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One development of the M_3 vs. M_1 problem

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1 The M_3 vs. M_1 problem

All spaces are assumed to be regular T_1. The term “CP” stands for “closure-preserving”.

To begin with, we give the definitions of M_i-spaces which were introduced by Ceder [2] in 1960 as generalized metric spaces:

Definition 1.1. A space is an *M_1-space* if it has a σ-CP base.

Definition 1.2. A space is an *M_2-space* if it has a σ-CP quasi-base.

Definition 1.3. A space is an *M_3-space* if it has a σ-cushioned pair-base.

Recalling the Nagata-Smirnov metrization theorem, we easily have the implication:

$$\text{Metric space} \to M_1 \to M_2 \to M_3$$

Later, Borges [1] renamed M_3-spaces “stratifiable spaces” in terms of the stratification as follows:

Definition 1.4. A space X is *stratifiable* if there exists a function S : {closed subsets of X} $\times \mathbb{N} \to \tau(X)$, called the *stratification* of X, satisfying the following:

(i) For each closed subset H,

$$\bigcap_n S(H,n) = \bigcap_n \overline{S(H,n)} = H;$$

(ii) if $H \subset K$, then $S(H,n) \subset S(K,n)$ for each n.

As for the reverse implications, Gruenhage [3] and Junnila [5] showed independently $M_3 \to M_2$. However, $M_3 \to M_1$? has not been answered yet and it has become one of the most outstanding open problems in general topology.

To this problem, many partial answers have been given, since Slaughter first showed that any Lašnev space is an M_1-space. In a sense, we could say that the history of this problem is one of partial answers. Diagram of those partial answers is given in the last part.

Since in order to state our discussion it seems better to be familiar with Ito's method which was adopted to show that Nagata spaces or M_3-spaces whose every point has a CP open neighborhood base are M_1, we give here the outline of his method:

Theorem 1.1 ([4]). *If each point of an M_3-space X has a CP open neighborhood base, then every closed subset of X has a CP open neighborhood base, necessarily X is an M_1-space.*

We represent from here the class of M_3-spaces whose each closed subset has a CP open neighborhood base by $\mathcal{P}$.

The following fact due to Ito above seems useful and applicable to our discussion: If M be a closed subset of an M_3-space, then there exists a CP closed neighborhood base $\mathcal{B}$ of M and at the same time there exists a dense subset $D = \bigcup\{D_n \mid n \in \mathbb{N}\}$, where each D_n is discrete and closed in X, such that

$$B = \overline{B \cap D}$$

for each $B \in \mathcal{B}$. By the assumption, each $p \in D_n$, $n \in \mathbb{N}$, has an open neighborhood base $\mathcal{U}(p)$. Then we can construct a CP open neighborhood base of M by expanding each point $p \in D_n \cap B$, $n \in \mathbb{N}$, to members in $\mathcal{U}(p)|S(\{p\},n)$.

On the other hand, Tamano pointed out that the assumption on points of an M_3-spaces can be weakened as follows:

Theorem 1.2 ([8]). *If each point p of an M_3-space X, there exists a CP family $\mathcal{U}$ of open subsets of X such that $\overline{\mathcal{U}} = \{\overline{U} \mid U \in \mathcal{U}\}$ forms a local network at p in X, then $X \in \mathcal{P}$.*

So, what kind of spaces satisfies the Tamano's criteria above? For this question, Tamano himself gave there the following positive result:

Theorem 1.3 ([8]). *If a space X is a Baire, Fréchet M_3-space, then X satisfies the Tamano's criteria, necessarily $X \in \mathcal{P}$.*

In the proof, Baire property is used to induce the fact that any Baire σ-space has a G_δ-dense metrizable subset, due to Douwen.

It is quite natural to ask the following problem: Can we delete the assumption "Baire" from the above theorem, that is, are Fréchet M_3-spaces M_1?

We gave the positive answer to this problem in our first paper [6]. To do this, we define property ($*$) as follows:

Definition 1.5 ([6]). A space X satisfies *property ($*$)* if for each $p \in \partial O$, $O \in \tau(X)$, there exists a CP closed local network $\mathcal{B}$ at p in X such that

$$(*) \qquad \begin{array}{c} B \subset \overline{O},\ B \cap \partial O = \{p\} \text{ and} \\ \overline{B \setminus \{p\}} = B \text{ for each } B \in \mathcal{B}. \end{array}$$

We note that every Fréchet space has this property, and also that this property generalized the Tamano's criteria in the sense of "without the term "open sets"".

With our criteria, we showed the following:

Theorem 1.4 ([6]). *An M_3-space with property ($*$) belongs to $\mathcal{P}$, necessarily is M_1.*

Whether the property could work well to the M_3 vs. M_1 problem or whether the property is useful to the M_3 vs. M_1 problem actually? For that matter, we have to clear the essential question whether property ($*$) is what M_3-spaces themselves have intrinsically as their property.

But unfortunately the answer is no, which was given by the following:

Example 1.1 ([6]). There exists an M_0-space which does not satisfy property ($*$).

This means that property ($*$) is too strong to apply to the M_3 vs. M_1 problem; so in the second paper, we weaken it to property (P):

Definition 1.6 ([7]). A space X satisfies *property (P)* if for each $p \in \partial O$, $O \in \tau(X)$, there exists a CP closed local network $\mathcal{B}$ at p in X such that $\overline{B \cap O} = B$ for each $B \in \mathcal{B}$.

Using a similar way to the case of property ($*$), we showed the following:

Theorem 1.5 ([7]). *An M_3-space with property (P) belongs to $\mathcal{P}$.*

What kind of M_3-spaces satisfies this property? Do M_3-spaces satisfy this property? With respect to these questions, we do not have the exact answers. Even if a space $X \in \mathcal{P}$, we do not know whether X satisfies property (P) or not. Rather, this question is equivalent with the M_3 vs. M_1 problem itself, as stated below, i.e., all M_3-space are M_1 if and only if all $X \in \mathcal{P}$ satisfy property (P).

Since the fact that in reality we do not know whether any space in $\mathcal{P}$ satisfies property (P) is one defect, in a final stage we relax the property (P) more to property (δ) as follows:

Definition 1.7. A space X satisfies *property (δ)* if for each nowhere dense, closed subset M of a space and each $p \in M$, there exists a CP closed local network $\mathcal{B}$ at p in X such that $B = \overline{B \setminus M}$ for each $B \in \mathcal{B}$.

Obviously, property ($*$) $\to$ property (P) $\to$ property (δ). This property is rather significant than the previous two in the sense that all spaces in $\mathcal{P}$ satisfy property (δ) in turn. Of course, to this case we can also show the following:

Theorem 1.6. *An M_3-space with property (δ) belongs to $\mathcal{P}$.*

From it, we can characterize the class $\mathcal{P}$ in terms of this property:

Corollary 1.1. *A space X belongs to $\mathcal{P}$ if and only if X is an M_3-space with property (δ).*

Here, we give lemmas needed for the proof.

Lemma 1.1 ([9]). *An M_3-space is a K_1-space in the sense of van Douwen.*

Lemma 1.2. *Let $\mathcal{B}$ be a CP family of closed subsets of an M_3-space X. Then there exists a pair $\langle \mathcal{F}, \mathcal{V} \rangle$ of families satisfying the following:*

(i) *$\mathcal{F}$ is a σ-discrete closed cover of X;*
(ii) *$\mathcal{V} = \{V(F) \mid F \in \mathcal{F}\}$ is a point-finite, σ-discrete open cover of X such that $F \subset V(F)$ for each $F \in \mathcal{F}$;*
(iii) *for each $F \in \mathcal{F}$ and $B \in \mathcal{B}$, $F \cap B \neq \emptyset$ implies $F \subset B$ and $F \cap B = \emptyset$ implies $V(F) \cap B = \emptyset$.*

Lemma 1.3. *Let $\mathcal{B}$ be a CP family of closed subsets of an M_3-space X. Then there exist families $\mathcal{B}(B)$, $B \in \mathcal{B}$, of subsets of X satisfying the following:*

(i) *For each $B \in \mathcal{B}$, $\mathcal{B}(B)$ is a closed neighborhood base of B in X;*
(ii) *$\bigcup\{\mathcal{B}(B) \mid B \in \mathcal{B}\}$ is CP in X.*

Lemma 1.4. *Let M be a closed subset of an M_3-space X and let $\mathcal{B}$ be a CP family of closed subsets of X. Then there exist families $\mathcal{W}(B)$, $B \in \mathcal{B}$, of subsets of X satisfying the following:*

(i) *$\bigcup\{\mathcal{W}(B) \mid B \in \mathcal{B}\}$ is a CP family of closed subsets of X;*

(ii) *for each* $B \in \mathcal{B}$, $\mathcal{W}(B)|M = \{B \cap M\}$ *and* $\mathcal{W}(B)|(X \setminus M)$ *is a closed neighborhood base of* $B \setminus M$ *in* $X \setminus M$.

Lemma 1.5. *Let* M *be a closed subset of an* M_3*-space* X *and let* $\mathcal{B}$ *be a CP family of closed subsets of* X. *Then there exists a family* $\{S(B) \mid B \in \mathcal{B}\}$ *of open subsets of* $X \setminus M$ *satisfying the following:*

(i) *For each* $B \in \mathcal{B}$, $B \setminus M \subset S(B)$;
(ii) *for any* $\mathcal{B}' \subset \mathcal{B}$,

$$\overline{\bigcup\{S(B) \mid B \in \mathcal{B}'\}} \cap M \subset \left(\bigcup \mathcal{B}'\right) \cap M.$$

Lemma 1.6. *Let* M *be a closed, nowhere dense subset of an* M_3*-space* X *with property* (δ) *and let* $\mathcal{B}$ *be a CP family of closed subsets of* X. *Then there exist families* $\{\mathcal{W}(B) \mid B \in \mathcal{B}\}$ *of subsets of* X *satisfying the following:*

(i) $\mathcal{W} = \bigcup\{\mathcal{W}(B) \mid B \in \mathcal{B}\}$ *is a CP family of closed subsets of* X;
(ii) *if* $B \subset O$, *where* $B \in \mathcal{B}$, $O \in \tau(X)$, *then there exists* $W \in \mathcal{W}(B)$ *such that* $B \subset W \subset O$;
(iii) *for each* $W \in \mathcal{W}$,

$$\overline{\mathrm{Int}(W \setminus M)} \cap M = W \cap M.$$

The next corollary shows how the properties (P) and (δ) are related to the M_3 vs. M_1 problems:

Corollary 1.2. *TFAE:*

(i) *All* M_3*-spaces are* M_1.
(ii) *All* M_3*-spaces belong to* $\mathcal{P}$.
(iii) *All spaces in* $\mathcal{P}$ *satisfy property (P).*
(iv) *All* M_3*-spaces satisfy property* (δ).

We consider what kind of spaces satisfy property (δ). To settle one possibility, we introduce the notion of δ-order, which is one variation of sequential order.

Definition 1.8. Let A be a subset of a space X. We introduce a operator $[\cdot]$ as follows:

$$[A] = \left\{p \in X \;\middle|\; \text{there exists } B \subset A \text{ such that } \overline{B} = B \cup \{p\}\right\}.$$

Let $A_0 = A$ and suppose A_β, $\beta < \alpha$, are defined. Then we define A_α as follows:
If α is a limit ordinal, then

$$A_\alpha = \bigcup\{A_\beta \mid \beta < \alpha\},$$

and if α is isolated, then

$$A_\alpha = [A_{\alpha-1}].$$

Define $\delta(X)$ to be the least α such that $\overline{A} = A_\alpha$ for any $A \subset X$ and call it the *δ-order of X*.

We remark two points.

Remark 1.1. If a space X satisfies property $(*)$, then $\overline{O} = [O]$ for any $O \in \tau(X)$.

Remark 1.2. If a space has the sequential order, then it has the δ-order. But the converse is not true.

We can show that every sequential M_3-space has property (δ). More strictly, we have the following:

Theorem 1.7. *If a space X is an M_3-space with the δ-order $\delta(X)$, then X satisfies property (δ); consequently $X \in \mathcal{P}$.*

Corollary 1.3. *If X is a k-M_3-space, then $X \in \mathcal{P}$.*

At this stage, this is the strongest result to the M_3 vs. M_1 problem among the known results.

To characterize spaces with the δ-order, we introduce the following definition:

Definition 1.9. A space X is a *δ-space* when a subset F is closed in X if and only if F has the property that $B \subset F$ and $\overline{B} = B \cup \{p\}$ imply $p \in F$. A space L is an *almost discrete space* if all points of L except one point are isolated in L.

Theorem 1.8. *TFAE:*

(i) *X is a δ-space.*
(ii) *X has the δ-order.*
(iii) *X is the image of $\oplus\{L_\lambda \mid \lambda \in \Lambda\}$ with each L_λ almost discrete under a quotient mapping φ such that $\varphi(L_\lambda)$ is a closed δ-space for each $\lambda \in \Lambda$.*

Theorem 1.9. *For a pointwise perfect space X, TFAE:*

(i) *X is a δ-space.*
(ii) *X has the δ-order.*
(iii) *X is the image of $\oplus\{L_\lambda \mid \lambda \in \Lambda\}$ with each L_λ an almost discrete, M_0-space under a quotient mapping φ such that $\varphi(L_\lambda)$ is a closed δ-space for each $\lambda \in \Lambda$.*

2 The diagrams

$$
\begin{tikzcd}[column sep=small]
 & & & & \text{Baire, Fréchet, } M_3\text{-space} \arrow[ddd] & \\
 & & \text{Lašnev space} \arrow[d] \arrow[ddrr] & & & \text{Nagata space} \arrow[ddl] \\
M_3,\ \sigma\text{-discrete space} \arrow[dd] & & L\text{-space} \arrow[dd] & & & \\
 & & & & \text{Fréchet } M_3\text{-space} \arrow[dd] & \\
M_3,\ F_\sigma\text{-metrizable space} \arrow[dr] & & \text{free } L\text{-space} \arrow[dl] & & & \\
 & M_3,\ \mu\text{-space} \arrow[d] & & & \text{sequential } M_3\text{-space} \arrow[ddl] & \\
 & \text{perfect image of an } M_0\text{-space} \arrow[drr] & & & & \\
 & & & M_3\text{-space with property (P)} \arrow[d] & & \\
 & & & M_1\text{-space} & &
\end{tikzcd}
$$

Diagram 1

$$
\begin{tikzcd}
\text{Nagata space} \arrow[d] & \\
\text{sequential } M_3\text{-space} \arrow[r] \arrow[d] & M_3\text{-space with property (P)} \arrow[d] \\
M_3\text{-space with the } \delta\text{-order} \arrow[r] \arrow[d, shift right] & M_3\text{-space with property } (\delta) \arrow[d, shift right] \\
M_3\text{-}\delta\text{-space} \arrow[r] \arrow[u, shift right] & \mathcal{P} \arrow[u, shift right]
\end{tikzcd}
$$

Diagram 2

3 (Added to the talk)

We list up here other open problems that are equivalent or related to the M_3 vs. M_1 problem. They are already proposed elsewhere by many researchers.

(P1) Is any M_3-space M_1?
(P2) Is any (closed) subspace of an M_1-space M_1?
(P3) Is any closed (perfect) image of an M_1-space M_1?
(P4) Does any point of an M_1-space have a CP open neighborhood base?
(P5) Is any adjunction space $X \cup_f Y$ for M_1-spaces X and Y M_1?
(P6) Is any M_1-space the perfect image of an M_1-space with $\mathrm{Ind} = 0$?
(P7) Is any M_3-space the perfect image of an M_3-space with $\mathrm{Ind} = 0$?
(P8) Does $\mathcal{EM}_3 \subset \{M_1\text{-spaces}\}$?
(P9) Does $\{M_1\text{-spaces}\} \subset \mathcal{EM}_3$?
(P10) For any M_1-space X, can we characterize $\dim X \leq n$ or $\mathrm{Ind}\, X \leq n$ by the fact that there exists a σ-CP base $\mathcal{W}$ for X such that $\dim \partial W \leq n-1$ or $\mathrm{Ind}\, \partial W \leq n-1$, respectively, for any $W \in \mathcal{W}$?
(P11) If an M_1-space X has $\mathrm{Ind} = 0$, then is X an M_0-space?
(P12) Does any M_3-space have an M-structure?
(P13) Does there exist a subclass $\mathcal{C}$ of M_1-spaces satisfying the following topological operations:
 (i) $\mathcal{C}$ is hereditary;
 (ii) $\mathcal{C}$ is countably productive;
 (iii) $\mathcal{C}$ is preserved by closed mappings?

Finally, we give a diagram among all the problems stated here, where A $\longrightarrow$ B means that if A is positively solved, then so is B.

$$
\begin{tikzcd}
\text{(P1)} \arrow[r, shift left] & \text{(P2)} \arrow[l, shift left] \arrow[r, shift left] & \text{(P3)} \arrow[l, shift left] \arrow[r] & \text{(P4)} \arrow[r] & \text{(P5)} \\
\text{(P7)} & \text{(P12)} \arrow[l] \arrow[u] \arrow[r] \arrow[d] \arrow[dr] & \text{(P6)} \arrow[r] & \text{(P9)} \\
\text{(P11)} & \text{(P10)} \arrow[l] & \text{(P8)} \arrow[r] & \text{(P13)}
\end{tikzcd}
$$

Diagram 3

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