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For arbitrary complex numbers \(x, y, z\), the inequality
\[
|x + y| + |y + z| + |z + x| \leq |x| + |y| + |z| + |x + y + z|
\]
is well known as Hlawka's inequality. Djoković [2] proved the following inequalities which contain the above one as a special case:

(1) \[
\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |x_{i_1} + \cdots + x_{i_k}| \leq \left(\frac{n - k}{k - 1}\right) \sum_{i=1}^{n} |x_i| + \left|\sum_{j=1}^{n} x_j\right|
\]
for complex numbers \(x_1, \ldots, x_n\) and \(2 \leq k \leq n\).

In this paper, we pay attention to the special case \(k = n - 1\) of (1), namely

(2) \[
\|a - \text{Tr } a\|_1 \leq \|a\|_1 + (n - 2)|\text{Tr } a|,
\]
for a vector \(a = (a_1, \ldots, a_n)\) in \(\mathbb{C}^n (n \geq 2)\), where \(\|\cdot\|_1\) is \(\ell_1\) norm on \(\mathbb{C}^n\) and \(\text{Tr } a = \sum_{i=1}^{n} a_i\). A weighted extension of the inequality (2) is known and is stated as follows:

**Proposition 1.** Let \(\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0\) and \(x_1, x_2, \ldots, x_n \in \mathbb{C} (n \geq 2)\). Then

(3) \[
\sum_{i=1}^{n} \alpha_i |x_i - \sum_{j=1}^{n} \alpha_j x_j| \leq \beta \sum_{i=1}^{n} \alpha_i |x_i| + (\sum_{i=1}^{n} \alpha_i - 2\alpha) \sum_{j=1}^{n} \alpha_j x_j|,
\]
where \(\alpha = \min\{\alpha_i : \alpha_i > 0\}\) and \(\beta = \max\{2\alpha - 1, 1\}\).

In view of the inequality (2), it might be natural to write the matrix version of Hlawka's inequality as follows:

(4) \[
\|A - \text{Tr } A\|_1 \leq \|A\|_1 + (n - 2)|\text{Tr } A|,
\]
for a complex \(n \times n\) matrix \(A\), where \(\text{Tr } A\) is the trace of \(A\) and \(\|A\|_1\) is the trace norm of \(A\), i.e. \(\|A\|_1 = \text{Tr } |A|\) with \(|A| = (A^*A)^{1/2}\). In this paper we prove the inequality (4) in a more general form stated in (5) of the following theorem. Indeed, the inequality (5) is not only a matrix extension of (3) but also a weighted extension of (4).

**Theorem 2.** Let \(n \in \mathbb{N}\) with \(n \geq 2\) and let \(A, B\) be complex \(n \times n\) matrices. If \(AB = BA\) and \(B \geq 0\), then

(5) \[
\text{Tr}(B |A - \text{Tr } BA|) \leq \max\{2\gamma(B) - 1, 1\} \text{Tr } B|A| + (\text{Tr } B - 2\gamma(B)) |\text{Tr } BA|,
\]
where \(\gamma(B)\) denotes the minimum positive eigenvalue of \(B\).
**Proof of Theorem 2**

**Proof.** Since $B \geq 0$ and $AB = BA$, we can write $B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with invertible $B_1$ after some unitary conjugation. So it is enough to assume that $B$ is invertible.

By approximation, we may assume that $\text{Tr} B \neq 1$. If we put $A_\epsilon = A + \epsilon I$, then $A_\epsilon - \text{Tr} B A_\epsilon = A - \text{Tr} BA + \epsilon(1 - \text{Tr} B)$ is invertible for small $\epsilon > 0$. So it suffices to prove the case where $A - \text{Tr} BA$ is invertible. Since $AB = BA$, there is a unique unitary matrix $U$ such that $U(A - \text{Tr} BA) = |A - \text{Tr} BA|$ and $UB = BU$. Hence there exists a unitary matrix $V$ and diagonal matrices $D_B$ and $D_U$ such that

$$B = V^* D_B V, \quad U = V^* D_U V.$$ 

So we have

$$\text{Tr}(B|A - \text{Tr} BA|) = \text{Tr}(BU(A - \text{Tr} BA))$$

$$= \text{Tr}(V^* D_B D_U V(A - \text{Tr} V^* D_B V A))$$

$$= \text{Tr}(D_B D_U(V A V^* - \text{Tr} D_B V A V^*)).$$

In this way, we can suppose that $B = \text{diag}(b_1, b_2, \ldots, b_n)$ with $b_i > 0$ and $U = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$. Then, by using Proposition 1, we have

$$\text{Tr}(B|A - \text{Tr} BA|) = \left| \sum_{i=1}^{n} e^{i\theta_i} b_i (A - \text{Tr} BA)_{ii} \right|$$

$$\leq \sum_{i=1}^{n} b_i |a_{ii} - b_j a_{jj}|$$

$$\leq \max\{2 \min b_i - 1, 1\} \sum_{i=1}^{n} b_i |a_{ii}| + (\sum_{i=1}^{n} b_i - 2 \min b_i) |\sum_{j=1}^{n} b_j a_{jj}|$$

$$\leq \max\{2 \gamma(B) - 1, 1\} \sum_{i=1}^{n} b_i |a_{ii}| + (\text{Tr} B - 2 \gamma(B)) |\text{Tr} BA|. $$

Moreover, note that $B$ commutes with $A$ and $A^*$ so that $|BA| = B|A|$. Since

$$\sum_{i=1}^{n} |\langle X e_i, e_i \rangle| \leq \text{Tr} |X|$$

for a general matrix $X$ with the canonical basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$, it follows that

$$\sum_{i=1}^{n} b_i |a_{ii}| = \sum_{i=1}^{n} |\langle B A e_i, e_i \rangle| \leq \text{Tr} |BA| = \text{Tr} |B| A|. $$

Therefore, the desired inequality (5) is obtained. \(\square\)

**Remark.** The inequality (5) fails to hold for some non-commuting pairs of matrices. For example, choose

$$A = \begin{pmatrix} 1 & 0.5 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}. $$
Corollary 3. For every $n \in \mathbb{N}$ with $n \geq 2$ and every complex $n \times n$ matrix $A$,

\begin{equation}
\|A - \text{Tr } A\|_1 \leq \|A\|_1 + (n-2)|\text{Tr } A| \tag{6}
\end{equation}

and

\begin{equation}
\|A\|_1 \leq \|A - \frac{1}{n-1} \text{Tr } A\|_1 + \frac{n-2}{n-1}|\text{Tr } A|. \tag{7}
\end{equation}

Proof. The inequality (6) is a specialization of (5) to the case $B = I$. Replace $A$ by $A - \frac{1}{n-1} \text{Tr } A$ in (6) to obtain the inequality (7). \hfill \Box

Let $A$ be a complex $n \times n$ matrix with $n \geq 2$, and consider the function $f(t) = \|A - t \text{Tr } A\|_1$ for $t \geq 0$. When $0 < t \leq 1$, it follows from the convexity of $f$ and (6) that

\begin{equation}
\frac{f(t) - f(0)}{t} \leq \frac{f(1) - f(0)}{1} \leq (n-2)|\text{Tr } A|.
\end{equation}

Therefore,

\begin{equation}
\|A - t \text{Tr } A\|_1 \leq \|A\|_1 + t(n-2)|\text{Tr } A| \quad (0 \leq t \leq 1).
\end{equation}

This inequality is also obtained by putting $B = tI$ in (5). Similarly, from the convexity of $f$ and (7), we can show that

\begin{equation}
\|A\|_1 \leq \|A - t \text{Tr } A\|_1 + t(n-2)|\text{Tr } A| \quad (t \geq \frac{1}{n-1}).
\end{equation}

3. Related inequalities

Let $M_n(\mathbb{C})$ be the space of complex $n \times n$ matrices. For $1 \leq p < \infty$ let $\|A\|_p$ denote the Schatten $p$-norm of $A \in M_n(\mathbb{C})$, i.e. $\|A\|_p = (\text{Tr } |A|^p)^{1/p}$. Also, the operator norm of $A$ is denoted by $\|A\|_\infty$.

In this section, we discuss some inequalities comparing $\|A - \text{Tr } A\|_p$ with $\|A\|_p$. To get such inequalities, we introduce some norms $\| \cdot \|_{(1,1)}$ and $\| \cdot \|_\infty$ on $M_n(\mathbb{C})$ and determine their dual norms.

Define the norm $\| \cdot \|_{(1,1)}$ on $M_n(\mathbb{C})$ by

\begin{equation}
\|A\|_{(1,1)} = \|A - \text{Tr } A\|_1 \quad (A \in M_n(\mathbb{C})).
\end{equation}

Note that the linear mapping $\Phi(A) = A - \text{Tr } A$ on $M_n(\mathbb{C})$ has the inverse $\Phi^{-1}(A) = A - \frac{1}{n-1} \text{Tr } A$, so $\| \cdot \|_{(1,1)}$ is actually a norm on $M_n(\mathbb{C})$. Consider the canonical duality $\langle A, B \rangle = \text{Tr } AB$ for $A, B \in M_n(\mathbb{C})$. Then we have

\begin{equation}
\max\{\|\langle A, B \rangle\|_{(1,1)} \leq 1\} = \max\{\|\langle A, B \rangle\| : \|A - \text{Tr } A\|_1 \leq 1\}
= \max\{\|\langle A - \frac{1}{n-1} \text{Tr } A, B \rangle\| : \|A\|_1 \leq 1\}
= \max\{\|\langle A, B - \frac{1}{n-1} \text{Tr } B \rangle\| : \|A\|_1 \leq 1\}
= \|B - \frac{1}{n-1} \text{Tr } B\|_\infty.
\end{equation}
This says that the dual norm of $\| \cdot \|_{(1, 1)}$ on $M_n(\mathbb{C})$ is equal to

\[(8) \quad \|A\|^{*}_{(1, 1)} = \|A - \frac{1}{n-1} \text{Tr } A\|_{\infty}.
\]

We next define the norm $\| \cdot \|$ on $M_n(\mathbb{C})$ by

\[
\|A\| = \|A\|_1 + (n - 2)\|\text{Tr } A\| \quad (A \in M_n(\mathbb{C})).
\]

Then the inequality (4) is rewritten as $\|A\|_{(1, 1)} \leq \|A\|$, so the dual form of (4) is given as

\[(9) \quad \|A\|^{*} \leq \|A\|^{*}_{(1, 1)},
\]

where $\| \cdot \|^{*}$ is the dual norm of $\| \cdot \|$ with respect to the canonical duality. Since $\|A\|^{*}_{(1, 1)} = \|A - \frac{1}{n-1} \text{Tr } A\|_{\infty}$ by (8), the above (9) is equivalent to

\[(10) \quad \|A - \text{Tr } A\|^{*} \leq \|A\|_{\infty}.
\]

Now let us determine the dual norm $\| \cdot \|^{*}$. To do so, define a semi-norm on the direct sum $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ by

\[
\|X \oplus Y\|_1 = \|X\|_1 + (n - 2)\|\text{Tr } Y\| \quad (X, Y \in M_n(\mathbb{C})).
\]

Clearly, the mapping $X \mapsto X \oplus X$ is isometric from $(M_n(\mathbb{C}), \| \cdot \|)$ into $(M_n(\mathbb{C}) \oplus M_n(\mathbb{C}), \| \cdot \|_1)$, so that $A \in M_n(\mathbb{C})$ satisfy $\|A\|^{*} \leq 1$. Then the norm of the functional $X \mapsto \langle X, A \rangle$ on the subspace $\{X \oplus X : X \in M_n(\mathbb{C})\}$ of $(M_n(\mathbb{C}) \oplus M_n(\mathbb{C}), \| \cdot \|_1)$ is equal to $\|A\|^{*}$. By the Hahn-Banach extension theorem, this can be extended to a linear functional $\varphi$ on $(M_n(\mathbb{C}) \oplus M_n(\mathbb{C}), \| \cdot \|_1)$ which has the norm $\leq 1$, namely

\[(11) \quad |\varphi(X \oplus Y)| \leq \|X \oplus Y\|_1 = \|X\|_1 + (n - 2)\|\text{Tr } Y\|.
\]

On the other hand, it is obvious that we can choose matrices $B, C \in M_n(\mathbb{C})$ such that $\varphi(X \oplus Y) = \langle X, B \rangle + \langle Y, C \rangle$. Then (11) implies that $B$ and $C$ must satisfy $\|B\|^{*}_{1} = \|B\|_{\infty} \leq 1$ and $|\langle Y, C \rangle| \leq (n - 2)\|\text{Tr } Y\|$. The latter implies $C = \zeta I$ for some $|\zeta| \leq n - 2$, so we have $\langle X, A \rangle = \langle X, B \rangle + \langle X, \zeta I \rangle$ for every $X \in M_n(\mathbb{C})$. This means that $A = B + \zeta I$ for some $|\zeta| \leq n - 2$. Conversely, it is immediate to see that if $A = B + \zeta I$ with $\|B\|_{\infty} \leq 1$ and $|\zeta| \leq n - 2$, then $\|A\|^{*} \leq 1$. In this way, we conclude the following proposition.

**Proposition 4.**

\[
\{A \in M_n(\mathbb{C}) : \|A\|^{*} \leq 1\} = \{X + \zeta I : |\zeta| \leq n - 2, X \in M_n(\mathbb{C}), \|X\|_{\infty} \leq 1\}.
\]

Put $t := \|A - \text{Tr } A\|^{*}$. From the above proposition, we have

\[
\frac{1}{t}(A - \text{Tr } A) - \zeta \|_{\infty} \leq 1,
\]
for some $\zeta$ with $|\zeta| \leq n - 2$. So we have
\[
\min_{|\zeta| \leq t} \|A - \text{Tr} A - (n - 2)\zeta\|_{\infty} \leq \||A - \text{Tr} A||^{*}
\]
Since the inequality (4) can be rewritten as $t \leq \|A\|_{\infty}$, we have
\[
\min_{|\zeta| \leq \|A\|_{\infty}} \|A - \text{Tr} A - (n - 2)\zeta\|_{\infty} \leq \||A - \text{Tr} A||^{*}.
\]
This and (6) imply that
\[
\|A - \text{Tr} A\|_{p} \leq (n - 1)\|A\|_{p} \quad (p = 1, \infty)
\]
for all $A \in M_{n}(\mathbb{C})$. So the following is a consequence of the complex interpolation method (cf. [5, Appendix to IX.4]).

**Proposition 5.** For every complex $n \times n$ matrix $A$ with $n \geq 2$,

(12) \[
\|A - \text{Tr} A\|_{p} \leq (n - 1)\|A\|_{p} \quad (1 \leq p \leq \infty).
\]

As in the last part of Sect. 1 we can get

(13) \[
\|A - t \text{Tr} A\|_{p} \leq (tn - 2t + 1)\|A\|_{p} \quad (1 \leq p \leq \infty, \ 0 \leq t \leq 1)
\]

from (12) and the convexity of $t \mapsto \|A - t \text{Tr} A\|_{p}$. According to (8) the dual form of (13) is given as
\[
\|A - \frac{t}{tn - 1} \text{Tr} A\|_{q} \geq \frac{1}{tn - 2t + 1}\|A\|_{q}
\]
for $1 \leq q \leq \infty$ and $0 \leq t \leq 1$ with $t \neq 1/n$. Rewriting this we obtain
\[
\|A - t \text{Tr} A\|_{p} \geq \frac{tn - 1}{2tn - 2t - 1}\|A\|_{p} \quad (1 \leq p \leq \infty, \ t \geq \frac{1}{n-1}).
\]

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