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Kyoto University
Powers of class \( wA(s, t) \) operators associated with generalized Aluthge transformation

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Abstract

This report is based on the following preprint:


An operator \( T = U|T| \) is said to belong to class \( wA(s, t) \) for \( s, t > 0 \) if

\[
|\tilde{T}_{s,t}|^\frac{2t}{s+t} \geq |T|^{2t} \quad \text{and} \quad |\tau|^{\frac{2s}{s+t}} \geq (|T|^{s}U|T|^{t})^\frac{s}{s+t},
\]

where \( \tilde{T}_{s,t} = |T|^s U |T|^t \).

We show that if \( T \) belongs to class \( wA(s, t) \), then \( T^n \) belongs to class \( wA(\frac{s}{n}, \frac{t}{n}) \) for every natural number \( n \).

1 Introduction

1.1 An order preserving operator inequality

In this report, an operator means a bounded linear operator on a Hilbert space \( H \). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( (Tx, x) \geq 0 \) for all \( x \in H \), and also \( T \) is said to be strictly positive (denoted by \( T > 0 \)) if \( T \) is positive and invertible.

We begin this report by introducing the following result which is quite useful for the study of the class of operators including normal operators \( (\iff T^*T = TT^*) \).

**Theorem F (Furuta inequality [12]).**

If \( A \geq B \geq 0 \), then for each \( r \geq 0 \),

(i) \[ (B^\frac{r}{2} A^p B^\frac{r}{2})^\frac{1}{q} \geq (B^\frac{r}{2} B^p B^\frac{r}{2})^\frac{1}{q} \]

and

(ii) \[ (A^\frac{r}{2} A^p A^\frac{r}{2})^\frac{1}{q} \geq (A^\frac{r}{2} B^p A^\frac{r}{2})^\frac{1}{q} \]

hold for \( p \geq 0 \) and \( q \geq 1 \) with \( (1+r)q \geq p+r \).
We remark that Theorem F yields Löwner-Heinz theorem “$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$” when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [10][23] and also an elementary one-page proof in [13]. It is shown in [25] that the domain drawn for $p, q$ and $r$ in Figure 1 is the best possible for Theorem F.

1.2 Aluthge transformation of $p$-hyponormal and log-hyponormal operators

An operator $T$ is said to be $p$-hyponormal for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$, and $T$ is said to be log-hyponormal if $T$ is invertible and $\log T^*T \geq \log TT^*$. $p$-Hyponormality and log-hyponormality were defined as extensions of hyponormality, that is, $T^*T \geq TT^*$. It is easily seen that every $q$-hyponormal operator is $p$-hyponormal for $q \geq p > 0$ by Löwner-Heinz theorem, and every invertible $p$-hyponormal operator for some $p > 0$ is log-hyponormal since $\log t$ is an operator-monotone function. We remark that $p$-hyponormality tends to log-hyponormality as $p \to +0$ since $\frac{X^p - I}{p} \to \log X$ as $p \to +0$ for every positive operator $X$.

The operator $\tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}$ is called Aluthge transformation of an operator $T$ whose polar decomposition is $T = U|T|$, where $|T| = (T^*T)^\frac{1}{2}$. Aluthge transformation was first introduced by Aluthge [1], and he showed the following result on Aluthge transformation of $p$-hyponormal operators as an application of Theorem F.

**Theorem A ([1]).** Let $T = U|T|$ be the polar decomposition of a $p$-hyponormal operator for $0 < p < 1$ and $U$ be unitary. Then

(i) $\tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}$ is $(p + \frac{1}{2})$-hyponormal if $0 < p \leq \frac{1}{2}$.

(ii) $\tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}$ is hyponormal if $\frac{1}{2} \leq p < 1$.

We remark that $\sigma(\tilde{T}) = \sigma(T)$ holds for any operator $T$ [4][7], and Theorem A states that $\tilde{T}$ belongs to a smaller class than a $p$-hyponormal operator $T$ for $0 < p < 1$. 
A generalization of Aluthge transformation of an operator $T = U|T|$ is $\tilde{T}_{s,t} = |T|^s U |T|^t$ for $s > 0$ and $t > 0$. In fact, it is clear that $\tilde{T}_{1/2,1/2} = \tilde{T}$. Huruya [19] and Yoshino [29] showed an extension of Theorem A on generalized Aluthge transformation of $p$-hyponormal operators. Tanahashi [26] showed a parallel result on generalized Aluthge transformation of log-hyponormal operators.

1.3 Classes of operators associated with Aluthge transformation

Recently, Aluthge and Wang introduced the class of $w$-hyponormal operators via Aluthge transformation $\tilde{T}$ in [4], and showed an equivalent condition to $w$-hyponormality in [5].

**Definition** ([4][5]).

$T : w$-hyponormal $\iff |\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$

$\iff (|T^*|^{1/2} |T||T^*|^{1/2})^{1/2} \geq |T^*|$ and $|T| \geq (|T|^{1/2} |T^*||T|^{1/2})^{1/2}$,

where $\tilde{T}$ is Aluthge transformation of $T$.

As a generalization of the class of $w$-hyponormal operators, Ito [20] introduced class $wA(s,t)$ for $s > 0$ and $t > 0$ via generalized Aluthge transformation $\tilde{T}_{s,t}$. In fact, it is clear that class $wA(1/2,1/2)$ coincides with the class of $w$-hyponormal operators.

**Definition** ([20]). For $s > 0$ and $t > 0$,

$T \in$ class $wA(s,t) \iff |\tilde{T}_{s,t}|^{2t/s+t} \geq |T|^{2t}$ and $|T|^{2s} \geq |(\tilde{T}_{s,t})^*|^{2t/s+t}$

$\iff (|T^*|^{t} |T^{2s}| |T^*|^t)^{1/t} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^{s} |T^*|^{2t} |T^*|^{s})^{1/s+t}$,

where $\tilde{T}_{s,t}$ is generalized Aluthge transformation of $T$. For the sake of convenience, we call class $wA(1,1)$ class $wA$ for short.

He also pointed out the following fact.

**Proposition B** ([20]). $T \in$ class $wA \iff |T^2| \geq |T|^2$ and $|T^*|^2 \geq |T^2|$.
1.4 Related classes and their inclusion relations

On the other hand, Furuta, Ito and Yamazaki [15] introduced a class of operators called class A.

**Definition ([15]).** $T \in \text{class } A \iff |T^2| \geq |T|^2$.

They showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal ($\iff \|T^2x\| \geq \|Tx\|^2$ for every unit vector $x$). This relations give another proof of the result by Ando [6].

As a generalization of class A, Fujii, D.Jung, S.H.Lee, M.Y.Lee and Nakamoto [11] introduced class $A(s, t)$ for $s > 0$ and $t > 0$. In fact, it was pointed out in [28] that class $A(1, 1)$ coincides with class A.

**Definition ([11]).** For $s > 0$ and $t > 0$,

(i) $T \in \text{class } A(s, t) \iff (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{s}{s+t}} \geq |T^*|^{2t}$.

(ii) $T \in \text{class } A_I(s, t) \iff T \in \text{class } A(s, t)$ and $T$ is invertible.

We remark the following inclusion relations:

$(\diamondsuit)$ class $A(s, t) \supseteq \text{class } wA(s, t) \supseteq \text{class } A_I(s, t)$

holds for each $s > 0$ and $t > 0$. The first relation of $(\diamondsuit)$ holds obviously, and the second holds by the following lemma.

**Lemma F ([14]).** Let $A > 0$ and $B$ be an invertible operator. Then

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}B^*B}A^{\frac{1}{2}})^{-1}A^{\frac{1}{2}}B^*$$

holds for any real number $\lambda$.

In fact, the first inequality in the definition of class $wA(s, t)$ yields the second by applying Lemma F in case $T$ is invertible as follows:

$$\begin{align*}
\frac{|T|^s|T^*|^t|T|^s}{|T^s|T^*|^t|T^s|}\leq |T|^s|T^*|^t|T|^s
\end{align*}$$

by the first inequality

$$= |T|^{2s}. $$

by Lemma F
We also remark the following results.

**Theorem C.1 ([20]).**

(i) *If an operator $T$ is $p$-hyponormal for some $p > 0$ or log-hyponormal, then $T$ belongs to class $wA(s, t)$ for all $s > 0$ and $t > 0.*

(ii) *Every class $wA(s_1, t_1)$ operator belongs to class $wA(s_2, t_2)$ for each $0 < s_1 \leq s_2$ and $0 < t_1 \leq t_2.*

**Theorem C.2 ([11]).**

(i) *An operator $T$ is log-hyponormal if and only if $T$ belongs to class $AI(s, t)$ for all $s > 0$ and $t > 0.*

(ii) *Every class $A(s, t_1)$ operator belongs to class $A(s, t_2)$ for each $0 < t_1 \leq t_2.*

The following diagram shows the inclusion relations among the classes of operators mentioned above.

![Diagram showing the inclusion relations among the classes of operators](image-url)
1.5 Results on powers of non-normal operators

Recently, Aluthge and Wang showed results on powers of $p$-hyponormal and log-hyponormal operators in [2][3]. Extensions of the results were shown by Furuta and Yanagida [16][17], Ito [22] and Yamazaki [27].

As continuation of this study, Aluthge and Wang [5] showed the following result on powers of invertible $w$-hyponormal operators. A simplified proof of Theorem D.1 was given by Y.O.Kim [24].

**Theorem D.1 ([5]).** Let $T$ be an invertible $w$-hyponormal operator. Then $T^2$ is also $w$-hyponormal.

Cho, Huruya and Y.O.Kim [8] showed the following result which states that Theorem D.1 remains valid with a weaker condition $N(T) = \{0\}$ than the invertibility of $T$.

**Theorem D.2 ([8]).** Let $T$ be a $w$-hyponormal operator with $N(T) = \{0\}$. Then $T^2$ is also $w$-hyponormal.

On the other hand, Ito [21] showed the following result on powers of invertible class A operators.

**Theorem D.3 ([21]).** Let $T$ be an invertible class A operator. Then the following assertions hold for all positive integer $n$:

(i) $|T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2$ and $|T^{n^*}|^2 \geq |T^{n+1^*}|^{\frac{2n}{n+1}}$.

(ii) $|T^n|^{\frac{2}{n}} \geq \cdots \geq |T^2| \geq |T|^2$ and $|T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^{n*}|^{\frac{2}{n}}$.

(iii) $|T^{2n}| \geq |T^n|^2$ and $|T^{n^*}|^2 \geq |T^{2n^*}|$, i.e., $T^n$ also belongs to class A.

As an extension of both Theorem D.1 and (iii) of Theorem D.3, Yamazaki [28] showed the following result on powers of class $AI(s, t)$ operators.

**Theorem D.4 ([28]).** Let $T$ be a class $AI(s, t)$ operator for $s \in (0, 1]$ and $t \in (0, 1]$. Then $T^n$ belongs to $AI(s, t)$ for all positive integer $n$. 
In fact, Theorem D.4 yields Theorem D.1 by putting $s = t = \frac{1}{2}$ and $n = 2$ since class $\text{AI}(\frac{1}{4}, \frac{1}{4}) \subseteq \text{AI}(\frac{1}{2}, \frac{1}{2})$ by (ii) of Theorem C.1. Theorem D.4 also yields (iii) of Theorem D.3 by putting $s = t = 1$ since class $\text{AI}(\frac{1}{n}, \frac{1}{n}) \subseteq \text{AI}(1,1)$ by (ii) of Theorem C.1. It is interesting to remark that Theorem D.4 states that $T^n$ belongs to a smaller class than a class $\text{AI}(s,t)$ operator $T$ for $s \in (0,1]$ and $t \in (0,1]$.

In this report, we shall show several results on powers of class $w\text{A}(s,t)$ operators as extensions of the results on powers of class $\text{AI}(s,t)$ operators and $w$-hyponormal operators mentioned above.

## 2 Results

Firstly, we show the following result on powers of class $w\text{A}$ operators.

**Theorem 1.** Let $T$ be a class $w\text{A}$ operator. Then the following assertions hold for all positive integer $n$:

(i) $|T^{n+1}|_{\frac{2n}{n+1}} \geq |T^n|^2$ and $|T^{n*}|^2 \geq |T^{n+1*}|_{\frac{2n}{n+1}}$.

(ii) $|T^n|_{\frac{2}{n}} \geq \cdots \geq |T^2| \geq |T|^{2}$ and $|T^{*}|^2 \geq |T^{2*}| \geq \cdots \geq |T^{n*}|_{\frac{2}{n}}$.

Secondly, we show the following result on powers of class $w\text{A}(s,t)$ operators.

**Theorem 2.** Let $T$ be a class $w\text{A}(s,t)$ operator for $s \in (0,1]$ and $t \in (0,1]$. Then $T^n$ belongs to $w\text{A}(\frac{s}{n}, \frac{t}{n})$ for all positive integer $n$.

Theorem 1 and Theorem 2 are extensions of Theorem D.3 and Theorem D.4, respectively, since every class $\text{AI}(s,t)$ operator belongs to class $w\text{A}(s,t)$ by (Φ). In other words, Theorem 1 and Theorem 2 state that Theorem D.3 and Theorem D.4 remain valid for class $w\text{A}$ and class $w\text{A}(s,t)$ operators without the invertibility of $T$, respectively.

Theorem 2 yields the following result as an immediate corollary which is an extension of Theorem D.2.

**Corollary 3.** Let $T$ be a $w$-hyponormal operator. Then $T^n$ is also $w$-hyponormal for all positive integer $n$. 


3 Proofs of the results

In order to give a proof of Theorem 1, we prepare the following results.

Proposition 4. Let $A$ and $B$ be positive operators. Then the following assertions hold:

(i) If $(B^\frac{\alpha_0}{2}A^\alpha B^\frac{\beta_0}{2})^\frac{\beta_0}{\alpha_0+\beta_0} \geq B^\beta_0$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then

\[(3.1) \quad (B^\frac{\alpha_0}{2}A^\alpha B^\frac{\beta_0}{2})^\frac{\beta_0}{\alpha_0+\beta_0} \geq B^\beta\]

holds for any $\beta \geq \beta_0$, and

\[(3.2) \quad A^\frac{\beta_1}{2}B^\beta A^\frac{\beta_0}{2} \geq (A^\frac{\alpha_0}{2}B^\beta_0 A^\frac{\alpha_0}{2})^\frac{\alpha_0+\beta_0}{\alpha_0+\beta_0} \]

holds for any $\beta_1$ and $\beta_2$ such that $\beta_2 \geq \beta_1 \geq \beta_0$.

(ii) If $A^\alpha_0 \geq (A^\frac{\alpha_0}{2}B^\beta_0 A^\frac{\alpha_0}{2})^\frac{\alpha_0}{\alpha_0+\beta_0}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then

\[A^\alpha \geq (A^\frac{\alpha_0}{2}B^\beta_0 A^\frac{\alpha_0}{2})^\frac{\alpha_0}{\alpha_0+\beta_0}\]

holds for any $\alpha \geq \alpha_0$, and

\[(B^\frac{\alpha_0}{2}A^\alpha B^\frac{\beta_0}{2})^\frac{\alpha_0+\beta_0}{\alpha_0+\beta_0} \geq B^\frac{\beta_0}{2}A^\alpha B^\frac{\beta_0}{2}\]

holds for any $\alpha_1$ and $\alpha_2$ such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

Lemma 5. Let $A$, $B$ and $C$ be positive operators. Then the following assertions holds for each $p \geq 0$ and $r \in (0, 1]$:

(i) If $(B^\frac{\alpha_0}{2}A^p B^\frac{\beta_0}{2})^\frac{r}{p+r} \geq B^r$ and $B \geq C$, then $(C^\frac{\alpha_0}{2}A^p C^\frac{\beta_0}{2})^\frac{r}{p+r} \geq C^r$.

(ii) If $A \geq B$, $B^r \geq (B^\frac{\alpha_0}{2}C^p B^\frac{\beta_0}{2})^\frac{r}{p+r}$ and the condition

\[(*) \quad \text{if } \lim_{n \to \infty} B^\frac{1}{2}x_n = 0 \text{ and } \lim_{n \to \infty} A^\frac{1}{2}x_n \text{ exists, then } \lim_{n \to \infty} A^\frac{1}{2}x_n = 0 \]

hold, then $A^r \geq (A^\frac{\alpha_0}{2}C^p A^\frac{\beta_0}{2})^\frac{r}{p+r}$. 
Proof of Proposition 4.

Proof of (i). Put $A_1 = (B^\frac{\beta_0}{\alpha_0} A^\alpha_0 B^\frac{\beta_0}{\alpha_0})^\frac{\beta_0}{\alpha_0 + \beta_0}$ and $B_1 = B^{\beta_0}$, then $A_1 \geq B_1 \geq 0$ by the hypothesis. By applying (i) of Theorem F to $A_1$ and $B_1$, we have

(3.3) \[ (B^{\frac{\beta_1}{2}} A_1^{p_1} B_1^{\frac{\beta_1}{2}})^{1+\frac{r_1}{p_1}} \geq B_1^{1+r_1} \] for any $p_1 \geq 1$ and $r_1 \geq 0$.

Put $p_1 = \frac{\alpha_0 + \beta_0}{\beta_0} \geq 1$ and $\beta = (1 + r_1)\beta_0 \geq \beta_0$ in (3.3), then we have

(3.1) \[ (B^\frac{\beta}{2} A^\alpha_0 B^\frac{\beta}{2})^\frac{\beta}{\alpha_0 + \beta} \geq B^{\beta} \] for any $\beta \geq \beta_0$.

By applying Löwner-Heinz theorem to (3.1), we have

(3.4) \[ (B^\frac{\beta}{2} A^\alpha_0 B^\frac{\beta}{2})^\frac{v}{\alpha_0 + \beta} \geq B^v \] for any $\beta \geq \beta_0$ and $v$ such that $\beta \geq v \geq 0$.

Put $f_{\beta_1}(\beta) = (A^\frac{\alpha_0}{2} B^\beta A^\frac{\alpha_0}{2})^\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta}$. For any $\beta$, $\beta_1$ and $v$ such that $\beta \geq \beta_1 \geq \beta_0$ and $\beta \geq v \geq 0$, we have

\[
\begin{align*}
\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta + v} & \in [0,1]. \\
\end{align*}
\]

Therefore for each $\beta_1 \geq \beta_0$, $f_{\beta_1}(\beta)$ is decreasing for $\beta \geq \beta_1$, so that

\[
A^\frac{\alpha_0}{2} B^{\beta_1} A^\frac{\alpha_0}{2} = f_{\beta_1}(\beta_1) \geq f_{\beta_1}(\beta_2) = (A^\frac{\alpha_0}{2} B^{\beta_2} A^\frac{\alpha_0}{2})^\frac{\alpha_0 + \beta_1}{\alpha_0 + \beta_2}
\]

holds for any $\beta_1$ and $\beta_2$ such that $\beta_2 \geq \beta_1 \geq \beta_0$, hence we have (3.2).

(ii) can be proved in the same way as (i), so that we omit the proof. □
Lemma 5 can be obtained as an application of the following results.

**Theorem E.1 ([9])**. Let $A$ and $B$ be bounded linear operators on a Hilbert space $H$. The following statements are equivalent;

1. $R(A) \subseteq R(B)$;
2. $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$; and
3. there exists a bounded linear operator $C$ on $H$ so that $A = BC$.

Moreover, if (1), (2) and (3) are valid, then there exists a unique operator $C$ so that

(a) $\|C\|^2 = \inf\{\mu | AA^* \leq \mu BB^*\}$;
(b) $N(A) = N(C)$; and
(c) $R(C) \subseteq \overline{R(B^*)}$.

**Theorem E.2 ([18])**. Let $X$ and $A$ be bounded linear operators on a Hilbert space $H$. We suppose that $X \geq 0$ and $\|A\| \leq 1$. If $f$ is an operator monotone function defined on $[0, \infty)$ such that $f(0) \leq 0$, then

$$A^* f(X) A \leq f(A^* X A).$$

We remark that the condition (c) in Theorem E.1 is equivalent to the condition $(c') \overline{R(C)} \subseteq \overline{R(B^*)}$. Here we consider when the equality of $(c')$ holds.

**Lemma 6.** Let $A$ and $B$ be operators which satisfy (1), (2) and (3) of Theorem E.1, and $C$ be the operator which is given in (3) and determined uniquely by (a), (b) and (c) of Theorem E.1. Then the following assertions are mutually equivalent:

(i) $\overline{R(C)} = \overline{R(B^*)}$.

(ii) If $\lim_{n \to \infty} A^* x_n = 0$ and $\lim_{n \to \infty} B^* x_n$ exists, then $\lim_{n \to \infty} B^* x_n = 0$. 
Proof. (i) is equivalent to $N(C^*) = N(B)$ and
\[ N(C^*) = N(B) \oplus (N(B)^\perp \cap N(C^*)) = N(B) \oplus (\overline{R(B^*)} \cap N(C^*)) \]
since $N(C^*) \supseteq N(B)$ by (c) of Theorem E.1, so that (i) is equivalent to the following (3.5):
\[ (3.5) \quad \overline{R(B^*)} \cap N(C^*) = \{0\}. \]
Noting that when $y = \lim_{n \to \infty} B^* x_n$ for some \( \{x_n\} \subseteq H \),
\[ C^* y = C^* \left( \lim_{n \to \infty} B^* x_n \right) = \lim_{n \to \infty} C^* B^* x_n = \lim_{n \to \infty} A^* x_n \]
holds by (3) of Theorem E.1, so that we have
\[ \overline{R(B^*)} \cap N(C^*) = \{y | \text{there exists } \{x_n\} \subseteq H \text{ such that } y = \lim_{n \to \infty} B^* x_n \text{ and } C^* y = 0\} \]
\[ = \{y | \text{there exists } \{x_n\} \subseteq H \text{ such that } y = \lim_{n \to \infty} B^* x_n \text{ and } \lim_{n \to \infty} A^* x_n = 0\}, \]
hence (3.5) is equivalent to (ii). \( \square \)

We also require the following lemma in order to give a proof of Lemma 5.

**Lemma 7.** Let $S$ be a positive operator and $\alpha \in (0, 1]$. If $\lim_{n \to \infty} S x_n = 0$ and $\lim_{n \to \infty} S^\alpha x_n$ exists, then $\lim_{n \to \infty} S^\alpha x_n = 0$.

**Proof.** $\lim_{n \to \infty} S^\alpha x_n \in \overline{R(S^\alpha)} \cap N(S^{1-\alpha}) = \overline{R(S)} \cap N(S) = \{0\}$ for $\alpha \in (0, 1)$ since $S^{1-\alpha} \left( \lim_{n \to \infty} S^\alpha x_n \right) = \lim_{n \to \infty} S x_n = 0$ by the hypothesis. \( \square \)

**Proof of Lemma 5.**

**Proof of (i).** $B \geq C$ ensures $B^r \geq C^r$ for $r \in (0, 1]$ by Löwner-Heinz theorem. By Theorem E.1, there exists an operator $X$ such that
\[ (3.6) \quad B^{\frac{r}{2}} X = X^* B^{\frac{r}{2}} = C^{\frac{r}{2}}, \]
\[ (3.7) \quad \|X\| \leq 1. \]
Then we have
\[(C^{\frac{r}{2}}A^{\frac{r}{2}}c^{p}A\frac{r}{2})^{\frac{r}{p+r}} = (X^{*}B^{\frac{r}{2}}A^{\frac{r}{2}}B^{\frac{r}{2}}X)^{\frac{r}{p+r}} \quad \text{by (3.6)}\]
\[\geq X^{*}(B^{\frac{r}{2}}A^{\frac{r}{2}}B^{\frac{r}{2}}X)^{\frac{r}{p+r}}X \quad \text{by Theorem E.2 and (3.7)}\]
\[\geq X^{*}B^{r}X \quad \text{by the hypothesis}\]
\[= C^{r} \quad \text{by (3.6)}.
\]

**Proof of (ii).** $A \geq B$ ensures $A^{r} \geq B^{r}$ for $r \in (0, 1]$ by Löwner-Heinz theorem. By Theorem E.1, there exists an operator $Y$ such that
\[(3.8) \quad A^{\frac{r}{2}}Y = Y^{*}A^{\frac{r}{2}} = B^{\frac{r}{2}},\]
\[(3.9) \quad \|Y\| \leq 1.\]

Then we have
\[Y^{*}(A^{\frac{r}{2}}C^{p}A^{\frac{r}{2}})^{\frac{r}{p+r}}Y \leq (Y^{*}A^{\frac{r}{2}}C^{p}A^{\frac{r}{2}}Y)^{\frac{r}{p+r}} \quad \text{by Theorem E.2 and (3.9)}\]
\[= (B^{\frac{r}{2}}C^{p}B^{\frac{r}{2}})^{\frac{r}{p+r}} \quad \text{by (3.8)}\]
\[\leq B^{r} \quad \text{by the hypothesis}\]
\[= Y^{*}A^{r}Y \quad \text{by (3.8)},\]
so that $A^{r} \geq (A^{\frac{r}{2}}C^{p}A^{\frac{r}{2}})^{\frac{r}{p+r}}$ holds on $\overline{R(Y)}$. On the other hand, $(\ast)$ implies the following condition:

\[(\ast\ast) \quad \text{if } \lim_{n \to \infty} B^{\frac{r}{2}}x_{n} = 0 \text{ and } \lim_{n \to \infty} A^{\frac{r}{2}}x_{n} \text{ exists, then } \lim_{n \to \infty} A^{\frac{r}{2}}x_{n} = 0\]

since if $\lim_{n \to \infty} B^{\frac{r}{2}}x_{n} = 0$ and $\lim_{n \to \infty} A^{\frac{r}{2}}x_{n}$ exists, then
\[\lim_{n \to \infty} B^{\frac{1}{2}}x_{n} = B^{\frac{1-\varepsilon}{2}} \left( \lim_{n \to \infty} B^{\frac{r}{2}}x_{n} \right) = 0\]
and $\lim_{n \to \infty} A^{\frac{1}{2}}x_{n} = A^{\frac{1-\varepsilon}{2}} \left( \lim_{n \to \infty} A^{\frac{r}{2}}x_{n} \right)$ exists, so that $\lim_{n \to \infty} A^{\frac{1}{2}}x_{n} = 0$ by $(\ast)$, and $\lim_{n \to \infty} A^{\frac{r}{2}}x_{n} = 0$ by Lemma 7. $(\ast\ast)$ ensures $\overline{R(Y)} = \overline{R(A^{\frac{r}{2}})}$ by Lemma 6, hence we have
\[N((A^{\frac{r}{2}}C^{p}A^{\frac{r}{2}})^{\frac{r}{p+r}}) = N(A^{\frac{r}{2}}C^{p}A^{\frac{r}{2}}) \supseteq N(A^{\frac{r}{2}}) = N(A^{r}) = N(Y^{*}),\]
so that $A^{r} = (A^{\frac{r}{2}}C^{p}A^{\frac{r}{2}})^{\frac{r}{p+r}} = 0$ on $N(Y^{*})$. Consequently the proof is complete since $H = \overline{R(Y)} \oplus N(Y^{*})$. □
**Proof of Theorem 1.** Put \(A_n = |T^n|^2\) and \(B_n = |T^{n*}|^2\) for each integer \(n\).

By the definition, \(T\) belongs to class \(wA\) if and only if

\[
(3.10) \quad \left(B_1^{\frac{1}{2}} A_1 B_1^{\frac{1}{2}} \right)^{\frac{1}{2}} = \left(|T^{*}||T|^2|T^{*}|\right)^{\frac{1}{2}} \geq |T^{*}|^2 = B_1
\]

and

\[
(3.11) \quad A_1 = |T|^2 \geq \left(|T||T^{*}|^2|T|\right)^{\frac{1}{2}} = (A_1^\frac{1}{2} B_1 A_1^\frac{1}{2})^{\frac{1}{2}}.
\]

We shall prove

\[
(3.12) \quad A_{n+1} = |T^{n+1}|^2 \geq |T^n|^2 = A_n
\]

and

\[
(3.13) \quad B_n = |T^{n*}|^2 = |T^{n+1*}|^2 \geq B_{n+1}^n
\]

hold for all positive integer \(n\) by induction. (3.12) and (3.13) hold for \(n = 1\) by Proposition B. Assume (3.12) holds for \(n = 1, 2, \cdots, k - 1\). Then \(A_{n+1} \geq A_n\) holds by Löwner-Heinz theorem for \(\frac{1}{n} \in [0, 1]\), so that we have

\[
(3.14) \quad A_k \geq A_{k-1} \geq \cdots \geq A_2 \geq A_1.
\]

We remark that \(A_1\) and \(A_k\) satisfy the condition

\[
(\star) \quad \text{if } \lim_{n \to \infty} A_{1}^{\frac{1}{2}} x_n = 0 \text{ and } \lim_{n \to \infty} A_{k}^{\frac{1}{2}} x_n \text{ exists, then } \lim_{n \to \infty} A_{k}^{\frac{1}{2}} x_n = 0
\]

since

\[
\lim_{n \to \infty} A_{1}^{\frac{1}{2}} x_n = 0 \iff \lim_{n \to \infty} |T| x_n = 0 \iff \lim_{n \to \infty} T x_n = 0 \iff \lim_{n \to \infty} T^k x_n = 0
\]

\[
\iff \lim_{n \to \infty} |T^k| x_n = 0 \iff \lim_{n \to \infty} A_{k}^{\frac{1}{2}} x_n = 0 \iff \lim_{n \to \infty} A_{k}^{\frac{1}{2}} x_n = 0.
\]

The last implication holds by Lemma 7. By applying (ii) of Lemma 5 to (3.11) and (3.14), we have

\[
(3.15) \quad A_k \geq (A_k^\frac{1}{2} B_1 A_k^\frac{1}{2})^{\frac{1}{2}}.
\]
By applying (ii) of Proposition 4 to (3.15),

\[(B_{1}^{\frac{k}{12}}A_{k}^{\alpha_{2}}B_{1}^{\frac{k}{12}})^{1} \geq B_{1}^{\frac{k}{12}}A_{k}^{\alpha_{1}}B_{1}^{\frac{k}{12}}\]

holds for any \(\alpha_{1}\) and \(\alpha_{2}\) such that \(\alpha_{2} \geq \alpha_{1} \geq 1\), so that we have

\[(B_{1}^{\frac{k}{12}}A_{k}^{\frac{k}{12}}B_{1}^{\frac{k}{12}})^{\frac{k}{k+1}} \geq B_{1}^{\frac{k}{12}}A_{k}^{\frac{k-1}{2}}A_{k}^{\frac{k}{12}} \geq B_{1}^{\frac{k}{12}}A_{k-1}^{\frac{k}{12}}B_{1}^{\frac{k}{12}},\]

since the first inequality is obtained by putting \(\alpha_{1} = k - 1\) and \(\alpha_{2} = k\) in (3.16), and the second holds since (3.12) holds for \(n = k - 1\) by the inductive assumption. (3.17) yields the following (3.18):

\[(|T^{*}|T^{k+1}|2|T^{*}|)^{\frac{k}{k+1}} \geq |T^{*}|T^{k-1}|2|T^{*}|.\]

Let \(T = U|T|\) be the polar decomposition of \(T\), then \(T^{*} = U^{*}|T^{*}|\) is the polar decomposition of \(T^{*}\). Here we have

\[|T^{k+1}|^{\frac{k}{k+1}} = (T^{*}|T^{k+1}|2T)^{\frac{k}{k+1}}\]

\[= (U^{*}|T^{*}|T^{k+1}|2|T^{*}|U)^{\frac{k}{k+1}}\]

\[= U^{*}(|T^{*}|T^{k+1}|2|T^{*}|)^{\frac{k}{k+1}}U\]

\[\geq U^{*}|T^{*}|T^{k-1}|2|T^{*}|U \quad \text{by (3.18)}\]

\[= T^{*}|T^{k-1}|2T\]

\[= |T^{k}|^{2},\]

so that it is proved that (3.12) holds for \(n = k\). (3.13) can be proved in the same way as (3.12), so that we omit the proof.

**Proof of (ii).** The first inequality of (ii) has been already proved in (3.14), and the second can be proved in the same way as the first. \(\Box\)

**Proof of Theorem 2.** Put \(A_{n} = |T^{n}|^{\frac{2}{n}}\) and \(B_{n} = |T^{n*}|^{\frac{2}{n}}\) for each integer \(n\), then \(T\) belongs to class \(wA(s,t)\) if and only if

\[(B_{1}^{\frac{k}{12}}A_{1}^{\frac{k}{12}}B_{1}^{\frac{k}{12}})^{\frac{k}{k+1}} = (|T^{*}|T^{k+1}|2|T^{*}|)^{\frac{k}{k+1}} \geq |T^{*}|^{2t} = B_{1}^{t}\]

and

\[A_{1}^{s} = |T|^{2s} \geq (|T^{s}|T^{*}|2t|T^{s}|)^{\frac{s}{s+t}} = (A_{1}^{\frac{s}{t}}B_{1}^{\frac{s}{t}}A_{1}^{\frac{s}{t}})^{\frac{s}{s+t}}\]
by the definition. Now $T$ belongs to class $wA$ since

$$\text{class } wA = \text{class } wA(1,1) \supseteq \text{class } wA(s,t)$$

for $s \in (0,1]$ and $t \in (0,1]$ by (ii) of Theorem C.1, so that by (ii) of Theorem 1,

(3.21) \quad \quad A_n \geq A_1

and

(3.22) \quad \quad B_1 \geq B_n

hold for all positive integer $n$. Hence we have

(3.23) \quad \quad A_n^s \geq (A_n^s B_1^t A_n^s)^{\frac{s}{s+t}} \geq (A_n^s B_n^t A_n^s)^{\frac{s}{s+t}}.

The first inequality in (3.23) is obtained by applying (ii) of Lemma 5 to (3.20) and (3.21) since $A_1$ and $A_n$ satisfy the condition

(★) \quad \quad \text{if } \lim_{k \to \infty} A_1^{\frac{1}{2}} x_k = 0 \text{ and } \lim_{k \to \infty} A_n^{\frac{1}{2}} x_k \text{ exists, then } \lim_{k \to \infty} A_n^{\frac{1}{2}} x_k = 0,

and the second holds by (3.22) and Löwner-Heinz theorem. (3.23) yields the following (3.24):

(3.24) \quad \quad |T^n|^{\frac{2s}{n}} \geq (|T^n|^\frac{s}{n} |T^{n*}|^{\frac{2t}{n}} |T^n|^\frac{s}{n})^{\frac{\frac{t}{n}}{\frac{s}{n} + \frac{t}{n}}}.

The following (3.25) can be obtained in the same way as (3.24):

(3.25) \quad \quad (|T^{n*}|^\frac{s}{n} |T^n|^{\frac{2s}{n}} |T^{n*}|^\frac{s}{n})^{\frac{\frac{t}{n}}{\frac{s}{n} + \frac{t}{n}}} \geq |T^{n*}|^{\frac{2t}{n}},

so that $T^n$ belongs to class $wA(\frac{s}{n}, \frac{t}{n})$ by the definition.

\[\square\]

Proof of Corollary 3. If $T$ belongs to class $wA(\frac{1}{2},\frac{1}{2})$, then $T^n$ belongs to class $wA(\frac{1}{2n},\frac{1}{2n})$ by Theorem 2, so that $T^n$ belongs to class $wA(\frac{1}{2},\frac{1}{2})$ by (ii) of Theorem C.1. Hence the proof is complete since class $wA(\frac{1}{2},\frac{1}{2})$ coincides with the class of $w$-hyponormal operators.

\[\square\]
4 Concluding remarks

Remark 1. \((B^\beta A^\alpha B^\beta)_{\alpha_0+\beta_0} \geq B^\beta\) and \(A^\alpha \geq (A^\alpha B^\beta A^\alpha)_{\alpha_0+\beta_0}\) in the assumptions of (i) and (ii) of Proposition 4 are mutually equivalent in case both \(A\) and \(B\) are invertible. In fact, by applying Lemma F to the right-hand side of the second inequality, we have

\[
A^\alpha \geq (A^\alpha B^\beta A^\alpha)_{\alpha_0+\beta_0} = A^\alpha B^\beta (B^\beta A^\alpha B^\beta)_{\alpha_0+\beta_0} B^\beta A^\alpha,
\]

so that the first inequality is obtained. But it is pointed out in [20] that they are not equivalent in general if either \(A\) or \(B\) are not invertible. In fact, \(A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\) and \(B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\) satisfy the second inequality, but do not satisfy the first.

Remark 2. Lemma 5 can be proved easily in case \(A\), \(B\) and \(C\) are invertible. In fact, (i) can be proved as follows: By Lemma F, \((B^\frac{r}{2} A^p B^\frac{r}{2})_{\frac{r}{p+r}} \geq B^r\) and \((C^\frac{r}{2} A^p C^\frac{r}{2})_{\frac{r}{p+r}} \geq C^r\) are equivalent to \(A^p \geq (A^\frac{r}{2} B^r A^\frac{r}{2})_{\frac{r}{p+r}} A^p \geq (A^\frac{r}{2} C^r A^\frac{r}{2})\), respectively, so that the first inequality implies the second by the assumption \(B \geq C\) and Löwner-Heinz theorem. (ii) can be proved similarly.

And one might expect that (ii) of Lemma 5 holds without the condition \((*)\). But there exists a counterexample. Put

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},
\]

then \(A \geq B\) and \(N(A) \subseteq N(B)\), so that \(A\) and \(B\) do not satisfy the condition \((*)\). And for each \(p > 0\) and \(r \in (0, 1]\),

\[
B^r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (B^\frac{r}{2} C^p B^\frac{r}{2})_{\frac{r}{p+r}}
\]

but

\[
A^r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \not\geq \begin{pmatrix} 0 & 0 \\ 0 & 2^{\frac{r}{p+r}} \end{pmatrix} = (A^\frac{r}{2} C^p A^\frac{r}{2})_{\frac{r}{p+r}}.
\]
References

[12] T. Furuta, $A \geq B \geq 0$ assures $(B^*A^pB^r)^{1/q} \geq B^{(p+2r)/4}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p + 2r$, Proc. Amer. Math. Soc. 101 (1987), 85–88.