Extensions of partitions of unity and covers

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1. Introduction

By a space we mean a topological space and $\gamma$ denotes an infinite cardinal. Let $X$ be a space and $A$ a subspace of $X$. By Shapiro [13], $A$ is said to be $P^\gamma$-embedded in $X$ if every $\gamma$-separable continuous pseudo-metric on $A$ can be extended to a continuous pseudo-metric on $X$. A subspace $A$ is said to be $P$-embedded in $X$ if $A$ is $P^\gamma$-embedded in $X$ for every $\gamma$. Recently, Dydak [5] defined that $A$ is $P^\gamma$(locally-finite)-embedded in $X$ if for every locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on $A$ with $|\Omega| \leq \gamma$, there exists a locally finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on $X$ such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$. A subspace $A$ is said to be $P$(locally-finite)-embedded in $X$ if $A$ is $P^\gamma$(locally-finite)-embedded in $X$ for every $\gamma$.

It was proved in [5] that $P^\gamma$(locally-finite)-embedding implies $P^\gamma$-embedding. This fact is also verified from characterizations of $P^\gamma$-embedding and $P^\gamma$(locally-finite)-embedding as the following. On Theorem 1.1, (1) $\iff$ (2) is well-known (cf. [1]), and (1) $\iff$ (3) is in [5] or [11].

**Theorem 1.1** ([1], [5], [11]). For a space $X$ and a subspace $A$ of $X$, the following statements are equivalent:

1. $A$ is $P^\gamma$-embedded in $X$;
2. for every locally finite cozero-set cover $\{U_\alpha : \alpha \in \Omega\}$ of $A$ with $|\Omega| \leq \gamma$, there exists a locally finite cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of $X$ such that $V_\alpha \cap A \subseteq U_\alpha$ for every $\alpha \in \Omega$;
3. for every locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on $A$ with $|\Omega| \leq \gamma$, there exists a (not necessarily locally finite) partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on $X$ such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$.

**Theorem 1.2** ([14]). For a space $X$ and a subspace $A$ of $X$, the following statements are equivalent:

1. $A$ is $P^\gamma$(locally-finite)-embedded in $X$;
2. for every locally finite cozero-set cover $\{U_\alpha : \alpha \in \Omega\}$ of $A$ with $|\Omega| \leq \gamma$, there exists a locally finite cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of $X$ such that $V_\alpha \cap A = U_\alpha$ for every $\alpha \in \Omega$.

Notice that the space $Z$ given in [11, Example 3] admits a $P$- but not $P^\omega$(locally-finite)-embedded subspace (cf. [14]).
The first purpose of this talk is to characterize $P^\gamma$-embedding under the viewpoint of exactly extending cozero-set covers such as in Theorem 1.2. The second one is to investigate for $P^\omega$(point-finite)-embedding (see Section 3 for the definition) under the same viewpoint to Theorem 1.2, and apply it to prove that the rationals $\mathbb{Q}$ of the Michael line $\mathbb{R}_\mathbb{Q}$ is not $P^\omega$(point-finite)-embedded in $\mathbb{R}_\mathbb{Q}$.

A collection $\{f_\alpha : \alpha \in \Omega\}$ of continuous functions $f_\alpha : X \to [0,1]$, $\alpha \in \Omega$, is said to be a partition of unity on $X$ if $\sum_{\alpha \in \Omega} f_\alpha(x) = 1$ for every $x \in X$. A partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on $X$ is said to be locally finite (resp. point-finite [5], or uniformly locally finite) if $\{f_\alpha^{-1}((0,1]) : \alpha \in \Omega\}$ is locally finite (resp. point-finite, or uniformly locally finite) in $X$. Here, a collection $\mathcal{F}$ of subsets of $X$ is said to be uniformly locally finite (resp. uniformly discrete) in $X$ if there exists a normal open cover $\mathcal{U}$ of $X$ such that every $U \in \mathcal{U}$ meets at most finitely many members (resp. at most one member) of $\mathcal{F}$ ([9], [10], [3]).

2. Exact extensions of cozero-set covers and $P$-embedding

Our main result in this section is the following; Alô-Shapiro proved in [1] the equivalence (1) $\Leftrightarrow$ (3) assuming that $X$ is normal and $A$ is closed in $X$.

**Theorem 2.1 (Main).** For a space $X$ and a subspace $A$ of $X$, the following statements are equivalent:

(1) $A$ is $P^\gamma$-embedded in $X$;

(2) for every uniformly locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on $A$ with $|\Omega| \leq \gamma$, there exists a uniformly locally finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on $X$ such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$;

(3) for every uniformly locally finite cozero-set cover $\{U_\alpha : \alpha \in \Omega\}$ of $A$ with $|\Omega| \leq \gamma$, there exists a uniformly locally finite cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of $X$ such that $V_\alpha \cap A = U_\alpha$ for every $\alpha \in \Omega$.

We apply Theorem 2.1 to give another characterization of $P$-embedding by exactly extending zero-set collections. Blair [3] essentially proved that: A subspace $A$ of a space $X$ is $P^\gamma$-embedded in $X$ if and only if for every uniformly discrete zero-set collection $\{Z_\alpha : \alpha \in \Omega\}$ of $A$ with $|\Omega| \leq \gamma$, there exists a uniformly discrete zero-set collection $\{F_\alpha : \alpha \in \Omega\}$ of $X$ such that $F_\alpha \cap A = Z_\alpha$ for every $\alpha \in \Omega$. In our case, we give the following:

**Theorem 2.2.** For a space $X$ and a subspace $A$ of $X$, the following statements are equivalent:

(1) $A$ is $P^\gamma$-embedded in $X$;
(2) every uniformly locally finite zero-set collection \( \{ Z_\alpha : \alpha \in \Omega \} \) of \( A \) with \( |\Omega| \leq \gamma \), there exists a uniformly locally finite zero-set collection \( \{ F_\alpha : \alpha \in \Omega \} \) of \( X \) such that \( F_\alpha \cap A = Z_\alpha \) for every \( \alpha \in \Omega \).

As another application of Theorem 2.1, we give some results concerning locations of spaces around functionally Katětov spaces. Let \( \gamma, \kappa \) be infinite cardinals. In [15], a space \( X \) is said to be \( (\gamma, \kappa) \)-Katětov if \( X \) is normal and for every closed subspace \( A \) of \( X \) and every locally finite \( \kappa^+ \)-open cover \( \{ U_\alpha : \alpha < \gamma \} \) of \( A \), there exists a locally finite \( \kappa^+ \)-open cover \( \{ V_\alpha : \alpha < \gamma \} \) of \( X \) such that \( V_\alpha \cap A = U_\alpha \) for every \( \alpha < \gamma \). Here, a subspace \( U \) of \( X \) is said to be \( \kappa^+ \)-open set if \( U \) can be expressed as the union of \( \kappa \) many cozero-sets of \( X \). When \( X \) is \( (\gamma, \omega) \)-Katětov for every \( \gamma \), \( X \) is said to be functionally Katětov (cf. [7], [11], [15]). Similarly, when \( X \) is \( (\gamma, \kappa) \)-Katětov for every \( \gamma \) and \( \kappa \) (resp. \( (\omega, \kappa) \)-Katětov for every \( \kappa \), or \( (\omega, \omega) \)-Katětov), \( X \) is said to be Katětov (resp. countably Katětov, or countably functionally Katětov).

Note that \( \gamma \)-collectionwise normal countably paracompactness implies being \( (\gamma, \kappa) \)-Katětov, and the latter implies \( \gamma \)-collectionwise normality (cf. [7], [15]). Moreover they were proved in [11] that every hereditarily normal space is countably Katětov, and that Rudin's Dowker space is functionally Katětov but not countably Katětov. In [11], they were essentially proved that every collectionwise normal \( P \)-space is functionally Katětov and that every normal \( P \)-space is countably functionally Katětov; here a space is said to be a \( P \)-space if every cozero-set is closed. A space \( X \) is said to be **hereditarily basically disconnected** if for every subspace \( A \) of \( X \), the closure of a cozero-set of \( A \) in \( A \) is open in \( A \).

With the aid of Theorem 2.1, we slightly generalize the result mentioned above in the following:

**Lemma 2.3.** Let \( X \) be a \( \gamma \)-collectionwise normal space. Assume that for every closed subspace \( A \) of \( X \), every locally finite \( \kappa^+ \)-open cover, with card \( \leq \gamma \), of \( A \) is uniformly locally finite in \( A \). Then, \( X \) is \( (\gamma, \kappa) \)-Katětov.

Hence we have:

**Theorem 2.4.** Every \( \gamma \)-collectionwise normal and hereditarily basically disconnected space is \( (\gamma, \omega) \)-Katětov.

It also follows from Lemma 2.3 that: If \( X \) is a collectionwise normal and hereditarily extremally disconnected space, then \( X \) is Katětov; where \( X \) is said to be **hereditarily extremally disconnected** if for every subspace \( A \) of \( X \), the closure of an open set of \( A \) in \( A \) is open in \( A \). The author does not know the assumption of \( X \) above implies countable paracompactness of \( X \).
3. \(P\text{(point-finite)}\)-embeddings and covers

Let \(X\) be a space and \(A\) a subspace of \(X\). On exactly extending partitions of unity, consider the following conditions:

(i) for every partition of unity \(\{f_\alpha : \alpha \in \Omega\}\) on \(A\) with \(|\Omega| \leq \gamma\), there exists a partition of unity \(\{g_\alpha : \alpha \in \Omega\}\) on \(X\) such that \(g_\alpha|A = f_\alpha\) for every \(\alpha \in \Omega\);

(ii) for every point-finite partition of unity \(\{f_\alpha : \alpha \in \Omega\}\) on \(A\) with \(|\Omega| \leq \gamma\), there exists a point-finite partition of unity \(\{g_\alpha : \alpha \in \Omega\}\) on \(X\) such that \(g_\alpha|A = f_\alpha\) for every \(\alpha \in \Omega\);

(iii) for every locally finite partition of unity \(\{f_\alpha : \alpha \in \Omega\}\) on \(A\) with \(|\Omega| \leq \gamma\), there exists a locally finite partition of unity \(\{g_\alpha : \alpha \in \Omega\}\) on \(X\) such that \(g_\alpha|A = f_\alpha\) for every \(\alpha \in \Omega\);

(iv) for every uniformly locally finite partition of unity \(\{f_\alpha : \alpha \in \Omega\}\) on \(A\) with \(|\Omega| \leq \gamma\), there exists a uniformly locally finite partition of unity \(\{g_\alpha : \alpha \in \Omega\}\) on \(X\) such that \(g_\alpha|A = f_\alpha\) for every \(\alpha \in \Omega\).

Dydak proved in [5] that (i) equals that \(A\) is \(P^\gamma\)-embedded in \(X\), and Theorem 2.1 shows that (iv) also equals that \(A\) is \(P^\gamma\)-embedded in \(X\). The condition (iii) is precisely the definition of \(P^\gamma\text{(locally-finite)}\)-embedding; as was already commented in the introduction, (iii) is strictly stronger than the \(P^\gamma\)-embedding. By Dydak [5], the above condition (ii) is said to be that \(A\) is \(P^\gamma\text{(point-finite)}\)-embedded in \(X\) and it is proved in [5] that this condition is also strictly stronger than the \(P^\gamma\)-embedding (cf. Theorem 3.4 below).

Recall Theorem 1.2 and (2) \(\Leftrightarrow\) (3) of Theorem 2.1. Then, we see that \(P^\gamma\)-embedding and \(P^\gamma\text{(locally-finite)}\)-embedding can be stated by extensions of cozero-set covers as well as extensions of partitions of unity. On the other hand, for \(P^\gamma\text{(point-finite)}\)-embedding, we have the following theorem and examples.

**Theorem 3.1 (Main).** For a space \(X\) and a subspace of \(A\), the following statements are equivalent:

1. \(A\) is \(P^\omega\text{(point-finite)}\)-embedded in \(X\);

2. for every point-finite countable cozero-set cover \(\{U_n : n \in \mathbb{N}\}\) of \(A\), there exists a point-finite countable cozero-set cover \(\{V_n : n \in \mathbb{N}\}\) of \(X\) such that \(V_n \cap A = U_n\) for every \(n \in \mathbb{N}\).

The following examples show that Theorem 3.1 need not hold on uncountable cardinal cases.

**Example 3.2.** Let \(\gamma\) be an uncountable cardinal. There exist a space \(X\) and a closed subspace \(A\) of \(X\) such that every point-finite cozero-set cover
of $A$ can be extended to a point-finite cozero-set cover of $X$, but $A$ is not $P^\gamma$-embedded in $X$.

Sketch of the construction. We use notations as in [2] and [8]. In particular, we assume the uncountable set $P$ in [2] as $|P| = \gamma$. Let $F$, $f_p$ and $F_p$ be the same as in [2]. Let $G$ be the space in [8], namely,

$$G = F_p \cup \{ f \in F : f(q) = 0 \text{ except for finitely many } q \in Q \}.$$

Consider the space introduced in the last part of [8, Example 2] and denote it $X$, namely,

$$X = (F_p \times \{0\}) \cup (G \times \{1/i : i \in \mathbb{N}\})$$

taking as a base at a point $(y, 0)$ the sets $\{(y, 0)\} \cup (U \times \{1/i : i \geq j\})$, where $U$ is a neighborhood of $y$ in $G$ and $j \in \mathbb{N}$, and other points be isolated. Let $A = F_p \times \{0\}$.

**Example 3.3.** There exist a space $X$ and a closed subspace $A$ of $X$ such that $A$ is $P(\text{point-finite})$-embedded in $X$, but that $A$ has a point-finite cozero-set cover which can not be extended to a cozero-set cover of $X$.

Sketch of the construction. Consider the product space $Z = L(\omega_1) \times (\omega + 1) \times (\omega_2 + 1)$, where $L(\omega_1)$ is the set $\omega_1 + 1$ taking a base at the point $\omega_1$ as $\{[\beta, \omega_1] : \beta < \omega_1\}$ and other points be isolated; and $\omega + 1$ and $\omega_2 + 1$ have the usual order topology. Let $X = Z - \{(\omega_1, \omega, \omega_2)\}$ and $A = L(\omega_1) \times (\omega + 1) \times \{\omega_2\} - \{(\omega_1, \omega, \omega_2)\}$ a subspace of $X$.

We give an application of Theorem 3.1. Let $\mathbb{R}_Q$ be the Michael line and $Q$ be the rationals. Dydak commented in [5] that "we do not know if $Q$ is $P(\text{point-finite})$-embedded in $\mathbb{R}_Q$" and constructed his own example of a $P$-embedding which is not $P(\text{point-finite})$-embedding. Answering his question, we have the following:

**Theorem 3.4.** $Q$ is not $P^\gamma(\text{point-finite})$-embedded in $\mathbb{R}_Q$.

Finally we give a result that three extension properties equal under a condition only for the subspace $A$.

**Theorem 3.5.** Let $X$ be a space, $A$ a subspace of $X$ and $\gamma$ an infinite cardinal. If $A$ is a $P$-space, then the following statements are equivalent:

1. $A$ is $P^\gamma$-embedded in $X$;
2. $A$ is $P^\gamma(\text{locally-finite})$-embedded in $X$;
3. $A$ is $P^\gamma(\text{point-finite})$-embedded in $X$.

Note that every closed subspace of Rudin's Dowker space is $P(\text{point-finite})$-embedded; it can be proved by combining some results in [5], [6] and [12]. This fact can also be seen by the above theorem directly.
References


