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Kyoto University
Powers of class \( wA(s, t) \) operators associated with generalized Aluthge transformation

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Abstract

This report is based on the following preprint:


An operator \( T = U|T| \) is said to belong to class \( wA(s, t) \) for \( s, t > 0 \) if
\[
|\tilde{T}_{s,t}|^\frac{2t}{s+t} \geq |T|^{2t} \quad \text{and} \quad |\tau|^{2s} \geq |(T_{s,t}^*)|^{2s},
\]
where \( \tilde{T}_{s,t} = |T|^{s}U|T|^{t} \).

We show that if \( T \) belongs to class \( wA(s, t) \), then \( T^n \) belongs to class \( wA(\frac{s}{n}, \frac{t}{n}) \) for every natural number \( n \).

1 Introduction

1.1 An order preserving operator inequality

In this report, an operator means a bounded linear operator on a Hilbert space \( H \). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( (Tx, x) \geq 0 \) for all \( x \in H \), and also \( T \) is said to be strictly positive (denoted by \( T > 0 \)) if \( T \) is positive and invertible.

We begin this report by introducing the following result which is quite useful for the study of the class of operators including normal operators \((\Leftrightarrow T^*T = TT^*)\).

**Theorem F (Furuta inequality [12]).**

If \( A \geq B \geq 0 \), then for each \( r \geq 0 \),

\[
\begin{align*}
(i) \quad & (B^\frac{r}{2}A^pB^\frac{r}{2})^\frac{1}{q} \geq (B^\frac{r}{2}B^pB^\frac{r}{2})^\frac{1}{q} \\
(ii) \quad & (A^\frac{r}{2}A^pA^\frac{r}{2})^\frac{1}{q} \geq (A^\frac{r}{2}B^pA^\frac{r}{2})^\frac{1}{q}
\end{align*}
\]

hold for \( p \geq 0 \) and \( q \geq 1 \) with \((1+r)q \geq p+r\).
We remark that Theorem F yields Löwner-Heinz theorem “\( A \geq B \geq 0 \) ensures \( A^\alpha \geq B^\alpha \) for any \( \alpha \in [0, 1] \)” when we put \( r = 0 \) in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [10][23] and also an elementary one-page proof in [13]. It is shown in [25] that the domain drawn for \( p, q \) and \( r \) in Figure 1 is the best possible for Theorem F.

1.2 Aluthge transformation of \( p \)-hyponormal and log-hyponormal operators

An operator \( T \) is said to be \( p \)-hyponormal for \( p > 0 \) if \( (T^*T)^p \geq (TT^*)^p \), and \( T \) is said to be log-hyponormal if \( T \) is invertible and \( \log T^*T \geq \log TT^* \). \( p \)-Hyponormality and log-hyponormality were defined as extensions of hyponormality, that is, \( T^*T \geq TT^* \). It is easily seen that every \( q \)-hyponormal operator is \( p \)-hyponormal for \( q \geq p > 0 \) by Löwner-Heinz theorem, and every invertible \( p \)-hyponormal operator for some \( p > 0 \) is log-hyponormal since \( \log t \) is an operator monotone function. We remark that \( p \)-hyponormality tends to log-hyponormality as \( p \to +0 \) since \( \frac{X^p-I}{p} \to \log X \) as \( p \to +0 \) for every positive operator \( X \).

The operator \( \tilde{T} = |T|^\frac{1}{2}U|T^\frac{1}{2} \) is called Aluthge transformation of an operator \( T \) whose polar decomposition is \( T = U|T| \), where \( |T| = (T^*T)^\frac{1}{2} \). Aluthge transformation was first introduced by Aluthge [1], and he showed the following result on Aluthge transformation of \( p \)-hyponormal operators as an application of Theorem F.

**Theorem A** ([1]). Let \( T = U|T| \) be the polar decomposition of a \( p \)-hyponormal operator for \( 0 < p < 1 \) and \( U \) be unitary. Then

(i) \( \tilde{T} = |T|^\frac{1}{2}U|T^\frac{1}{2} \) is \( (p + \frac{1}{2}) \)-hyponormal if \( 0 < p \leq \frac{1}{2} \).

(ii) \( \tilde{T} = |T|^\frac{1}{2}U|T^\frac{1}{2} \) is hyponormal if \( \frac{1}{2} \leq p < 1 \).

We remark that \( \sigma(\tilde{T}) = \sigma(T) \) holds for any operator \( T \) [4][7], and Theorem A states that \( \tilde{T} \) belongs to a smaller class than a \( p \)-hyponormal operator \( T \) for \( 0 < p < 1 \).
A generalization of Aluthge transformation of an operator \( T = U|T| \) is \( \tilde{T}_{s,t} = |T|^s U|T|^t \) for \( s > 0 \) and \( t > 0 \). In fact, it is clear that \( \tilde{T}_{\frac{1}{2},\frac{1}{2}} = \tilde{T} \). Huruya [19] and Yoshino [29] showed an extension of Theorem A on generalized Aluthge transformation of \( p \)-hyponormal operators. Tanahashi [26] showed a parallel result on generalized Aluthge transformation of log-hyponormal operators.

1.3 Classes of operators associated with Aluthge transformation

Recently, Aluthge and Wang introduced the class of \( w \)-hyponormal operators via Aluthge transformation \( \tilde{T} \) in [4], and showed an equivalent condition to \( w \)-hyponormality in [5].

Definition ([4][5]).

\( T : w \)-hyponormal \( \iff |\tilde{T}| \geq |T| \geq |(\tilde{T})^*| \)
\( \iff (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})\frac{1}{2} \geq |T^*| \) and \( |T| \geq (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})\frac{1}{2} \),

where \( \tilde{T} \) is Aluthge transformation of \( T \).

As a generalization of the class of \( w \)-hyponormal operators, Ito [20] introduced class \( wA(s, t) \) for \( s > 0 \) and \( t > 0 \) via generalized Aluthge transformation \( \tilde{T}_{s,t} \). In fact, it is clear that class \( wA(\frac{1}{2}, \frac{1}{2}) \) coincides with the class of \( w \)-hyponormal operators.

Definition ([20]). For \( s > 0 \) and \( t > 0 \),

\( T \in \text{class } wA(s, t) \iff |\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t} \) and \( |T|^{2s} \geq |(\tilde{T}_{s,t})^*|^{\frac{2t}{s+t}} \)
\( \iff (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{1}{2t}} \geq |T^*|^{2t} \) and \( |T|^{2s} \geq (|T^*|^t|T|^{2t}|T^*|^t)^{\frac{1}{2t}} \),

where \( \tilde{T}_{s,t} \) is generalized Aluthge transformation of \( T \). For the sake of convenience, we call class \( wA(1, 1) \) class \( wA \) for short.

He also pointed out the following fact.

Proposition B ([20]). \( T \in \text{class } wA \iff |T^2| \geq |T|^2 \) and \( |T^*|^2 \geq |T^2|^s \).
1.4 Related classes and their inclusion relations

On the other hand, Furuta, Ito and Yamazaki [15] introduced a class of operators called class $A$.

**Definition ([15]).** $T \in \text{class } A \iff |T^2| \geq |T|^2$.

They showed that every log-hyponormal operator belongs to class $A$ and every class $A$ operator is paranormal ($\iff \|T^2x\| \geq \|Tx\|^2$ for every unit vector $x$). This relations give another proof of the result by Ando [6].

As a generalization of class $A$, Fujii, D.Jung, S.H.Lee, M.Y.Lee and Nakamoto [11] introduced class $A(s, t)$ for $s > 0$ and $t > 0$. In fact, it was pointed out in [28] that class $A(1, 1)$ coincides with class $A$.

**Definition ([11]).** For $s > 0$ and $t > 0$,

(i) $T \in \text{class } A(s, t) \iff (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{s}{s+t}} \geq |T^*|^{2t}$.

(ii) $T \in \text{class } AI(s, t) \iff T \in \text{class } A(s, t)$ and $T$ is invertible.

We remark the following inclusion relations:

$(\star)$ class $A(s, t) \supseteq \text{class } wA(s, t) \supseteq \text{class } AI(s, t)$ holds for each $s > 0$ and $t > 0$. The first relation of $(\star)$ holds obviously, and the second holds by the following lemma.

**Lemma F ([14]).** Let $A > 0$ and $B$ be an invertible operator. Then

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}B^*B}A^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number $\lambda$.

In fact, the first inequality in the definition of class $wA(s, t)$ yields the second by applying Lemma F in case $T$ is invertible as follows:

$$(|T|^s|T^*|^t|T|^{2s}|T^*|^t)^{\frac{s}{s+t}}$$

$= |T|^s|T^*|^t(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{s}{s+t}}|T^*|^t|T|^s$ by Lemma F

$\leq |T|^s|T^*|^t |T^*|^{-2t} |T^*|^t|T|^s$ by the first inequality

$= |T|^{2s}$.
We also remark the following results.

Theorem C.1 ([20]).

(i) If an operator $T$ is $p$-hyponormal for some $p > 0$ or log-hyponormal, then $T$ belongs to class $wA(s, t)$ for all $s > 0$ and $t > 0$.

(ii) Every class $wA(s_1, t_1)$ operator belongs to class $wA(s_2, t_2)$ for each $0 < s_1 \leq s_2$ and $0 < t_1 \leq t_2$.

Theorem C.2 ([11]).

(i) An operator $T$ is log-hyponormal if and only if $T$ belongs to class $AI(s, t)$ for all $s > 0$ and $t > 0$.

(ii) Every class $A(s, t_1)$ operator belongs to class $A(s, t_2)$ for each $0 < t_1 \leq t_2$.

The following diagram shows the inclusion relations among the classes of operators mentioned above.

\[ \text{Figure 2} \]
1.5 Results on powers of non-normal operators

Recently, Aluthge and Wang showed results on powers of \( p \)-hyponormal and log-hyponormal operators in [2][3]. Extensions of the results were shown by Furuta and Yanagida [16][17], Ito [22] and Yamazaki [27].

As continuation of this study, Aluthge and Wang [5] showed the following result on powers of invertible \( w \)-hyponormal operators. A simplified proof of Theorem D.1 was given by Y.O.Kim [24].

**Theorem D.1** ([5]). Let \( T \) be an invertible \( w \)-hyponormal operator. Then \( T^2 \) is also \( w \)-hyponormal.

Cho, Huruya and Y.O.Kim [8] showed the following result which states that Theorem D.1 remains valid with a weaker condition \( N(T) = \{0\} \) than the invertibility of \( T \).

**Theorem D.2** ([8]). Let \( T \) be a \( w \)-hyponormal operator with \( N(T) = \{0\} \). Then \( T^2 \) is also \( w \)-hyponormal.

On the other hand, Ito [21] showed the following result on powers of invertible class A operators.

**Theorem D.3** ([21]). Let \( T \) be an invertible class A operator. Then the following assertions hold for all positive integer \( n \):

(i) \( |T^{n+1}| \frac{2n}{n+1} \geq |T^n|^2 \) and \( |T^{n*}|^2 \geq |T^{n+1*}| \frac{2n}{n+1} \).

(ii) \( |T^n|^\frac{2}{n} \geq \cdots \geq |T^2|^2 \geq |T|^2 \) and \( |T^*|^2 \geq |T^2*| \geq \cdots \geq |T^{n*}|^\frac{2}{n} \).

(iii) \( |T^{2n}| \geq |T^n|^2 \) and \( |T^{n*}|^2 \geq |T^{2n*}| \), i.e., \( T^n \) also belongs to class A.

As an extension of both Theorem D.1 and (iii) of Theorem D.3, Yamazaki [28] showed the following result on powers of class \( AI(s,t) \) operators.

**Theorem D.4** ([28]). Let \( T \) be a class \( AI(s,t) \) operator for \( s \in (0,1] \) and \( t \in (0,1] \). Then \( T^n \) belongs to \( AI(\frac{s}{n}, \frac{1}{n}) \) for all positive integer \( n \).
In fact, Theorem D.4 yields Theorem D.1 by putting \( s = t = \frac{1}{2} \) and \( n = 2 \) since class \( \text{AI}(\frac{1}{4}, \frac{1}{4}) \subseteq \text{AI}(\frac{1}{2}, \frac{1}{2}) \) by (ii) of Theorem C.1. Theorem D.4 also yields (iii) of Theorem D.3 by putting \( s = t = 1 \) since class \( \text{AI}(\frac{1}{n}, \frac{1}{n}) \subseteq \text{AI}(1, 1) \) by (ii) of Theorem C.1. It is interesting to remark that Theorem D.4 states that \( T^n \) belongs to a smaller class than a class \( \text{AI}(s, t) \) operator \( T \) for \( s \in (0, 1) \) and \( t \in (0, 1) \).

In this report, we shall show several results on powers of class \( w\text{A}(s, t) \) operators as extensions of the results on powers of class \( \text{AI}(s, t) \) operators and \( w \)-hyponormal operators mentioned above.

2 Results

Firstly, we show the following result on powers of class \( w\text{A} \) operators.

**Theorem 1.** Let \( T \) be a class \( w\text{A} \) operator. Then the following assertions hold for all positive integer \( n \):

(i) \[ |T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2 \text{ and } |T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2. \]

(ii) \[ |T^n|^{\frac{2}{n}} \geq \cdots \geq |T^2| \geq |T| |T^2|^n \geq \cdots \geq |T^n|^2. \]

Secondly, we show the following result on powers of class \( w\text{A}(s, t) \) operators.

**Theorem 2.** Let \( T \) be a class \( w\text{A}(s, t) \) operator for \( s \in (0, 1] \) and \( t \in (0, 1] \). Then \( T^n \) belongs to \( w\text{A}(\frac{s}{n}, \frac{t}{n}) \) for all positive integer \( n \).

Theorem 1 and Theorem 2 are extensions of Theorem D.3 and Theorem D.4, respectively, since every class \( \text{AI}(s, t) \) operator belongs to class \( w\text{A}(s, t) \) by (\. In other words, Theorem 1 and Theorem 2 state that Theorem D.3 and Theorem D.4 remain valid for class \( w\text{A} \) and class \( w\text{A}(s, t) \) operators without the invertibility of \( T \), respectively.

Theorem 2 yields the following result as an immediate corollary which is an extension of Theorem D.2.

**Corollary 3.** Let \( T \) be a \( w \)-hyponormal operator. Then \( T^n \) is also \( w \)-hyponormal for all positive integer \( n \).
3 Proofs of the results

In order to give a proof of Theorem 1, we prepare the following results.

Proposition 4. Let $A$ and $B$ be positive operators. Then the following assertions hold:

(i) If \((B^{\frac{\alpha_0}{2}} A^{\alpha_0} B^{\frac{\alpha_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \geq B^{\beta_0}\) holds for fixed \(\alpha_0 > 0\) and \(\beta_0 > 0\), then

\[
(B^{\frac{\beta}{2}} A^{\alpha_0} B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha_0+\beta}} \geq B^{\beta}
\]

holds for any \(\beta \geq \beta_0\), and

\[
A^{\frac{\alpha_0 + \beta_0}{2}} B^{\beta_1} A^{\frac{\alpha_0}{2}} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0 + \beta_0}{2}}
\]

holds for any \(\beta_1\) and \(\beta_2\) such that \(\beta_2 \geq \beta_1 \geq \beta_0\).

(ii) If \(A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0 + \beta_0}{2}}\) holds for fixed \(\alpha_0 > 0\) and \(\beta_0 > 0\), then

\[
A^{\alpha} \geq (A^{\frac{\alpha}{2}} B^{\beta_0} A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha_0+\beta_0}}
\]

holds for any \(\alpha \geq \alpha_0\), and

\[
(B^{\frac{\alpha_0}{2}} A^{\alpha_2} B^{\frac{\alpha_0}{2}})^{\frac{\alpha_1 + \beta_0}{\alpha_2 + \beta_0}} \geq B^{\frac{\beta_0}{2}} A^{\alpha_1} B^{\frac{\alpha_2}{2}}
\]

holds for any \(\alpha_1\) and \(\alpha_2\) such that \(\alpha_2 \geq \alpha_1 \geq \alpha_0\).

Lemma 5. Let $A$, $B$ and $C$ be positive operators. Then the following assertions holds for each $p \geq 0$ and $r \in (0, 1]$:

(i) If \((B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{r}{p+r}} \geq B^r\) and $B \geq C$, then \((C^{\frac{p}{2}} A^p C^{\frac{p}{2}})^{\frac{r}{p+r}} \geq C^r\).

(ii) If $A \geq B$, $B^r \geq (B^{\frac{p}{2}} C^p B^{\frac{p}{2}})^{\frac{r}{p+r}}$ and the condition

\[(*) \quad \text{if } \lim_{n \to \infty} B^{\frac{1}{2}} x_n = 0 \text{ and } \lim_{n \to \infty} A^{\frac{1}{2}} x_n \text{ exists, then } \lim_{n \to \infty} A^{\frac{1}{2}} x_n = 0\]

hold, then $A^r \geq (A^{\frac{p}{2}} C^p A^{\frac{p}{2}})^{\frac{r}{p+r}}$. 

Proof of Proposition 4.

Proof of (i). Put \( A_1 = (B^\beta A^\alpha B^\beta)^{\alpha_0+\beta_0} \) and \( B_1 = B^{\beta_0}, \) then \( A_1 \geq B_1 \geq 0 \) by the hypothesis. By applying (i) of Theorem F to \( A_1 \) and \( B_1, \) we have

\[
(B_1^{p_1} A_1^{r_1} B_1^{r_1})^{1+r_1} \geq B_1^{1+r_1} \quad \text{for any } p_1 \geq 1 \text{ and } r_1 \geq 0.
\]

Put \( p_1 = \frac{\alpha_0+\beta_0}{\beta_0} \geq 1 \) and \( \beta = (1+r_1)\beta_0 \geq \beta_0 \) in (3.3), then we have

\[
(B_1^{p_1} A_1^{r_1} B_1^{r_1})^{\frac{\beta_0}{\alpha_0+\beta}} \geq B_{10}^{\beta} \quad \text{for any } \beta \geq \beta_0.
\]

By applying Löwner-Heinz theorem to (3.1), we have

\[
(B_1^{p_1} A_1^{r_1} B_1^{r_1})^{\frac{v}{\alpha_0+\beta}} \geq B^v \quad \text{for any } \beta \geq \beta_0 \text{ and } v \text{ such that } \beta \geq v \geq 0.
\]

Put \( f_{\beta_1}(\beta) = (A^{\alpha_0} B^{\beta} A^{\alpha_0} B^{\beta})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta}}. \) For any \( \beta, \beta_1 \) and \( v \) such that \( \beta \geq \beta_1 \geq \beta_0 \) and \( \beta \geq v \geq 0, \) we have

\[
f_{\beta_1}(\beta) = (A^{\alpha_0} B^{\beta} A^{\alpha_0} B^{\beta})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta}} = \{(A^{\alpha_0} B^{\beta} A^{\alpha_0} B^{\beta})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta}}\}_{\alpha_0+\beta}
\]

\[
= \{(A^{\alpha_0} B^{\beta} A^{\alpha_0} B^{\beta})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta}}\}_{\alpha_0+\beta+v}
\]

\[
\geq \{A^{\alpha_0} B^{\beta} A^{\alpha_0} B^{\beta}\}_{\alpha_0+\beta+v}
\]

\[
= (A^{\alpha_0} B^{\beta+v} A^{\alpha_0})^{\frac{\alpha_0+\beta}{\alpha_0+\beta+v}}
\]

\[
= f_{\beta_1}(\beta+v).
\]

The above inequality holds by (3.4) and Löwner-Heinz theorem since \( \frac{\alpha_0+\beta_1}{\alpha_0+\beta+v} \in [0, 1]. \) Therefore for each \( \beta_1 \geq \beta_0, \) \( f_{\beta_1}(\beta) \) is decreasing for \( \beta \geq \beta_1, \) so that

\[
A^{\alpha_0} B^{\beta_1} A^{\alpha_0} = f_{\beta_1}(\beta_1) \geq f_{\beta_1}(\beta_2) = (A^{\alpha_0} B^{\beta_2} A^{\alpha_0})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta_2}}
\]

holds for any \( \beta_1 \) and \( \beta_2 \) such that \( \beta_2 \geq \beta_1 \geq \beta_0, \) hence we have (3.2).

(ii) can be proved in the same way as (i), so that we omit the proof. \( \square \)
Lemma 5 can be obtained as an application of the following results.

**Theorem E.1** ([9]). Let $A$ and $B$ be bounded linear operators on a Hilbert space $H$. The following statements are equivalent;

1. $R(A) \subseteq R(B)$;
2. $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$; and
3. there exists a bounded linear operator $C$ on $H$ so that $A = BC$.

Moreover, if (1), (2) and (3) are valid, then there exists a unique operator $C$ so that

(a) $\|C\|^2 = \inf \{\mu : AA^* \leq \mu BB^*\}$;
(b) $N(A) = N(C)$; and
(c) $R(C) \subseteq \overline{R(B^*)}$.

**Theorem E.2** ([18]). Let $X$ and $A$ be bounded linear operators on a Hilbert space $H$. We suppose that $X \geq 0$ and $\|A\| \leq 1$. If $f$ is an operator monotone function defined on $[0, \infty)$ such that $f(0) \leq 0$, then

$$A^* f(X) A \leq f(A^* X A).$$

We remark that the condition (c) in Theorem E.1 is equivalent to the condition (c') $\overline{R(C)} \subseteq \overline{R(B^*)}$. Here we consider when the equality of (c') holds.

**Lemma 6.** Let $A$ and $B$ be operators which satisfy (1), (2) and (3) of Theorem E.1, and $C$ be the operator which is given in (3) and determined uniquely by (a), (b) and (c) of Theorem E.1. Then the following assertions are mutually equivalent:

(i) $\overline{R(C)} = \overline{R(B^*)}$.

(ii) If $\lim_{n \to \infty} A^* x_n = 0$ and $\lim_{n \to \infty} B^* x_n$ exists, then $\lim_{n \to \infty} B^* x_n = 0$. 
Proof. (i) is equivalent to $N(C^*) = N(B)$ and

$$N(C^*) = N(B) \oplus (N(B)^\perp \cap N(C^*)) = N(B) \oplus (\overline{R(B^*)} \cap N(C^*))$$

since $N(C^*) \supseteq N(B)$ by (c) of Theorem E.1, so that (i) is equivalent to the following (3.5):

(3.5) $\overline{R(B^*)} \cap N(C^*) = \{0\}$.

Noting that when $y = \lim_{narrow \infty} B^* x_n$ for some $\{x_n\} \subseteq H$,

$$C^* y = C^* \left( \lim_{narrow \infty} B^* x_n \right) = \lim_{narrow \infty} C^* B^* x_n = \lim_{narrow \infty} A^* x_n$$

holds by (3) of Theorem E.1, so that we have

$$\overline{R(B^*)} \cap N(C^*) = \{y \mid \text{there exists } \{x_n\} \subseteq H \text{ such that } y = \lim_{narrow \infty} B^* x_n \text{ and } C^* y = 0\}$$

$$= \{y \mid \text{there exists } \{x_n\} \subseteq H \text{ such that } y = \lim_{narrow \infty} B^* x_n \text{ and } \lim_{narrow \infty} A^* x_n = 0\},$$

hence (3.5) is equivalent to (ii). \(\square\)

We also require the following lemma in order to give a proof of Lemma 5.

**Lemma 7.** Let $S$ be a positive operator and $\alpha \in (0,1]$. If $\lim_{narrow \infty} S x_n = 0$ and $\lim_{narrow \infty} S^\alpha x_n$ exists, then $\lim_{narrow \infty} S^\alpha x_n = 0$.

**Proof.** $\lim_{narrow \infty} S^\alpha x_n \in \overline{R(S^\alpha)} \cap N(S^{1-\alpha}) = \overline{R(S)} \cap N(S) = \{0\}$ for $\alpha \in (0,1)$ since $S^{1-\alpha} \left( \lim_{narrow \infty} S^\alpha x_n \right) = \lim_{narrow \infty} S x_n = 0$ by the hypothesis. \(\square\)

**Proof of Lemma 5.**

**Proof of (i).** $B \geq C$ ensures $B^r \geq C^r$ for $r \in (0,1]$ by Löwner-Heinz theorem. By Theorem E.1, there exists an operator $X$ such that

(3.6) $B^\frac{r}{2} X = X^* B^\frac{r}{2} = C^\frac{r}{2}$,

(3.7) $\|X\| \leq 1$. 

Then we have
\[
(C^\frac{r}{2}A^{p}C^\frac{r}{2})^\frac{r}{p+r} = (X^*B^\frac{r}{2}A^{p}B^\frac{r}{2}X)^\frac{r}{p+r}
\]
by (3.6)
\[
\geq X^*(B^\frac{r}{2}A^{p}B^\frac{r}{2})^\frac{r}{p+r}X
\]
by Theorem E.2 and (3.7)
\[
\geq X^*B^r X
\]
by the hypothesis
\[
= C^r
\]
by (3.6).

**Proof of (ii).** \( A \geq B \) ensures \( A^r \geq B^r \) for \( r \in (0, 1] \) by Löwner-Heinz theorem. By Theorem E.1, there exists an operator \( Y \) such that
\[
(3.8) \quad A^\frac{r}{2}Y = Y^*A^\frac{r}{2} = B^\frac{r}{2},
\]
\[
(3.9) \quad \|Y\| \leq 1.
\]
Then we have
\[
Y^*(A^\frac{r}{2}CpA^\frac{r}{2})^\frac{r}{p+r}Y \leq (Y^*A^\frac{r}{2}CpA^\frac{r}{2}Y)^\frac{r}{p+r}
\]
by Theorem E.2 and (3.9)
\[
= (B^\frac{r}{2}CpB^\frac{r}{2})^\frac{r}{p+r}
\]
by (3.8)
\[
\leq B^r
\]
by the hypothesis
\[
= Y^*A^r Y
\]
by (3.8),
so that \( A^r \geq (A^\frac{r}{2}CpA^\frac{r}{2})^\frac{r}{p+r} \) holds on \( \overline{R(Y)} \). On the other hand, (*) implies the following condition:
\[
(\ast\ast) \quad \text{if } \lim_{n \to \infty} B^\frac{r}{2}x_n = 0 \text{ and } \lim_{n \to \infty} A^\frac{r}{2}x_n \text{ exists, then } \lim_{n \to \infty} A^\frac{r}{2}x_n = 0
\]
since if \( \lim_{n \to \infty} B^\frac{r}{2}x_n = 0 \) and \( \lim_{n \to \infty} A^\frac{r}{2}x_n \) exists, then
\[
\lim_{n \to \infty} B^\frac{1}{2}x_n = B^\frac{1-r}{2}\left(\lim_{n \to \infty} B^\frac{r}{2}x_n\right) = 0
\]
and \( \lim_{n \to \infty} A^\frac{1}{2}x_n = A^\frac{1-r}{2}\left(\lim_{n \to \infty} A^\frac{r}{2}x_n\right) \) exists, so that \( \lim_{n \to \infty} A^\frac{1}{2}x_n = 0 \) by (*), and \( \lim_{n \to \infty} A^\frac{r}{2}x_n = 0 \) by Lemma 7. (\ast\ast) ensures \( \overline{R(Y)} = R(A^\frac{r}{2}) \) by Lemma 6, hence we have
\[
N((A^\frac{r}{2}CpA^\frac{r}{2})^\frac{r}{p+r}) = N(A^\frac{r}{2}CpA^\frac{r}{2}) \supseteq N(A^\frac{r}{2}) = N(A^r) = N(Y^*),
\]
so that \( A^r = (A^\frac{r}{2}CpA^\frac{r}{2})^\frac{r}{p+r} = 0 \) on \( N(Y^*) \). Consequently the proof is complete since \( H = \overline{R(Y)} \oplus N(Y^*) \). \( \square \)
Proof of Theorem 1. Put \( A_n = |T^n|^\frac{2}{n} \) and \( B_n = |T^{n*}|^\frac{2}{n} \) for each integer \( n \).

By the definition, \( T \) belongs to class \( wA \) if and only if

\[
(B_1^\frac{1}{2} A_1 B_1^\frac{1}{2})^\frac{1}{2} = (|T^*||T|^2|T^*|)^\frac{1}{2} \geq |T^*|^2 = B_1
\]

and

\[
A_1 = |T|^2 \geq (|T||T^*|^2|T|)^\frac{1}{2} = (A_1^\frac{1}{2} B_1 A_1^\frac{1}{2})^\frac{1}{2}.
\]

We shall prove

\[
A_n = |T^n|^\frac{2n}{n+1} \geq |T^n|^2 = A_n
\]

and

\[
B_n = |T^{n*}|^2 = |T^{n+1*}|^\frac{2n}{n+1} = B_n
\]

hold for all positive integer \( n \) by induction. (3.12) and (3.13) hold for \( n = 1 \) by Proposition B. Assume (3.12) holds for \( n = 1, 2, \ldots, k-1 \). Then \( A_{n+1} \geq A_n \) holds by Löwner-Heinz theorem for \( \frac{1}{n} \in [0, 1] \), so that we have

\[
A_k \geq A_{k-1} \geq \cdots \geq A_2 \geq A_1.
\]

We remark that \( A_1 \) and \( A_k \) satisfy the condition

\[
\text{if } \lim_{n \to \infty} A_1^\frac{1}{2} x_n = 0 \text{ and } \lim_{n \to \infty} A_k^\frac{1}{2} x_n \text{ exists, then } \lim_{n \to \infty} A_k^\frac{1}{2} x_n = 0
\]

since

\[
\lim_{n \to \infty} A_1^\frac{1}{2} x_n = 0 \iff \lim_{n \to \infty} |T| x_n = 0 \iff \lim_{n \to \infty} T x_n = 0 \iff \lim_{n \to \infty} T^k x_n = 0
\]

\[
\iff \lim_{n \to \infty} |T^k| x_n = 0 \iff \lim_{n \to \infty} A_k^\frac{1}{2} x_n = 0 \iff \lim_{n \to \infty} A_k^\frac{1}{2} x_n = 0.
\]

The last implication holds by Lemma 7. By applying (ii) of Lemma 5 to (3.11) and (3.14), we have

\[
A_k \geq (A_k^\frac{1}{2} B_1 A_k^\frac{1}{2})^\frac{1}{2}.
\]
By applying (ii) of Proposition 4 to (3.15),

\[(B_1^{\frac{k}{12}}A_k^{\alpha_2}B_1^{\frac{k}{12}})^{\frac{\alpha_2+1}{\alpha_2+1}} \geq B_1^{\frac{k}{12}}A_k^{\alpha_1}B_1^{\frac{k}{12}}\]

holds for any \(\alpha_1\) and \(\alpha_2\) such that \(\alpha_2 \geq \alpha_1 \geq 1\), so that we have

\[(B_1^{\frac{k}{12}}A_k^{\alpha_k}B_1^{\frac{k}{12}})^{\frac{k+1}{k+1}} \geq B_1^{\frac{k}{12}}A_k^{\alpha_k-1}B_1^{\frac{k}{12}} \geq B_1^{\frac{k}{12}}A_k^{-1}B_1^{\frac{k}{12}},\]

since the first inequality is obtained by putting \(\alpha_1 = k - 1\) and \(\alpha_2 = k\) in (3.16), and the second holds since (3.12) holds for \(n = k - 1\) by the inductive assumption. (3.17) yields the following (3.18):

\[(|T^*||T^k|2|T^*|)^{\frac{k}{k+1}} \geq |T^*||T^k-1|2|T^*|.\]

Let \(T = U|T|\) be the polar decomposition of \(T\), then \(T^* = U^*|T^*|\) is the polar decomposition of \(T^*\). Here we have

\[|T^{k+1}|^{\frac{2k}{k+1}} = (T^*|T^k|^2T)^{\frac{k}{k+1}} \]
\[= (U^*|T^*||T^k|^2|T^*|U)^{\frac{k}{k+1}} \]
\[= U^* (|T^*||T^k|^2|T^*|)^{\frac{k}{k+1}} U \]
\[\geq U^*|T^*||T^k-1|2|T^*|U \quad \text{by (3.18)} \]
\[= T^*|T^{k-1}|2T \]
\[= |T^k|^2, \]

so that it is proved that (3.12) holds for \(n = k\). (3.13) can be proved in the same way as (3.12), so that we omit the proof.

**Proof of (ii).** The first inequality of (ii) has been already proved in (3.14), and the second can be proved in the same way as the first. \(\square\)

**Proof of Theorem 2.** Put \(A_n = |T^n|^\frac{2}{n}\) and \(B_n = |T^n*|^\frac{2}{n}\) for each integer \(n\), then \(T\) belongs to class \(wA(s, t)\) if and only if

\[(B_1^{\frac{s}{12}}A_1^{-1}B_1^{\frac{s}{12}})^{\frac{t}{s+t}} = (|T^*|^t|T^s|^2|T^*|^t)^{\frac{1}{s+t}} \geq |T^*|^2t = B_1^t \]

and

\[A_1^s = |T|^2s \geq (|T|^s|T^*|^2t|T|^s)^{\frac{1}{s+t}} = (A_1^{\frac{s}{12}}B_1^{t}A_1^{\frac{s}{12}})^{\frac{s}{s+t}}\]
by the definition. Now \( T \) belongs to class \( wA \) since

\[
\text{class } wA = \text{class } wA(1, 1) \supseteq \text{class } wA(s, t)
\]

for \( s \in (0, 1] \) and \( t \in (0, 1] \) by (ii) of Theorem C.1, so that by (ii) of Theorem 1,

\[
(3.21) \quad A_n \geq A_1
\]

and

\[
(3.22) \quad B_1 \geq B_n
\]

hold for all positive integer \( n \). Hence we have

\[
(3.23) \quad A_n^s \geq (A_n^s B_1^t A_n^s)^{\frac{s}{s+t}} \geq (A_n^s B_1^t A_n^s)^{\frac{s}{s+t}}.
\]

The first inequality in (3.23) is obtained by applying (ii) of Lemma 5 to (3.20) and (3.21) since \( A_1 \) and \( A_n \) satisfy the condition

\[
(\star) \quad \lim_{k \to \infty} A_1^\frac{1}{2} x_k = 0 \quad \text{and} \quad \lim_{k \to \infty} A_n^\frac{1}{2} x_k \text{ exists, then } \lim_{k \to \infty} A_n^\frac{1}{2} x_k = 0,
\]

and the second holds by (3.22) and Löwner-Heinz theorem. (3.23) yields the following (3.24):

\[
(3.24) \quad |T^n|^{\frac{2s}{n}} \geq (|T^n|^{\frac{s}{n}} |T^n^*|^{\frac{2t}{n}} |T^n|^{\frac{s}{n})}^{\frac{\frac{t}{n}}{\frac{s}{n}+\frac{t}{n}}}.
\]

The following (3.25) can be obtained in the same way as (3.24):

\[
(3.25) \quad (|T^n^*|^{\frac{t}{n}} |T^n|^{\frac{2s}{n}} |T^n^*|^{\frac{t}{n}})^{\frac{\frac{s}{n}}{\frac{s}{n}+\frac{t}{n}}} \geq |T^n^*|^{\frac{2t}{n}},
\]

so that \( T^n \) belongs to class \( wA(\frac{s}{n}, \frac{t}{n}) \) by the definition.

**Proof of Corollary 3.** If \( T \) belongs to class \( wA(\frac{1}{2}, \frac{1}{2}) \), then \( T^n \) belongs to class \( wA(\frac{1}{2n}, \frac{1}{2n}) \) by Theorem 2, so that \( T^n \) belongs to class \( wA(\frac{1}{2}, \frac{1}{2}) \) by (ii) of Theorem C.1. Hence the proof is complete since class \( wA(\frac{1}{2}, \frac{1}{2}) \) coincides with the class of \( w \)-hyponormal operators.
4 Concluding remarks

Remark 1. \((B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\alpha_0 + \beta_0} \geq B^\beta\) and \(A^\alpha \geq (A^\alpha B^\beta A^\alpha)^{\frac{\alpha_0}{\alpha_0 + \beta_0}}\) in the assumptions of (i) and (ii) of Proposition 4 are mutually equivalent in case both \(A\) and \(B\) are invertible. In fact, by applying Lemma \(F\) to the right-hand side of the second inequality, we have

\[
A^\alpha \geq (A^\alpha B^\beta A^\alpha)^{\frac{\alpha_0}{\alpha_0 + \beta_0}} = A^\frac{\alpha_0}{\alpha_0 + \beta_0} B^\frac{\alpha_0}{\alpha_0 + \beta_0} B^\frac{\alpha_0}{\alpha_0 + \beta_0} A^\frac{\alpha_0}{\alpha_0 + \beta_0},
\]

so that the first inequality is obtained. But it is pointed out in [20] that they are not equivalent in general if either \(A\) or \(B\) are not invertible. In fact, \(A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) and \(B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\) satisfy the second inequality, but do not satisfy the first.

Remark 2. Lemma 5 can be proved easily in case \(A, B\) and \(C\) are invertible. In fact, (i) can be proved as follows: By Lemma \(F\), \((B^\frac{r}{2} A^p B^\frac{r}{2})^{\frac{r}{p+r}} \geq B^r\) and \((C^\frac{r}{2} A^p C^\frac{r}{2})^{\frac{r}{p+r}} \geq C^r\) are equivalent to \(A^p \geq (A^\frac{r}{2} B^r A^\frac{r}{2})^{\frac{p}{p+r}}\) respectively, so that the first inequality implies the second by the assumption \(B \geq C\) and Löwner-Heinz theorem. (ii) can be proved similarly.

And one might expect that (ii) of Lemma 5 holds without the condition (*) But there exists a counterexample. Put

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\text{ and } C = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},
\]

then \(A \geq B\) and \(N(A) \subset N(B)\), so that \(A\) and \(B\) do not satisfy the condition (*). And for each \(p > 0\) and \(r \in (0, 1]\),

\[
B^r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (B^\frac{r}{2} C^p B^\frac{r}{2})^{\frac{r}{p+r}}
\]

but

\[
A^r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \not \geq \begin{pmatrix} 0 & 0 \\ 0 & 2^{\frac{p}{p+r}} \end{pmatrix} = (A^\frac{r}{2} C^p A^\frac{r}{2})^{\frac{r}{p+r}}.
\]
References


