 0 \).

For \( \xi = (x_n)_n, \eta = (y_n)_n \in \Omega \) define
\[ d(\xi, \eta) = \begin{cases} \ell(P(\xi, \eta)) & \text{if } \xi \neq \eta, \\ 0 & \text{if } \xi = \eta. \end{cases} \]
Then \( d \) is a metric on \( \Omega \), and \( \Omega \) is a compact space. For \( x \in X \) let \( B(x) = \{ \xi \in \Omega : x \in [\xi] \} \). If we take \( \eta \in \Omega \) with \( x \in [\eta] \), we have that \( B(x) = \{ \xi \in \Omega : d(\xi, \eta) \leq \ell(x) \} \).

The set \( B(x) \) is both open and closed in \( \Omega \).

For \( K \subset \Omega \) and \( \alpha > 0 \) we define
\[ \Lambda_\alpha^r(K, \ell) = \inf \left\{ \sum_j (\ell(z_j))^\alpha : K \subset \bigcup_j B(z_j), \ell(z_j) < r \right\} \quad \text{for } r > 0, \]
and
\[ \Lambda_\alpha(K, \ell) = \lim_{r \to +0} \Lambda_\alpha^r(K, \ell) = \sup_{r > 0} \Lambda_\alpha^r(K, \ell). \]
We have that \( 0 \leq \Lambda_\alpha(K, \ell) \leq \infty \). The value \( \Lambda_\alpha(K, \ell) \) is called the \( \alpha \)-dimensional Hausdorff measure of \( (K, \ell) \). Define the Hausdorff dimension of \( K \) with a distance function \( \ell \) as
\[ \dim_H(K, \ell) = \inf \{ \alpha : \Lambda_\alpha(K, \ell) = 0 \} = \sup \{ \alpha : \Lambda_\alpha(K, \ell) = \infty \}. \]
Note that \( 0 \leq \dim_H(K, \ell) \leq \infty \).

Now we define a function \( \varphi(x) \) as follows. Let \( \varphi(o) = 1 \). For \( x \in X_n, n > 1 \), we take \( y \in X_{n-1} \) such that \( \rho(x, y) = 1 \) and let
\[ \varphi(x) = \frac{\varphi(y)}{\# \{ z \in X_n : \rho(y, z) = 1 \}}. \]
§2 A construction of a Cantor set with variable ratios of contraction in each inductive step

In this section we construct a Cantor set $E$ with variable ratios of contraction in each inductive step.

For any number $n \geq 1$, let \( \{c_j^{(n)}\}_{j=0,1,2,\ldots,2^n-1} \) be a sequence of real numbers with the properties:

(C1) \( 0 < c_j^{(n)} < 1 \) for each \( n \geq 1 \),

(C2) \( \lim_{n \to \infty} a^{(1)}a^{(2)}\cdots a^{(n)} = 0 \) where \( a^{(n)} = \max \{c_j^{(n)} : j = 0, 1, 2, \ldots, 2^n - 1\} \) for \( n \geq 1 \).

Let \( E_0 \) be a bounded closed interval in \( \mathbb{R}^1 \). Denote the diameter of a set \( E \subset \mathbb{R}^1 \) by \( |E| \). Note that a natural number \( j \) can be written by \( i_1i_2\cdots i_n \) as a number of \( n \) figures in a binary notation. For example,

Case \( n = 2 \): \( 0=00, 1=01, 2=10, 3=11 \), in a binary notation ;

Case \( n = 3 \): \( 0=000, 1=001, 2=010, 3=011 \), in a binary notation.

Put \( c_{i_1i_2\cdots i_n} = c_j^{(n)} \) if \( j = i_1i_2\cdots i_n \) in a binary notation. Define a family \( \{M_{i_1i_2\cdots i_n}\}_{i_1i_2\cdots i_n} \) of subintervals of \( E_0 \) indexed by a finite sequence of figures 0, 1 as follows by induction :

(i) For \( n = 1 \), let \( M_0 \) and \( M_1 \) be two closed subintervals of \( E_0 \) such that

\[ E_0 \setminus \text{(a middle open interval)} = M_0 \cup M_1, \]

where \( \min M_0 = \min E_0, \max M_1 = \max E_0 \) and \( |M_{i_1}| = |E_0| c_{i_1} \) for \( i_1 = 0, 1 \).

(ii) If \( M_{i_1i_2\cdots i_n} \) is defined, let \( M_{i_1i_2\cdots i_n0} \) and \( M_{i_1i_2\cdots i_n1} \) be two closed subintervals of \( M_{i_1i_2\cdots i_n} \) such that

\[ M_{i_1i_2\cdots i_n} \setminus \text{(a middle open subinterval)} = M_{i_1i_2\cdots i_n0} \cup M_{i_1i_2\cdots i_n1}, \]

where \( \min M_{i_1i_2\cdots i_n0} = \min M_{i_1i_2\cdots i_n}, \max M_{i_1i_2\cdots i_n1} = \max M_{i_1i_2\cdots i_n} \) and

\[ |M_{i_1i_2\cdots i_nj}| = |M_{i_1i_2\cdots i_n}| c_{i_1i_2\cdots i_ni_{n+1}} \quad \text{for} \ j = i_1i_2\cdots i_ni_{n+1} \text{ in a binary notation}. \]

Then the family \( \{M_{i_1i_2\cdots i_n}\}_{i_1i_2\cdots i_n} \) satisfies the following :

(M1) For any infinite sequence \( i_1i_2\cdots i_n \cdots \) in \( \{0, 1\} \),
\[ M_{i_1} \supset M_{i_1i_2} \supset \cdots \supset M_{i_1i_2\cdots i_n} \supset M_{i_1i_2\cdots i_ni_{n+1}} \supset \cdots. \]

(M2) If \( i_1i_2\cdots i_n \neq k_1k_2\cdots k_n \), then \( M_{i_1i_2\cdots i_n} \cap M_{k_1k_2\cdots k_n} = \emptyset \).

(M3) \( |M_{i_1i_2\cdots i_n}| = |E_0|c_{i_1}c_{i_1i_2}\cdots c_{i_1i_2\cdots i_n}. \)

(M4) For any infinite sequence \( i_1i_2\cdots i_n \) in \( \{0,1\} \),

\[ \lim_{n \to \infty} |M_{i_1i_2\cdots i_n}| = 0. \]

Hence, \( \bigcap_{n=1}^{\infty} M_{i_1i_2\cdots i_n} = \) one point.

Let

\[ E_n = \bigcup_{n=1}^{\infty} M_{i_1i_2\cdots i_n} \] if \( i_1i_2\cdots i_n \) is a sequence in \( \{0,1\} \) with length \( n \) for \( n \geq 1 \).

Then the set \( E = \bigcap_{n=1}^{\infty} E_n \) is a Cantor set in \( \mathbb{R}^1 \).

**Remark.** The 1/3-Cantor set is a set \( E \) with

\[ c_j^{(n)} = \begin{cases} 
\frac{1}{3} & \text{for } n \geq 1 \text{ and } j = 0, 1, \ldots, 2^n - 1.
\end{cases} \]

Next we define a tree \( (X, \mathcal{A}, o) \) corresponding to the Cantor set \( E \) as follows:

(T1) \( X = X_0 \cup \bigcup_{n=1}^{\infty} X_n \), where \( X_0 = \{o\}, X_1 = \{0,1\}, \ldots \), and

\( X_n = \{i_1i_2\cdots i_n : \text{a sequence in } \{0,1\} \text{ with length } n \} \) for \( n \geq 1 \).

(T2) \( \mathcal{A} = \{[o,0], [o,1]\} \cup \bigcup_{n=1}^{\infty} \{[x_n, y_{n+1}] : x_n \in X_n, y_{n+1} \in X_{n+1}, x_n = i_1i_2\cdots i_n, y_{n+1} = i_1i_2\cdots i_ni_{n+1}\} \),

where \([x,y]\) means the arc joining \( x \) and \( y \) in \( X \).

Then \( \varphi(x_n) = \frac{1}{2^n} \) for \( x_n \in X_n \).

Define \( \ell(x_n) = c_{i_1}c_{i_1i_2}\cdots c_{i_1i_2\cdots i_n} \) for \( x_n = i_1i_2\cdots i_n \).

Then the function \( \ell \) satisfies the requirements in the definition of the boundary of a tree.
We have a bijection \( g : \Omega \rightarrow E \) defined by
\[
g(\xi) = s \quad \text{where } \{s\} = \bigcap_{n=1}^{\infty} M_{i_1 i_2 \cdots i_n}
\]
for \( \xi = (a, y_1, y_2, \cdots, y_n, \cdots) \) with \( y_n = i_1 i_2 \cdots i_n, \ n \geq 1 \).
Then \( \dim_H(\Omega, \ell) = \dim_H E \).

\section*{3 Proofs}

Example 1 in the following shows Theorem 2.

**Example 1.** For each \( n \), define
\[
c_j^{(n)} = \begin{cases} 
\frac{1}{3} & : j = 0, 2, \cdots, 2^n - 2, \\
\frac{1}{9} & : j = 1, 3, \cdots, 2^n - 1,
\end{cases}
\]
and
\[
\ell(y_n) = c_{i_1} c_{i_2} \cdots c_{i_1 i_2 \cdots i_n} \quad \text{for } y_n = i_1 i_2 \cdots i_n
\]
\[
= c_{j_1}^{(1)} c_{j_2}^{(2)} \cdots c_{j_n}^{(n)},
\]
where \( j_r = 2^{r-1} i_1 + 2^{r-2} i_2 + \cdots + 2 i_{r-1} + i_r, \ r = 1, 2, \cdots, n \).

Then, the resulting Cantor set \( E \) gives an example of Theorem 2 (see Fig. 1).

(1) The right side of Kurata's formula \( = \frac{\log 2}{\log 3} \).

In fact, take a path \( \xi = (a, y_1, y_2, \cdots, y_n, \cdots) \in \Omega \) with \( y_n = 00 \cdots 0 \) for any \( n \).
We have that
\[
\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{\log 3} \quad \text{for any } n.
\]
Hence,
\[
\lim_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{\log 3}.
\]

(2) The left side of Kurata's formula \( = \frac{\log 2}{2 \log 3} \).
In fact, take any \( x \in X \) with \( x = i_1 i_2 \cdots i_n \). Let \( y_n \) be any point in \( X \) such that \( B(y_n) \subset B(x) \). For any \( n > m \), set \( y_n = i_1 i_2 \cdots i_m i_{m+1} \cdots i_n \) and \( i_{m+1} = \cdots = i_n = 1 \). Then, for any \( n > m \),

\[
\ell(y_n) = c_j^{(1)} c_j^{(2)} \cdots c_j^{(m)} \left(\frac{1}{3}\right)^{n-m}
\]

and

\[
\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{2(1 - \frac{m}{n}) \log 3 - \frac{1}{n} \log c_j^{(1)} c_j^{(2)} \cdots c_j^{(n)}}.
\]

Hence,

\[
\liminf_{B(y_n) \subset B(x)} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{2 \log 3}. \quad \square
\]

Theorem 3 is established by Examples 2 - 6 in the following.

**Example 2. Case :** \( \gamma = 0 \). For each \( n \), define

\[
c_j^{(n)} = \left(\frac{1}{3}\right)^n \quad \text{for } j = 0, 1, \cdots, 2^n - 1.
\]

Then, the resulting Cantor set \( E \) has Hausdorff dimension 0.

In fact, take any \( y_n \in \Omega \) with \( y_n = i_1 i_2 \cdots i_n \). Then

\[
\ell(y_n) = \left(\frac{1}{3}\right)^1 \left(\frac{1}{3}\right)^2 \cdots \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^{\frac{1}{2}n(n+1)}.
\]
The function \( \ell \) satisfies the conditions (L1) - (L2).

Since 
\[
\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{2 \log 2}{(n+1) \log 3} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]
we have that \( \dim_{H} E = 0 \) from Theorem 1. \( \square \)

**Example 3.** *Case:* \( \gamma = 1 \). For each \( n \), define
\[
c_{j}^{(n)} = \frac{1}{4} \frac{2^n + 1}{2^{n-1} + 1} \quad \text{for} \quad j = 0, 1, \ldots, 2^n - 1.
\]
Then, the resulting Cantor set \( E \) has Hausdorff dimension 1.

In fact, take any \( y_n \in \Omega \) with \( y_n = i_1 i_2 \cdots i_n \). Then
\[
\ell(y_n) = \left(\frac{1}{4}\right)^n \frac{2^n + 1}{2} = \frac{2^n + 1}{2^{2n+1}}.
\]
The \( \ell \) satisfies the conditions (L1) - (L2).

Since 
\[
\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{1}{(2 + \frac{1}{n}) - \frac{\log(2^n+1)}{\log 2^n}} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty,
\]
we have that \( \dim_{H} E = 1 \) from Theorem 1. \( \square \)

**Example 4.** *Case:* \( 0 < \gamma < 1 \). For each \( n \), define
\[
c_{j}^{(n)} = \left(\frac{1}{2}\right)^{n} \quad \text{for} \quad j = 0, 1, \ldots, 2^n - 1.
\]
Then, the resulting Cantor set \( E \) has Hausdorff dimension \( \gamma \).

In fact, the both sides of Kurata's formula are equal to \( \gamma \). \( \square \)

**Example 5.** *Case:* \( 1 < \gamma < \infty \). For some integer \( N \geq 2 \) with \( \gamma \leq N \), we can obtain a Cantor set \( E \) in \( \mathbb{R}^N \) with \( \dim_{H} E = \gamma \) by appropriate modifications to that of §2. We explain how to construct such a Cantor set \( E \) in \( \mathbb{R}^2 \) for \( N = \gamma = 2 \).

Let \( E_0 \) be a closed regular square in \( \mathbb{R}^2 \). For each \( n \), define
\[
c_{j}^{(n)} = \frac{1}{4} \frac{2^n + 1}{2^{n-1} + 1} \quad \text{for} \quad j = 0, 1, \ldots, 2^n - 1,
\]
and
\[ c_{i_1 i_2 \cdots i_n} = c_j^{(n)} \quad \text{for} \quad j = i_1 i_2 \cdots i_n \quad \text{in a 4-ary notation}. \]

Define a family \( \{ M_{i_1 i_2 \cdots i_n} \} \) of closed subsquares of \( E_0 \) indexed by a finite sequence of figures 0, 1, 2, 3 with the properties (M1) - (M4). Analogously in § 2 we have a Cantor set \( E \subset \mathbb{R}^2 \) with \( \dim_H E = 2 \) (Fig. 2). □

![Fig. 2](image)

**Example 6.** *Case: \( \gamma = \infty \).* We construct a Cantor set \( E \) in the Hilbert cube \( Q \) with \( \dim_H E = \infty \). The Hilbert cube means a space

\[ Q = \{(t_i) : 0 \leq \frac{1}{t_i} \leq \frac{1}{i} \quad \text{for} \quad i = 1, 2, 3, \cdots \} \]

with the metric

\[ d(s, t) = \sqrt{\sum_{n=1}^{\infty} (s_i - t_i)^2} \quad \text{for} \quad s = (s_i), \ t = (t_i). \]

Define a set \( E \subset Q \) as follows:

\[ E = \bigcup_{n=1}^{\infty} A_n \cup \{a_0\}, \]

where \( a_0 = (0, 0, 0, \cdots) \), and for any \( n, A_n \) is a Cantor set such that

(A1) \( A_n \subset \left[ \frac{1}{n+1}, \frac{1}{n} \right]^n \times \{0\} \times \{0\} \times \cdots \),
(A2) \( \dim_H A_n = n \),

(A3) \( A_m \cap A_n = \emptyset \) if \( m \neq n \).

Since \( E \) is a totally disconnected compact metric space with no isolated points, it is a Cantor set. We have that

\[
\dim_H E = \sup_n \dim_H A_n = \infty. \quad \square
\]

References


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