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An Elementary Construction of a Cantor Set with Arbitrary Hausdorff Dimension

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Abstract

Recently H. Kurata gave an evaluation formula of the Hausdorff dimension of the boundary of a tree with a distance function as follows.

Theorem 1 (Kurata).

\[ \sup_{x \in X} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right) \leq \dim_H(\Omega, \ell) \leq \sup_{\xi \in \Omega} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right) \]

In the paper we shall investigate the usefulness of Kurata's formula and obtain the following results.

Theorem 2. There exists a Cantor set for which both sides of Kurata's formula do not coincide.

Theorem 3. For each \( \gamma, 0 \leq \gamma \leq \infty \), there exists a Cantor set \( E \) with Hausdorff dimension \( \gamma \).

§1 Introduction

Recently H. Kurata gave an evaluation formula of the Hausdorff dimension of the boundary of a tree and calculated the Hausdorff dimension of certain sets of \( \mathbb{R}^n \) by using it.
Theorem 1 (Kurata's formula [7]). Let $\Omega$ be the boundary of a tree $(X, A, o)$ with a distance function $\ell$. Then
\[
\sup_{x \in X} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right) \leq \dim_H(\Omega, \ell) \leq \sup_{\xi \in \Omega} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right).
\]

In the present paper we shall show the following:

Theorem 2. There exists a Cantor set for which the both sides of Kurata's formula do not coincide.

Theorem 3. For each $\gamma$, $0 \leq \gamma \leq \infty$, there exists a Cantor set $E$ with Hausdorff dimension $\gamma$.

Our Cantor sets satisfying the condition in each of Theorems 2 and 3 are not self-similar, in general. So we cannot apply the formula $c_1^D + c_2^D = 1$ of the Hausdorff dimension $D$, where each $c_i$ denotes the ratio of similarity. We use the Kurata's formula to calculate the Hausdorff dimension of our Cantor sets.

Theorem 3 is known, for example [6], but our construction of required Cantor sets is elementary and geometrical. The ratios of contraction vary in each inductive step in the construction.

Let us recall a tree and the Hausdorff dimension of its boundary with a distance function.

Definition (Kurata [7]). Let $(X, A, o)$ be a tree, i.e. simply connected and locally finite graph. The set $X$ is an infinite set of points and the collection $A$ is a set of arcs. The point $o \in X$ is called the root point. For $x, y \in X$ with $x \neq y$ let $\rho(x, y)$ be the least number of arcs which join $x$ and $y$, and $\rho(x, x) = 0$. Then $\rho$ is a metric on $X$. We assume that $\# \{y \in X : \rho(x, y) = 1\} \geq 2$ for each $x \in X$. We set $X_n = \{x \in X : \rho(o, x) = n\}$ for $n = 0, 1, 2, \cdots$.

Let $\Omega$ be the set of all paths from $o$. A path is a sequence of points $(x_0, x_1, x_2, \cdots)$ such that $x_0 = o$, and $\rho(x_n, x_{n+1}) = 1$ for any $x_n \in X_n$, $n = 0, 1, 2, \cdots$.

For $\xi = (x_n)_n$, $\eta = (y_n)_n \in \Omega$ we define $[\xi] = \{x_0, x_1, x_2, \cdots\}$ where $x_0 = o$. 

\[
\sup_{x \in X} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right) \leq \dim_H(\Omega, \ell) \leq \sup_{\xi \in \Omega} \left( \liminf_{n \to \infty} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} \right).
\]
and 
\[ P(\xi, \eta) = x_n \quad \text{if} \quad x_0 = y_0, x_1 = y_1, \ldots, x_n = y_n, x_{n+1} \neq y_{n+1}. \]

Now \( P(\xi, \xi) \) is not defined. The space \( \Omega \) is called the boundary of a tree \( (X, A, o) \).

Let \( \ell \) be a positive function from \( X \) to \( \mathbb{R}^1 \) with the following properties:

For any path \( \xi = (x_n)_n \),

(L1) \( \ell(x_n) \) is strictly decreasing in \( n \),

(L2) \( \lim_{n \to \infty} \ell(x_n) = 0 \).

For \( \xi = (x_n)_n, \eta = (y_n)_n \in \Omega \) define

\[ d(\xi, \eta) = \begin{cases} \ell(P(\xi, \eta)) & \text{if} \ \xi \neq \eta, \\ 0 & \text{if} \ \xi = \eta. \end{cases} \]

Then \( d \) is a metric on \( \Omega \), and \( \Omega \) is a compact space. For \( x \in X \) let \( B(x) = \{ \xi \in \Omega : x \in [\xi] \} \). If we take \( \eta \in \Omega \) with \( x \in [\eta] \), we have that \( B(x) = \{ \xi \in \Omega : d(\xi, \eta) \leq \ell(x) \} \).

The set \( B(x) \) is both open and closed in \( \Omega \).

For \( K \subset \Omega \) and \( \alpha > 0 \) we define

\[ \Lambda_{\alpha}^r(K, \ell) = \inf \left\{ \sum_j (\ell(z_j))^\alpha : K \subset \bigcup_j B(z_j), \ \ell(z_j) < r \right\} \quad \text{for} \ r > 0, \]

and

\[ \Lambda_{\alpha}(K, \ell) = \lim_{r \to 0^+} \Lambda_{\alpha}^r(K, \ell) = \sup_{r > 0} \Lambda_{\alpha}^r(K, \ell). \]

We have that \( 0 \leq \Lambda_{\alpha}(K, \ell) \leq \infty \). The value \( \Lambda_{\alpha}(K, \ell) \) is called the \( \alpha \)-dimensional Hausdorff measure of \( (K, \ell) \). Define the Hausdorff dimension of \( K \) with a distance function \( \ell \) as

\[ \dim_H(K, \ell) = \inf \{ \alpha : \Lambda_{\alpha}(K, \ell) = 0 \} = \sup \{ \alpha : \Lambda_{\alpha}(K, \ell) = \infty \}. \]

Note that \( 0 \leq \dim_H(K, \ell) \leq \infty \).

Now we define a function \( \varphi(x) \) as follows. Let \( \varphi(o) = 1 \). For \( x \in X_n, \ n > 1 \), we take \( y \in X_{n-1} \) such that \( \rho(x, y) = 1 \) and let

\[ \varphi(x) = \frac{\varphi(y)}{\# \{ z \in X_n : \rho(y, z) = 1 \}}. \]
§2 A construction of a Cantor set with variable ratios of contraction in each inductive step

In this section we construct a Cantor set $E$ with variable ratios of contraction in each inductive step.

For any number $n \geq 1$, let $\{c_{j}^{(n)}\}_{j=0,1,2,\ldots,2^{n}-1}$ be a sequence of real numbers with the properties:

(C1) $0 < c_{j}^{(n)} < 1$ for each $n \geq 1$,

(C2) $\lim_{n \to \infty}a^{(1)}a^{(2)}\cdots a^{(n)} = 0$ where $a^{(n)} = \max\{c_{j}^{(n)} : j = 0, 1, 2, \ldots, 2^{n} - 1\}$ for $n \geq 1$.

Let $E_{0}$ be a bounded closed interval in $\mathbb{R}^{1}$. Denote the diameter of a set $E \subset \mathbb{R}^{1}$ by $|E|$. Note that a natural number $j$ can be written by $i_{1}i_{2}\cdots i_{n}$ as a number of $n$ figures in a binary notation. For example,

Case $n = 2$: 0=00, 1=01, 2=10, 3=11, in a binary notation;

Case $n = 3$: 0=000, 1=001, 2=010, 3=011, in a binary notation.

Put $c_{i_{1}i_{2}\cdots i_{n}} = c_{j}^{(n)}$ if $j = i_{1}i_{2}\cdots i_{n}$ in a binary notation. Define a family $\{M_{i_{1}i_{2}\cdots i_{n}}\}_{i_{1}i_{2}\cdots i_{n}}$ of subintervals of $E_{0}$ indexed by a finite sequence of figures 0, 1 as follows by induction:

(i) For $n = 1$, let $M_{0}$ and $M_{1}$ be two closed subintervals of $E_{0}$ such that

$E_{0} \setminus$ (a middle open interval) = $M_{0} \cup M_{1}$,

where $\min M_{0} = \min E_{0}$, $\max M_{1} = \max E_{0}$ and $|M_{i_{1}}| = |E_{0}|c_{i_{1}}$ for $i_{1} = 0, 1$.

(ii) If $M_{i_{1}i_{2}\cdots i_{n}}$ is defined, let $M_{i_{1}i_{2}\cdots i_{n}0}$ and $M_{i_{1}i_{2}\cdots i_{n}1}$ be two closed subintervals of $M_{i_{1}i_{2}\cdots i_{n}}$ such that

$M_{i_{1}i_{2}\cdots i_{n}0} \setminus$ (a middle open subinterval) = $M_{i_{1}i_{2}\cdots i_{n}0} \cup M_{i_{1}i_{2}\cdots i_{n}1}$,

where $\min M_{i_{1}i_{2}\cdots i_{n}0} = \min M_{i_{1}i_{2}\cdots i_{n}}$, $\max M_{i_{1}i_{2}\cdots i_{n}1} = \max M_{i_{1}i_{2}\cdots i_{n}}$ and $|M_{i_{1}i_{2}\cdots i_{n}j}| = |M_{i_{1}i_{2}\cdots i_{n}}|c_{i_{1}i_{2}\cdots i_{n}i_{n+1}}$ for $j = i_{1}i_{2}\cdots i_{n}i_{n+1}$ in a binary notation.

Then the family $\{M_{i_{1}i_{2}\cdots i_{n}}\}_{i_{1}i_{2}\cdots i_{n}}$ satisfies the following:

(M1) For any infinite sequence $i_{1}i_{2}\cdots i_{n} \cdots$ in $\{0, 1\}$,
\[ M_{i_{1}} \supset M_{i_{1}i_{2}} \supset \cdots \supset M_{i_{1}i_{2} \cdots i_{n}} \supset M_{i_{1}i_{2} \cdots i_{n}i_{n+1}} \supset \cdots. \]

(M2) If \( i_{1}i_{2} \cdots i_{n} \neq k_{1}k_{2} \cdots k_{n}, \) then \( M_{i_{1}i_{2} \cdots i_{n}} \cap M_{k_{1}k_{2} \cdots k_{n}} = \emptyset. \)

(M3) \( |M_{i_{1}i_{2} \cdots i_{n}}| = |E_{0}| c_{i_{1}} c_{i_{2}} \cdots c_{i_{1}i_{2} \cdots i_{n}}. \)

(M4) For any infinite sequence \( i_{1}i_{2} \cdots i_{n} \cdots \) in \( \{0,1\}, \)

\[
\lim_{n \to \infty} |M_{i_{1}i_{2} \cdots i_{n}}| = 0.
\]

Hence, \( \bigcap_{n=1}^{\infty} M_{i_{1}i_{2} \cdots i_{n}} = \) one point.

Let

\[
E_{n} = \bigcup_{n=1}^{\infty} \left\{ M_{i_{1}i_{2} \cdots i_{n}} : i_{1}i_{2} \cdots i_{n} \text{ is a sequence in } \{0,1\} \text{ with length } n \right\} \text{ for } n \geq 1.
\]

Then the set \( E = \bigcap_{n=1}^{\infty} E_{n} \) is a Cantor set in \( \mathbb{R}^{1}. \)

**Remark.** The 1/3-Cantor set is a set \( E \) with

\[
c^{(n)}_{j} = \frac{1}{3} \quad \text{for } n \geq 1 \text{ and } j = 0, 1, \ldots, 2^{n} - 1.
\]

Next we define a tree \( (X, \mathcal{A}, o) \) corresponding to the Cantor set \( E \) as follows:

(T1) \( X = X_{0} \cup \bigcup_{n=1}^{\infty} X_{n}, \) where \( X_{0} = \{o\}, X_{1} = \{0,1\}, \cdots, \) and

\[
X_{n} = \{ i_{1}i_{2} \cdots i_{n} : \text{a sequence in } \{0,1\} \text{ with length } n \} \quad \text{for } n \geq 1.
\]

(T2) \( \mathcal{A} = \{ [0,0], [0,1] \} \cup \)

\[
\bigcup_{n=1}^{\infty} \left\{ [x_{n}, y_{n+1}] : x_{n} \in X_{n}, y_{n+1} \in X_{n+1}, x_{n} = i_{1}i_{2} \cdots i_{n}, y_{n+1} = i_{1}i_{2} \cdots i_{n}i_{n+1} \right\},
\]

where \([x,y]\) means the arc joining \( x \) and \( y \) in \( X. \)

Then \( \varphi(x_{n}) = \frac{1}{2^{n}} \) for \( x_{n} \in X_{n}. \)

Define \( \ell(x_{n}) = c_{i_{1}}c_{i_{1}i_{2}} \cdots c_{i_{1}i_{2} \cdots i_{n}} \) for \( x_{n} = i_{1}i_{2} \cdots i_{n}. \)

Then the function \( \ell \) satisfies the requirements in the definition of the boundary of a tree.
We have a bijection $g : \Omega \rightarrow E$ defined by

$$g(\xi) = s \quad \text{where} \quad \{s\} = \bigcap_{n=1}^{\infty} M_{i_{1}i_{2}\cdots i_{n}}$$

for $\xi = (o, y_{1}, y_{2}, \cdots, y_{n}, \cdots)$ with $y_{n} = i_{1}i_{2}\cdots i_{n}$, $n \geq 1$.

Then $\dim_{H}(\Omega, \ell) = \dim_{H} E$.

§3 Proofs

Example 1 in the following shows Theorem 2.

**Example 1.** For each $n$, define

$$c_{j}^{(n)} = \begin{cases} \frac{1}{3} : & j = 0, 2, \cdots, 2^{n} - 2, \\ \frac{1}{9} : & j = 1, 3, \cdots, 2^{n} - 1, \end{cases}$$

and

$$\ell(y_{n}) = c_{i_{1}}c_{i_{2}}\cdots c_{i_{1}i_{2}\cdots i_{n}} \quad \text{for} \quad y_{n} = i_{1}i_{2}\cdots i_{n}$$

$$= c_{j_{1}}^{(1)}c_{j_{2}}^{(2)}\cdots c_{j_{n}}^{(n)},$$

where $j_{r} = 2^{r-1}i_{1} + 2^{r-2}i_{2} + \cdots + 2i_{r-1} + i_{r}$, $r = 1, 2, \cdots, n$.

Then, the resulting Cantor set $E$ gives an example of Theorem 2 (see Fig. 1).

(1) The right side of Kurata's formula $= \frac{\log 2}{\log 3}$. In fact, take a path $\xi = (o, y_{1}, y_{2}, \cdots, y_{n}, \cdots) \in \Omega$ with $y_{n} = 00\cdots 0$ for any $n$.

We have that

$$\frac{\log 1/\varphi(y_{n})}{\log 1/\ell(y_{n})} = \frac{\log 2}{\log 3} \quad \text{for any} \quad n.$$

Hence,

$$\lim_{n \rightarrow \infty} \inf_{y_{n} \in [\xi]} \frac{\log 1/\varphi(y_{n})}{\log 1/\ell(y_{n})} = \frac{\log 2}{\log 3}.$$

(2) The left side of Kurata's formula $= \frac{\log 2}{2\log 3}$.
In fact, take any \( x \in X \) with \( x = i_1i_2 \cdots i_n \). Let \( y_n \) be any point in \( X \) such that \( B(y_n) \subset B(x) \). For any \( n > m \), set \( y_n = i_1i_2 \cdots i_m i_{m+1} \cdots i_n \) and \( i_{m+1} = \cdots = i_n = 1 \). Then, for any \( n > m \),

\[
\ell(y_n) = c_{j_1}^{(1)} c_{j_2}^{(2)} \cdots c_{j_m}^{(n)} \left( \frac{1}{9} \right)^{n-m}
\]

and

\[
\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{2(1 - \frac{m}{n}) \log 3 - \frac{1}{n} \log c_{j_1}^{(1)} c_{j_2}^{(2)} \cdots c_{j_m}^{(n)}}.
\]

Hence,

\[
\liminf_{\substack{B(y_n) \subset B(x) \\ n \to \infty}} \frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{\log 2}{2 \log 3}.
\]

\[\square\]

---

**Fig. 1**

Theorem 3 is established by Examples 2 - 6 in the following.

**Example 2. Case**: \( \gamma = 0 \). For each \( n \), define

\[
c_j^{(n)} = \left( \frac{1}{3} \right)^n \quad \text{for} \quad j = 0, 1, \cdots, 2^n - 1.
\]

Then, the resulting Cantor set \( E \) has Hausdorff dimension 0.

In fact, take any \( y_n \in \Omega \) with \( y_n = i_1i_2 \cdots i_n \). Then

\[
\ell(y_n) = \left( \frac{1}{3} \right)^1 \left( \frac{1}{3} \right)^2 \cdots \left( \frac{1}{3} \right)^n = \left( \frac{1}{3} \right)^{\frac{1}{2}n(n+1)}.
\]
The function $\ell$ satisfies the conditions (L1) - (L2).

Since

$$\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{2\log 2}{(n+1)\log 3} \to 0 \quad \text{as} \quad n \to \infty,$$

we have that $\dim_H E = 0$ from Theorem 1. \(\square\)

**Example 3.** Case: $\gamma = 1$. For each $n$, define

$$c_j^{(n)} = \frac{1}{4} \frac{2^n + 1}{2^{n-1} + 1} \quad \text{for } j = 0, 1, \ldots, 2^n - 1.$$

Then, the resulting Cantor set $E$ has Hausdorff dimension 1.

In fact, take any $y_n \in \Omega$ with $y_n = i_1i_2 \cdots i_n$. Then

$$\ell(y_n) = \left(\frac{1}{4}\right)^n \frac{2^n + 1}{2} = \frac{2^n + 1}{2^{2n+1}}.$$

The $\ell$ satisfies the conditions (L1) - (L2).

Since

$$\frac{\log 1/\varphi(y_n)}{\log 1/\ell(y_n)} = \frac{1}{(2 + \frac{1}{n}) - \frac{\log(2^n + 1)}{\log 2^n}} \to 1 \quad \text{as} \quad n \to \infty,$$

we have that $\dim_H E = 1$ from Theorem 1. \(\square\)

**Example 4.** Case: $0 < \gamma < 1$. For each $n$, define

$$c_j^{(n)} = \left(\frac{1}{2}\right)^\gamma \quad \text{for } j = 0, 1, \ldots, 2^n - 1.$$

Then, the resulting Cantor set $E$ has Hausdorff dimension $\gamma$.

In fact, the both sides of Kurata's formula are equal to $\gamma$. \(\square\)

**Example 5.** Case: $1 < \gamma < \infty$. For some integer $N \geq 2$ with $\gamma \leq N$, we can obtain a Cantor set $E$ in $\mathbb{R}^N$ with $\dim_H E = \gamma$ by appropriate modifications to that of §2. We explain how to construct such a Cantor set $E$ in $\mathbb{R}^2$ for $N = \gamma = 2$.

Let $E_0$ be a closed regular square in $\mathbb{R}^2$. For each $n$, define

$$c_j^{(n)} = \frac{1}{4} \frac{2^n + 1}{2^{n-1} + 1} \quad \text{for } j = 0, 1, \ldots, 2^n - 1,$$
and
\[ c_{i_{1}i_{2}\cdots i_{n}} = c_{j}^{(n)} \quad \text{for} \quad j = i_{1}i_{2}\cdots i_{n} \quad \text{in a 4-ary notation}. \]

Define a family \( \{M_{i_{1}i_{2}\cdots i_{n}}\}_{i_{1}i_{2}\cdots i_{n}} \) of closed subsquares of \( E_{0} \) indexed by a finite sequence of figures 0, 1, 2, 3 with the properties (M1) - (M4). Analogously in §2 we have a Cantor set \( E \subset \mathbb{R}^{2} \) with \( \dim_{H}E = 2 \) (Fig. 2). \( \square \)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example6.png}
\caption{Fig. 2}
\end{figure}

**Example 6.** Case: \( \gamma = \infty \). We construct a Cantor set \( E \) in the Hilbert cube \( Q \) with \( \dim_{H}E = \infty \). The Hilbert cube means a space
\[ Q = \{(t_{i}) : 0 \leq \frac{1}{t_{i}} \leq \frac{1}{i} \quad \text{for} \quad i = 1, 2, 3, \cdots \} \]
with the metric
\[ d(s, t) = \sqrt{\sum_{n=1}^{\infty} (s_{i} - t_{i})^{2}} \quad \text{for} \quad s = (s_{i}), \ t = (t_{i}). \]

Define a set \( E \subset Q \) as follows:
\[ E = \bigcup_{n=1}^{\infty} A_{n} \cup \{a_{0}\}, \]
where \( a_{0} = (0, 0, 0, \cdots) \), and for any \( n \), \( A_{n} \) is a Cantor set such that
\[ (A1) \quad A_{n} \subset \left[ \frac{1}{n+1}, \frac{1}{n} \right]^{n} \times \{0\} \times \{0\} \times \cdots, \]
(A2) $\dim_H A_n = n,$

(A3) $A_m \cap A_n = \emptyset$ \hspace{1em} if \hspace{1em} $m \neq n.$

Since $E$ is a totally disconnected compact metric space with no isolated points, it is a Cantor set. We have that

$$\dim_H E = \sup_n \dim_H A_n = \infty.$$ \hspace{1em} $\square$

References


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