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<th>On some classes of operators by Fujii and Nakamoto related to $p$-hyponormal and paranormal operators (Development of Operator Theory and Problems)</th>
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<td>Author(s)</td>
<td>Ito, Masatoshi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1189: 122-133</td>
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<td>Issue Date</td>
<td>2001-02</td>
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<td>URL</td>
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<td>Type</td>
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Kyoto University
On some classes of operators by Fujii and Nakamoto related to $p$-hyponormal and paranormal operators

東京理科大学 伊藤 公智 (Masatoshi Ito)
(Faculty of Science, Science University of Tokyo)

This report is based on the following paper:


**Abstract**

Recently, we introduced class A as a new class of operators in [18]. Class A is defined by an operator inequality, and also the definition of class A is similar to that of paranormality defined by a norm inequality. We showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal in [18]. As generalizations of class A and paranormality, class $A(p, r)$ was introduced in [11] and absolute-$(p, r)$-paranalormality was introduced in [30]. Moreover, Fujii-Nakamoto [12] introduced class $F(p, r, q)$ and $(p, r, q)$-paranormality which are further generalizations of these classes.

In this report, we shall show some inclusion relations among the families of class $F(p, r, q)$ and $(p, r, q)$-paranormality, and we shall show the result on powers of class $F(p, r, q)$ operators.

1 Introduction

In this report, a capital letter means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also an operator $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

As extensions of hyponormal operators, i.e., $T^*T \geq TT^*$, $p$-hyponormal operators for $p > 0$ defined by $(T^*T)^p \geq (TT^*)^p$ and log-hyponormal operators defined by $\log T^*T \geq \log TT^*$ for an invertible operator $T$ are well known. And also an operator $T$ is $p$-quasi-hyponormal for $p > 0$ if $T$ is $p$-hyponormal on $\overline{R(T)}$. It is easily obtained that every $p$-hyponormal operator is $q$-hyponormal for $p > q > 0$ by Löwner-Heinz theorem $"A \geq B \geq 0 \text{ ensures } A^\alpha \geq B^\alpha \text{ for any } \alpha \in [0,1],"$ and every invertible $p$-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function. We remark that log-hyponormality is sometimes regarded as $0$-hyponormality since $\frac{X^p-1}{p} \rightarrow \log X$ as $p \rightarrow +0$ for $X > 0$. 
An operator $T$ is paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. It has been studied by many authors, so there are too many to cite their references, for instance, [3] [13] [17] and [21]. Ando [3] showed that every $p$-hyponormal operator for $p > 0$ and log-hyponormal operator is paranormal.

Recently, in [18], we introduced class $A$ defined by $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every log-hyponormal operator belongs to class $A$ and every class $A$ operator is paranormal. It turns out that these results contain another proof of Ando's result stated above. We remark that class $A$ is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms.

And also we introduced two families of classes of operators based on class $A$ and paranormality in [18] as follows: An operator $T$ belongs to class $A(k)$ for $k > 0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$, and also an operator $T$ is absolute-$k$-paranormal for $k > 0$ if $\|T^kTx\| \geq \|Tx\|^{k+1}$ for every unit vector $x \in H$. Particularly an operator $T$ is a class $A$ (resp. paranormal) operator if and only if $T$ is a class $A(1)$ (resp. absolute-1-paranormal) operator. It was shown in [18] that the classes of invertible class $A(k)$ operators and absolute-$k$-paranormal operators constitute parallel and increasing lines, that is, invertible class $A(k) \subseteq$ invertible class $A(l)$ and absolute-$k$-paranormal $\subseteq$ absolute-$l$-paranormal for $0 < k \leq l$.

On the other hand, Fujii-Izumino-Nakamoto [7] introduced $p$-paranormality for $p > 0$ defined by $\|T^pU|T|^px\| \geq \|T^px\|^2$ for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of $T$. We remark that $1$-paranormality equals paranormality. As generalizations of class $A(k)$, absolute-$k$-paranormality and $p$-paranormality, Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto [11] introduced class $A(p,r)$ and Yamazaki-Yanagida [30] introduced absolute-$(p,r)$-paranormality as follows:

**Definition.**

(1) For each $p > 0$ and $r > 0$, an operator $T$ belongs to class $A(p,r)$ if

$$ (|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{p+r}} \geq |T|^2r, $$

and let class $AI(p,r)$ be the class of all invertible class $A(p,r)$ operators.

(2) For each $p > 0$ and $r > 0$, an operator $T$ is absolute-$(p,r)$-paranormal if

$$ \|T^p|T^*|^rx\|^r \geq \|T^*|^rx\|^{p+r} $$

(1.1)

for every unit vector $x \in H$.

It was pointed out that class $A(k,1)$ equals class $A(k)$ in [28]. And also, in [30], it was shown that absolute-$(k,1)$-paranormality equals absolute-$k$-paranormality and absolute-$(p,p)$-paranormality equals $p$-paranormality. Moreover class $AI(\frac{1}{2}, \frac{1}{2})$ equals the class
of invertible and $w$-hyponormal operators ($|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$ where $T = U|T|$ is the polar decomposition of $T$ and $\tilde{T} = |T|^\frac{1}{2} U |T|^\frac{1}{2}$) introduced by Aluthge-Wang [2]. We should remark that the families of class $A_I(p, r)$ determined by operator inequalities and absolute-$(p, r)$-paranormality determined by norm inequalities constitute two increasing lines on $p > 0$ and $r > 0$ whose origin is log-hyponormality (see section 2).

Moreover, as a continuation of the discussion in [11], Fujii-Nakamoto [12] introduced the following classes of operators.

**Definition.**

(1) For each $p > 0$, $r \geq 0$ and $q \geq 1$, an operator $T$ belongs to class $F(p, r, q)$ if

$$
(|T^*|^r |T|^{2p} |T^*|^r)^\frac{1}{q} \geq |T^*|^\frac{2(p+r)}{q}.
$$

(2) For each $p > 0$, $r \geq 0$ and $q > 0$, an operator $T$ is $(p, r, q)$-paranormal if

$$
|||T|^p U |T^*|^r x||^\frac{1}{q} \geq |||T^*||^\frac{p+r}{q} || x||
$$

for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of $T$.

We remark that class $F(p, r, \frac{p+r}{r})$ equals class $A(p, r)$, and we obtain that $(p, r, \frac{p+r}{r})$-paranormality equals absolute-$(p, r)$-paranormality in the next section. Thus many researchers have been discussed parallel families of classes of operators which are generalizations of class $A$ and paranormality.

In this report, firstly, we obtain more precise inclusion relations among the families of class $F(p, r, q)$ and $(p, r, q)$-paranormality from the view of monotonicity of class $A(p, r)$ and absolute-$(p, r)$-paranormality. Secondly, we give a characterization of log-hyponormal operators via class $F(p, r, q)$ and $(p, r, q)$-paranormality. Lastly, we obtain the result on powers of class $F(p, r, q)$ operators.

## 2 Background and preliminaries

Firstly, we obtain another expression of $(p, r, q)$-paranormality without using $U$ which appears in the polar decomposition of $T$, and it causes that $(p, r, \frac{p+r}{r})$-paranormality equals absolute-$(p, r)$-paranormality.

**Proposition 1.** For each $p > 0$, $r > 0$ and $q \geq 1$, an operator $T$ is $(p, r, q)$-paranormal if and only if

$$
|||T|^p |T^*|^r x||^\frac{1}{q} \geq |||T^*||^\frac{p+r}{q} || x||
$$

for every unit vector $x \in H$. 

Corollary 2. For each $p > 0$ and $r > 0$, $(p, r, \frac{p+r}{r})$-paranormality equals absolute-$(p, r)$-paranormality.

Next, to explain the background of the classes of operators discussed in this paper, we have to state the following celebrated order preserving operator inequality.

**Theorem F (Furuta inequality [14]).**

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^\frac{r}{2}A^pB^\frac{r}{2})^{\frac{1}{q}} \geq (B^\frac{r}{2}B^pB^\frac{r}{2})^{\frac{1}{q}}$$

and

(ii) $$(A^\frac{r}{2}A^pA^\frac{r}{2})^{\frac{1}{q}} \geq (A^\frac{r}{2}B^pA^\frac{r}{2})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

We remark that Theorem F yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F were given in [5] and [24] and also an elementary one page proof in [15]. It was shown in [25] that the domain drawn for $p, q$ and $r$ in the Figure 1 is the best possible one for Theorem F.

Fujii-Nakamoto [12] observed that class $F(p, r, q)$ derives from Theorem F and $(p, r, q)$-paranormality corresponds to class $F(p, r, q)$, and also they showed the following Theorem A.1.

**Theorem A.1 ([12]).**

(i) For a fixed $k > 0$, $T$ is $k$-hyponormal if and only if $T$ belongs to class $F(2kp, 2kr, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(1+2r)q \geq 2(p+r)$, i.e., $T$ belongs to class $F(p, r, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(k+r)q \geq p+r$.

(ii) If $T$ belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then $T$ belongs to class $F(p_0, r, q_0)$ for any $r \geq r_0$.

(iii) If $T$ belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then $T$ belongs to class $F(p_0, r_0, q)$ for any $q \geq q_0$.

(iv) If $T$ belongs to class $F(p, r, q)$ for $p > 0$, $r \geq 0$ and $q \geq 1$, then $T$ is $(p, r, q)$-paranormal.

(v) If $T$ is $(p_0, r_0, q_0)$-paranormal for $p_0 > 0$, $r_0 \geq 0$ and $q_0 > 0$, then $T$ is $(p_0, r_0, q)$-paranormal for any $q \geq q_0$. 
(vi) If $T$ is $(p_0, r_0, 1)$-paranormal for $p_0 > 0$ and $r_0 \geq 0$, then $T$ is $(p_0, r, 1)$-paranormal for any $r \geq r_0$.

(vii) If $T$ is $(p, r, 1)$-paranormal for $p > 0$ and $r \geq 0$, then $T$ is $\max\{p, r\}$-paranormal.

On the other hand, chaotic order is defined by $\log A \geq \log B$ for positive and invertible operators $A$ and $B$. Chaotic order is weaker than usual order $A \geq B$ since $\log t$ is an operator monotone function. As a characterization of chaotic order, the following Theorem B.1 was obtained by using Theorem F.

Theorem B.1 ([6][8][16][26]). Let $A$ and $B$ be positive invertible operators. Then the following properties are mutually equivalent:

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{1}{2}}A^{p}B^{\frac{1}{2}})^{\frac{1}{2}} \geq B^{p}$ for all $p \geq 0$.

(iii) $(B^{\frac{1}{2}}A^{p}B^{\frac{1}{2}})^{\frac{r}{p+r}} \geq B^{r}$ for all $p \geq 0$ and $r \geq 0$.

We remark that the equivalence between (i) and (ii) was shown in [4].

Noting that class $F(p, r, \frac{p+r}{r})$ equals class $A(p, r)$, we can verify that class $A(p, r)$ derives from Theorem B.1. On class $A(p, r)$ and absolute-$(p, r)$-paranormality, the following Theorem A.2 and Theorem A.3 were shown in [11] and [30], respectively. We remark that Figure 2 expresses the inclusion relations shown in Theorem A.2 and Theorem A.3.

Theorem A.2 ([11]).

(i) $T$ is log-hyponormal if and only if $T$ belongs to class $A(p, r)$ for all $p > 0$ and $r > 0$.

(ii) If $T$ belongs to class $A(p_0, r_0)$ for $p_0 > 0$ and $r_0 > 0$, then $T$ belongs to class $A(p, r)$ for any $p \geq p_0$ and $r \geq r_0$.

(iii) If $T$ belongs to class $A(p_0, r_0)$ for $p_0 > 0$ and $r_0 > 0$, then $T$ belongs to class $A(p_0, r)$ for any $r \geq r_0$.

Theorem A.3 ([30]).

(i) $T$ is log-hyponormal if and only if $T$ is invertible and absolute-$(p, r)$-paranormal for all $p > 0$ and $r > 0$. 
(ii) If $T$ is absolute-$(p_0, r_0)$-paranormal for $p_0 > 0$ and $r_0 > 0$, then $T$ is absolute-$(p, r)$-paranormal for any $p \geq p_0$ and $r \geq r_0$.

(iii) If $T$ belongs to class $A(p, r)$ for $p > 0$ and $r > 0$, then $T$ is absolute-$(p, r)$-paranormal.

(iv) If $T$ is absolute-$(p, r)$-paranormal for $p > 0$ and $r > 0$, then $T$ is normaloid, i.e., $\|T\| = r(T)$ where $r(T)$ is the spectral radius of $T$.

Theorem A.2 and Theorem A.3 state that the families of class $AI(p, r)$ determined by operator inequalities and absolute-$(p, r)$-paranormality determined by norm inequalities have monotonicity on $p > 0$ and $r > 0$, and log-hyponormality regarded as class $AI(0, 0)$ or absolute-$(0, 0)$-paranormality, namely they constitute two increasing lines whose origin is log-hyponormality.

3 Inclusion relations

In this section, we discuss monotonicity of class $F(p, r, q)$ and $(p, r, q)$-paranormality.
In section 2, we verified that class $A(p, r)$ derives from Theorem B.1, and also we explained that Theorem A.2 and Theorem A.3 state that the families of class $A(p, r)$ and absolute-$(p, r)$-paranormality constitute two increasing lines on $p > 0$ and $r > 0$ whose origin is log-hyponormality.

On the other hand, as a parallel result to Theorem B.1, Theorem F also leads to the following Theorem B.2.

**Theorem B.2 ([9][10]).** For positive operators $A$ and $B$, $A^{\delta} \geq B^{\delta}$ for a fixed $\delta > 0$ if and only if

$$(B^{\frac{\delta}{r}}A^{\frac{p}{r}}B^{\frac{\delta}{r}})^{\frac{\delta+r}{p+r}} \geq B^{\delta+r}$$

holds for all $p \geq \delta$ and $r \geq 0$.

Considering these matters, it seems natural that we rewrite class $F(p, r, q)$ and $(p, r, q)$-paranormality by class $F(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality when we discuss monotonicity of class $F(p, r, q)$ and $(p, r, q)$-paranormality on $p$ and $r$. In fact, we obtain the following results on monotonicity of class $F(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality. And also the following Figure 3 represents the inclusion relations shown in this section.

**Proposition 3.** The following assertions hold for each $p > 0$ and $r > 0$:

(i) $T$ is $p$-quasihyponormal if and only if $T$ belongs to class $F(p, r, 1)$ if and only if $T$ is $(p, r, 1)$-paranormal.

(ii) $T$ is $p$-quasihyponormal if and only if $T$ is $(p, 0, 1)$-paranormal.

**Theorem 4.** Let $T$ be a class $F(p_{0}, r_{0}, \frac{p_{0}+r_{0}}{\delta+r_{0}})$ operator for $p_{0} > 0$, $r_{0} \geq 0$ and $-r_{0} < \delta \leq p_{0}$. Then the following assertions hold:

(i) $T$ belongs to class $F(p_{0}, r_{0}, \frac{p_{0}+r_{0}}{\delta+r_{0}})$ for any $r \geq r_{0}$.

(ii) If $T$ is invertible and $0 \leq \delta \leq p_{0}$, then $T$ belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_{0}$ and $r \geq r_{0}$.

**Theorem 5.** Let $T$ be a $(p_{0}, r_{0}, \frac{p_{0}+r_{0}}{\delta+r_{0}})$-paranormal operator for $p_{0} > 0$, $r_{0} \geq 0$ and $\delta > -r_{0}$. Then the following assertions hold:

(i) If $-r_{0} < \delta \leq p_{0}$, then $T$ is $(p_{0}, r, \frac{p_{0}+r_{0}}{\delta+r_{0}})$-paranormal for any $r \geq r_{0}$.

(ii) If $0 \leq \delta$, then $T$ is $(p, r, \frac{p_{0}+r_{0}}{\delta+r_{0}})$-paranormal for any $p \geq p_{0}$.

(iii) If $0 \leq \delta \leq p_{0}$, then $T$ is $(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p \geq p_{0}$ and $r \geq r_{0}$.
Proposition 3, Theorem 4 and Theorem 5 assert that invertible class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality for $\delta > 0$ constitute two increasing lines for $p \geq \delta > 0$ and $r \geq r_0 > 0$ which have $\delta$-quasihyponormality as the origin since $\delta$-quasihyponormality equals class $\mathrm{F}(\delta, r_0, 1)$ or $(\delta, r_0, 1)$-paranormality. And also, in case $\delta = 0$, (i) and (ii) of Theorem 4 means (iii) and (ii) of Theorem A.2, respectively, and Theorem 5 means (ii) of Theorem A.3. Therefore monotonicity of invertible class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality for $\delta > 0$ is parallel to monotonicity of class $\mathrm{A}(p, r)$ and absolute-$(p, r)$-paranormality since invertible $\delta$-quasihyponormality (i.e., $\delta$-hyponormality) approaches log-hyponormality as $\delta \to +0$.

**Remark.** We remark that Proposition 1 does not hold for $r = 0$ and $q = 1$ since (2.1) holds for $p > 0$, $r = 0$ and $q = 1$, i.e., $\|T^p x\| \geq \|T^* p x\|$ for every unit vector $x \in H$ if and only if $T$ is $p$-hyponormal, but $T$ is $(p, 0, 1)$-paranormal for $p > 0$ if and only if $T$ is $p$-quasihyponormal by (ii) of Proposition 3.
4 Log-hyponormality

As a characterization of log-hyponormal operators, the following Theorem D.1 was obtained.

**Theorem D.1 ([11][29][30]).** Let $T$ be an invertible operator. Then the following assertions are mutually equivalent:

(i) $T$ is log-hyponormal.

(ii) $T$ belongs to class $A(p, p)$, i.e., class $AI(p, p)$ for all $p > 0$.

(iii) $T$ belongs to class $A(p, r)$, i.e., class $AI(p, r)$ for all $p > 0$ and $r > 0$.

(iv) $T$ is $p$-paranormal for all $p > 0$.

(v) $T$ is absolute-$(p, r)$-paranormal for all $p > 0$ and $r > 0$.

(i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) was obtained in [11], and also (i)$\Leftrightarrow$(iv) and (i)$\Leftrightarrow$(iv)$\Leftrightarrow$(v) were obtained in [29] and [30], respectively.

As an extension of Theorem D.1 via class $F(p, r, q)$ and $(p, r, q)$-paranormality, we have the following Theorem 6.

**Theorem 6.** Let $T$ be an invertible operator. Then the following assertions are mutually equivalent for any fixed $\alpha \in (0, 1]$:

(i) $T$ is log-hyponormal.

(ii) $T$ belongs to class $F(p, p, \frac{2}{\alpha})$ for all $p > 0$.

(iii) $T$ belongs to class $F(p, r, \frac{p+r}{r\alpha})$ for all $p > 0$ and $r > 0$.

(iv) $T$ is $(p, p, \frac{2}{\alpha})$-paranormal for all $p > 0$.

(v) $T$ is $(p, r, \frac{p+r}{r\alpha})$-paranormal for all $p > 0$ and $r > 0$.

We remark that Theorem 6 ensures Theorem D.1 by putting $\alpha = 1$. 
5 Powers of class $F(p,r,q)$ operators

On powers of $p$-hyponormal and log-hyponormal operators, Aluthge-Wang [1] and Yamazaki [27] showed the following results (see also [19][20][23]).

**Theorem E.1** ([1]). Let $T$ be a $p$-hyponormal operator for $0 < p \leq 1$. Then $T^n$ is $\frac{p}{n}$-hyponormal for all positive integer $n$.

**Theorem E.2** ([27]). Let $T$ be a log-hyponormal operator. Then $T^n$ is also log-hyponormal for all positive integer $n$.

On the other hand, on powers of class $A(p,r)$ operators, Yamazaki [28] showed the following Theorem E.3 (see also [22]).

**Theorem E.3** ([28]). Let $T$ be a class $A(p,r)$ operator for $0 < p \leq 1$ and $0 < r \leq 1$. Then $T^n$ belongs to class $A(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer $n$.

In this section, we obtain the following result on powers of class $F(p,r,q)$ operators.

**Theorem 7.** Let $T$ be an invertible class $F(p,r,q)$ operator for $0 < p \leq 1, 0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$. Then $T^n$ belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer $n$.

Theorem 7 interpolates Theorem E.1 and Theorem E.3 in case $T$ is invertible. In fact, Theorem 7 yields Theorem E.1 by putting $q = 1$ and $r = 0$, and also Theorem 7 yields Theorem E.3 by putting $q = \frac{p+r}{r}$.

**References**


[14] T.Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p + 2r$, Proc. Amer. Math. Soc., **101** (1987), 85–88.


