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Several properties on Aluthge transformation

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**Abstract**

In 1990, Aluthge defined an operator transformation $\overline{T}$ of $T$ by $\overline{T} = |T|^{|\frac{1}{2}}U|\tau|^{|\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of $T$. This transformation has very interesting properties, and many authors call $\overline{T}$ Aluthge transformation and have studied properties of this transformation.

In this paper, firstly, we shall show properties of Aluthge transformation on operator norm, and a characterization of normaloid operators by giving a definition to $n$-th Aluthge transformation $\overline{T}_n = (\overline{T}_{n-1})$.

Secondly, we shall point out that there are parallelisms between Aluthge transformation and powers of operators. Moreover we shall show $\lim_{n \to \infty} ||\overline{T}_n|| = r(T)$ which is a parallel result to $\lim_{n \to \infty} ||T^n||^{\frac{1}{n}} = r(T)$.

Lastly, we shall discuss relations between the orders $|\overline{T}|^p \geq |T|^p$ and $|T|^{p-1} \geq |T^*|^{p-1}$ for some positive number $p$.

**1. Introduction**

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $H$. For each $p > 0$, an operator $T$ is said to be $p$-hyponormal if $|T|^{2p} \geq |T^*|^{2p}$, where $|T| = (T^*T)^{\frac{1}{2}}$. Especially, an operator $T$ is said to be hyponormal if $T$ is 1-hyponormal. It is well known that "every $p$-hyponormal operator is $q$-hyponormal for $p \geq q > 0."$ And it is also well known that "for each $q > 0$, there
exists a q-hyponormal and non-p-hyponormal operator for any \( p > q > 0 \). Especially, there exists a \( \frac{1}{2} \)-hyponormal and non-hyponormal operator. Relating to these facts, many authors have studied some operator transformations from \( \frac{1}{2} \)-hyponormal operator to hyponormal operator. And the following two operator transformations were obtained:

Let \( T = U|T| \) be the polar decomposition of \( T \).

(i) \( S = U|T|^\frac{1}{2} \).

(ii) \( \overline{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2} \) (Aluthge transformation [1]).

If \( T \) is \( \frac{1}{2} \)-hyponormal, then both \( S \) and \( \overline{T} \) are hyponormal. Moreover, it was shown that \( \sigma(T) = \sigma(\overline{T}) \) in [3, 4, 11], where \( \sigma(T) \) is the spectrum of \( T \). So we understand that Aluthge transformation is a better transformation than (i).

In this paper, we shall show several properties of Aluthge transformation as follows: Firstly, it is well known that \( \|T\| \geq \|\overline{T}\| \) holds for all operator \( T \). Relating to this fact, we shall show a characterization of the condition \( \|T\| = \|\overline{T}\| \), and generalize this result by giving a definition to “n-th Aluthge transformation”. An operator \( T \) is said to be normaloid if \( \|T\| = r(T) \), where \( r(T) \) is the spectral radius of \( T \). It is well known that “for each \( p > 0 \), every p-hyponormal operator is normaloid.” Moreover we shall show a characterization of normaloid operators via Aluthge transformation.

Secondly, we shall show a parallel result to powers of p-hyponormal operators for \( p \in [0,1] \) via n-th Aluthge transformation. And we shall show a new expression of spectral radius via Aluthge transformation.

Lastly, we shall discuss relations between the orders \( |T|^p \geq |T^*|^p \) and \( |\overline{T}|^{p-1} \geq |T|^{p-1} \) for some positive number \( p \).

2. A CHARACTERIZATION OF NORMALOID OPERATORS

Fujii, Izumino and Nakamoto [6] showed the following characterization of normaloid operators via Aluthge transformation as follows:

**Theorem A** ([6]). Let \( T \in B(H) \). Then the following assertions are mutually equivalent:

(1) \( T \) is normaloid.

(2) \( \|T\| = \|\overline{T}\| \) and \( \overline{T} \) is normaloid (i.e., \( \|\overline{T}\| = r(\overline{T}) \)).

In this section, we shall discuss the condition under which \( \|T\| = \|\overline{T}\| \) and normaloidness of \( T \) via Aluthge transformation. First, we obtain the following result:
Theorem 1 ([Y1]). Let $T \in B(H)$. Then for each natural number $n$, the following assertions are equivalent:

1. $\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}}$.
2. $\|T\| = \|	ilde{T}^n\|^{\frac{1}{n}}$.

Remark 1. Put $n = 1$ in Theorem 1, we obtain the following equivalence relation:

\[ \|T\| = \|T^2\|^{\frac{1}{2}} \iff \|T\| = \|	ilde{T}\|. \]

To prove Theorem 1, we cite the following norm inequality:

**Theorem B** ([10]). Let $A$ and $B$ be positive operators, and $X \in B(H)$. Then the following inequalities hold:

1. $\|A^rXB^r\| \leq \|AXB\|^r\|X\|^{1-r}$ for $r \in [0, 1]$.
2. $\|A^rXB^r\| \geq \|AXB\|^r\|X\|^{1-r}$ for $r > 1$.

**Proof of Theorem 1.** Let $T = U|T|$ be the polar decomposition of $T$.

Proof of (1) $\Rightarrow$ (2). Assume that $\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}}$. Then we have

\[
\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}} = \|\|T|^{\frac{1}{2}} (U|T|^n U|T|)^{\frac{1}{2}}\|^{\frac{1}{n+1}} = \|\|	ilde{T}^n\|^{\frac{1}{n}} \leq \|T\|^{\frac{1}{n+1}} \|	ilde{T}^n\|^{\frac{1}{n+1}}.
\]

Hence $\|T\| \leq \|	ilde{T}^n\|^{\frac{1}{n}} \leq \|	ilde{T}\| \leq \|T\|$ hold.

Proof of (2) $\Rightarrow$ (1). Assume that $\|T\| = \|	ilde{T}^n\|^{\frac{1}{n}}$. Then by (i) of Theorem B for $\frac{1}{2} \in [0, 1]$, we have

\[
\|T\| = \|	ilde{T}^n\|^{\frac{1}{n}} = \|\|T|^{\frac{1}{2}} (U|T|^n U|T|)^{\frac{1}{2}}\|^{\frac{1}{n}} = \|\|T|^{\frac{1}{2}} (U|T|^{n-1} U|T|)^{\frac{1}{2}}\|^{\frac{1}{n}} \leq \left\{ \|T| (U|T|)^{n-1} U|T|\|^{\frac{1}{2}} \cdot \|(U|T|)^{n-1} U\|^{\frac{1}{2}} \right\}^{\frac{1}{n}} \leq \|T^{n+1}\|^{\frac{1}{2n}} \cdot \|T^{n-1}\|^{\frac{1}{2n}} \leq \|T^{n+1}\|^{\frac{1}{2n}} \cdot \|T\|^{\frac{n-1}{2n}}.
\]

Hence we obtain

\[
\|T\| \leq \|T^{n+1}\|^{\frac{1}{n+1}} \leq \|T\|.
\]

Therefore the proof of Theorem 1 is complete. \qed
By considering the following “n-th Aluthge transformation”, we obtain another generalization of (2.1).

**Definition 1** (n-th Aluthge transformation [Y1]). Let $T \in B(H)$ and $T = U|T|$ be the polar decomposition of $T$. Then for each natural number $n$, n-th Aluthge transformation $\tilde{T}_n$ of $T$ is defined by $\tilde{T}_n = \tilde{(T_{n-1})}$ and $\tilde{T}_1 = \tilde{T}$.

**Theorem 2** ([Y1]). Let $T \in B(H)$. Then for each natural number $n$, the following assertions are equivalent:

1. $\|T\| = \|T^{k+1}\|^\frac{1}{k+1}$ for all $k = 1, 2, \ldots, n$.
2. $\|T\| = \|\tilde{T}_k\|$ for all $k = 1, 2, \ldots, n$.

**Proof.** We shall prove Theorem 2 by induction on $n$.

In case $n = 1$. Theorem 2 holds by (2.1).

Assume that Theorem 2 holds in case $n = m$.

In case $n = m + 1$.

Proof of (1) $\Rightarrow$ (2). Suppose that

\[ \|T\| = \|T^2\|^\frac{1}{2} = \cdots = \|T^{m+2}\|^\frac{1}{m+2}. \]

Then we have

(2.2) \[ \|T\| = \|\tilde{T}\| = \cdots = \|\tilde{T}^{m+1}\|^\frac{1}{m+1} \]

by Theorem 1. Put $S = \tilde{T}$ in (2.2). Then (2.2) asserts

\[ \|S\| = \|S^2\|^\frac{1}{2} = \cdots = \|S^{m+1}\|^\frac{1}{m+1}. \]

By the induction hypothesis for the case $n = m$, we have

(2.3) \[ \|\tilde{T}\| = \|S\| = \|\tilde{S}\| = \cdots = \|\tilde{S}_m\| = \|(T_m)\| = \|\tilde{T}_{m+1}\|. \]

Hence we obtain

\[ \|T\| = \|\tilde{T}\| = \cdots = \|\tilde{T}_{m+1}\| \]

by (2.2) and (2.3).

Proof of (2) $\Rightarrow$ (1). Suppose that

(2.4) \[ \|T\| = \|\tilde{T}\| = \cdots = \|\tilde{T}_{m+1}\| = \|(T_m)\|. \]

Put $S = \tilde{T}$ in (2.4). Then we have

\[ \|S\| = \|\tilde{S}\| = \cdots = \|\tilde{S}_m\| \]

By the induction hypothesis for the case $n = m$, we have

\[ \|T\| = \|\tilde{T}\| = \|S\| = \|S^2\|^\frac{1}{2} = \cdots = \|S^{m+1}\|^\frac{1}{m+1} = \|\tilde{T}^{m+1}\|^\frac{1}{m+1}. \]
Hence we have
\[ ||T|| = ||T^k||^{\frac{1}{k}} \text{ for all } k = 1, 2, \ldots, m + 2 \]
by Theorem 1.

Therefore the proof of Theorem 2 is complete. \qed

By Theorem 2, we obtain the following Corollary 3 which is a characterization of normaloid operators, immediately.

**Corollary 3 ([Y1]).** Let \( T \in B(H) \). Then the following assertions are equivalent:
1. \( T \) is normaloid.
2. \( ||T|| = ||T_n|| \) for all natural number \( n \).

By Corollary 3, we can obtain Theorem A, easily as follows:

\( T \) is normaloid
\[ \iff ||T|| = ||\overline{T}|| = ||T_n|| \text{ for all natural number } n \text{ by Corollary 3} \]
\[ \iff ||T|| = ||\overline{T}|| \text{ and } \overline{T} \text{ is normaloid by Corollary 3}. \]

**Proof of Corollary 3.** We recall the following well-known result:

\( T \) is normaloid \( \iff ||T|| = ||T^n||^{\frac{1}{n}} \text{ for all positive integer } n. \)

Hence we obtain Corollary 3 by Theorem 2. \qed

### 3. Parallel results between Aluthge transformation and powers of operators

It was shown that “there exists a hyponormal operator \( T \) such that \( T^2 \) is not hyponormal” in [9]. Relating to this fact, Aluthge and Wang [2] showed that “if \( T \) is a \( p \)-hyponormal operator for \( p \in (0, 1] \), then \( T^n \) is \( \frac{p}{n} \)-hyponormal for all natural number \( n \).” As an extension of this result, the following result was shown in [8]:

**Theorem C ([8]).** Let \( T \) be a \( p \)-hyponormal operator for \( p \in (0, 1] \). Then for each natural number \( n \), the following inequalities hold:

(i) \( |T|^{2(p+1)} \leq |T^n|^{2(p+1)} \leq \cdots \leq |T^n|^{\frac{2(p+1)}{n}} \).
(ii) \( |T^*|^{2(p+1)} \geq |T^{n*}|^{2(p+1)} \geq \cdots \geq |T^{n*}|^{\frac{2(p+1)}{n}} \).

We remark that as a generalization of the result by Aluthge and Wang, Ito [12] showed that “if \( T \) is a \( p \)-hyponormal operator for \( p > 0 \), then \( T^n \) is \( \min\{1, \frac{p}{n}\} \)-hyponormal for all natural number \( n \).” And he showed an extension of Theorem C. As a parallel result to Theorem C, we obtain the following result:
Theorem 4 ([Y2]). Let $T$ be a $\frac{p}{2}$-hyponormal operator for $p \in (0, 1]$. Then for each
natural number $n$, the following inequalities hold:

(i) $|T|^p \leq |\tilde{T}|^p \leq \cdots \leq |\tilde{T}_n|^p$.
(ii) $|T|^p \geq |\tilde{T}|^p \geq \cdots \geq |\tilde{T}_n|^p$.

To prove Theorem 4, we prepare the following results and definition:

Theorem D ([1, 7, 11, 16]). Let $T$ be a $\frac{p}{2}$-hyponormal operator for $p > 0$ (i.e., $|T|^p \geq |T^*|^p$). Then the following inequalities hold:

(i) In case $p \in (0, 1]$, $|\tilde{T}|^p \geq |T|^p \geq |(\tilde{T})^*|^p$ (i.e., $\tilde{T}$ is $\frac{p+1}{2}$-hyponormal).
(ii) In case $p \in [1, 2]$, $|\tilde{T}|^2 \geq |T|^2 \geq |(\tilde{T})^*|^2$ (i.e., $\tilde{T}$ is hyponormal).

Theorem D was shown in [1] when $U$ is unitary, where $T = U|T|$ is the polar decomposi-
tion of $T$. And Theorem D was shown in [11, 16]. Moreover, a generalization of Theorem D was shown in [7, 11, 16].

Definition 2 (*-Aluthge transformation [Y2]). Let $T = U|T|$ be the polar decom-
pition of an operator $T$. Then $*$-Aluthge transformation of $T$ is defined as follows:

(i) $\overline{T}^{(*)} \overset{\text{def}}{=} (\overline{T}^*)^* = |T^*|^\frac{1}{2}U|T^*|^\frac{1}{2}$ ($*$-Aluthge transformation).
(ii) For each natural number $n$,

$\overline{T}_n^{(*)} \overset{\text{def}}{=} (\overline{T}_{n-1}^{(*)})^* = (\overline{T}_n^{(*)})^*$,

and $\overline{T}_1^{(*)} \overset{\text{def}}{=} \overline{T}^{(*)}$ ($n$-th $*$-Aluthge transformation).

As relations between $\tilde{T}$ and $\overline{T}^{(*)}$, we obtain the following results, immediately.

Theorem 5 ([Y2]). Let $T \in B(H)$. Then the following assertions hold:

(i) $\sigma(\tilde{T}) = \sigma(\overline{T}^{(*)}) = \sigma(T)$.
(ii) $w(\tilde{T}) = w(\overline{T}^{(*)})$, where $w(T)$ is the numerical radius of $T$.
(iii) $\|\tilde{T}\| = \|\overline{T}^{(*)}\|$.

We remark that (i) of Theorem 5 asserts more generalization form of $\sigma(\tilde{T}) - \{0\} = \sigma(\overline{T}^{(*)}) - \{0\} = \sigma(T) - \{0\}$. And $\sigma(\tilde{T}) = \sigma(T)$ has been already shown in [3, 4, 11].

Proposition 6 ([Y2]). Let $T \in B(H)$. Then for each $p > 0$,

$\tilde{T}$ is $p$-hyponormal $\iff$ $\overline{T}^{(*)}$ is $p$-hyponormal.

Remark 2. If $T$ is $\frac{p}{2}$-hyponormal for $p \in (0, 2]$, then we obtain the following assertions, easily. $\overline{T}_n$ is $\frac{p}{2}$-hyponormal for all natural number $n$ by using Theorem D
several times, and also $\widetilde{T}_n^{(*)}$ is $\frac{p}{2}$-hyponormal for all natural number $n$ by Theorem D and Proposition 6 several times.

**Proof of Theorem 4.** We shall prove Theorem 4 by induction on $n$.

Proof of (i). (a) By (i) of Theorem D, we have $|\widetilde{T}|^{p+1} \geq |T|^{p+1}$.

(b) Assume that $|\widetilde{T}_{n-1}|^{p+1} \geq \cdots \geq |\widetilde{T}|^{p+1} \geq |T|^{p+1}$.

(c) Proof of $|\widetilde{T}_n|^{p+1} \geq |\widetilde{T}_{n-1}|^{p+1}$.

Put $S = \widetilde{T}_{n-1}$. Then $S$ is also $\frac{p}{2}$-hyponormal by Remark 2. Hence we have

$$|\widetilde{T}_n|^{p+1} = |S|^{p+1} \geq |S|^{p+1} = |\widetilde{T}_{n-1}|^{p+1}$$

by (i) of Theorem D.

Proof of (ii). Let $T = U|T|$ be the polar decomposition of $T$.

(a) Proof of $|T^*|^{p+1} \geq |\widetilde{T}^*|^{p+1}$.

$$|T^*|^{p+1} = U|T|^{p+1}U^*$$

$$\geq U|(\widetilde{T})^*|^{p+1}U^*$$

by (i) of Theorem D

$$= U(|T|^{\frac{1}{2}}U|T^*|U|T|^{\frac{1}{2}})^{p+1}U^*$$

$$= (U|T|^{\frac{1}{2}}U|T^*|U|T|^{\frac{1}{2}})^{p+1}$$

$$= |T^*|^{p+1}$$

(b) Assume that $|T^*|^{p+1} \geq |\widetilde{T}^*|^{p+1} \geq \cdots \geq |\widetilde{T}_{n-1}^*|^{p+1}$.

(c) Proof of $|\widetilde{T}_{n-1}^*|^{p+1} \geq |\widetilde{T}_n^*|^{p+1}$.

Put $S = (T_{n-1}^*)^* = (\widetilde{T}_{n-1})^*$. Then $S$ is also $\frac{p}{2}$-hyponormal by Remark 2. By (a), we obtain

$$|\widetilde{T}_{n-1}^*|^{p+1} = |S^*|^{p+1} \geq |S^*|^{p+1} = |\widetilde{T}_n^*|^{p+1}$$

Therefore the proof of Theorem 4 is complete. \(\square\)

By considering Theorem 4, we can understand that $n$-th Aluthge transformation and powers of operators have similar properties. On the other hand, \(\lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = r(T)\) is a very famous and useful result. So we shall show the parallel result to this one as follows:

**Theorem 7 ([Y3]).** Let $T \in B(H)$. Then \(\lim_{n \to \infty} \|\widetilde{T}_n\| = r(T)\).

To prove Theorem 7, we prepare the following lemmas:
Lemma 8 ([Y3]). For a natural number \( n \) and \( k = 0, 1, \cdots, n + 1 \), let
\[
D_k = \frac{n!(n-2k+1)}{k!(n-k+1)!}.
\]
Then the following assertions hold:

(i) \( D_0 = 1 \) for all natural number \( n \).

(ii) \( D_k + D_{k+1} = D_{k+1} + D_{k+1} \) for all natural number \( n \)
    and \( k = 0, 1, \cdots, n \).

(iii) \( 2D_{n+1} = 2(n+1)D_{n+1} \) for all natural number \( n \).

(iv) \( \sum_{k=0}^{[\frac{n}{2}]} (n-2k+1)D_k = 2^n \),
     where \( [\frac{n}{2}] \) is the largest integer satisfying \( \frac{n}{2} \leq \frac{n}{2} \).

(v) \( \lim_{n \to \infty} \frac{n(n-2k+1)}{2^n}D_k = 0 \) for all positive integer \( k \).

Lemma 9 ([Y3]). Let \( T \in B(H) \). Then
\[
\| \tilde{T}^n \| \leq \| T^n+1 \|^{\frac{1}{2}} \| T^n-1 \|^{\frac{1}{2}}
\]
holds for all natural number \( n \).

Proof. Let \( T = U|T| \) be the polar decomposition of \( T \). Then we have
\[
\| \tilde{T}^n \| = \| (|T|^\frac{1}{2}U|T|^\frac{1}{2})^n \| = \| |T|^\frac{1}{2} (U|T|)^{n-1} U|T|^{\frac{1}{2}} \| 
\leq \| U|T|^{n-1} U \|^{\frac{1}{2}} \| (U|T|)^{n-1} U \|^{\frac{1}{2}} \quad \text{by (i) of Theorem B}
\leq \| T^{n+1} \|^{\frac{1}{2}} \| T^{n-1} \|^{\frac{1}{2}}.
\]

Lemma 10 ([Y3]). Let \( T \in B(H) \) and \( m = [\frac{n}{2}] \). Then
\[
\| \tilde{T}_n \| \leq \| T^{n+1} \|^{\frac{nD_0}{2^n}} \| T^{n-1} \|^{\frac{nD_1}{2^n}} \cdots \| T^{n-2k+1} \|^{\frac{nD_k}{2^n}} \cdots \| T^{n-2m+1} \|^{\frac{nD_m}{2^n}}.
\]
Proof. We shall prove Lemma 10 by induction on \( n \).

(a) \( \| \tilde{T} \| \leq \| T^2 \|^{\frac{1}{2}} \) holds by Lemma 9.

(b) Assume that
\[
\| \tilde{T}_{n-1} \| \leq \| T^n \|^{\frac{nD_0}{2^n-1}} \| T^{n-2} \|^{\frac{n-1D_1}{2^n-1}}
\times \cdots \times \| T^{n-2k} \|^{\frac{n-1D_k}{2^n-1}} \cdots \| T^{n-2m} \|^{\frac{n-1D_m}{2^n-1}},
\]
where \( m = [\frac{n-1}{2}] \).
In case $n = 2m + 1$ for $m = 1, 2, \cdots$. Then $\left[ \frac{n}{2} \right] = \left[ \frac{n-1}{2} \right] = m$. Hence by (3.2), we have
\[
\|\tilde{T}_n\| = \|(\tilde{T})_{n-1}\|
\leq \|\tilde{T}^n\| \left( \frac{n-1}{2^n-1} \right) \|\tilde{T}^{n-2}\| \left( \frac{n-1}{2^n-1} \right) \cdots \|\tilde{T}^3\| \left( \frac{n-1}{2^n-1} \right) \|\tilde{T}\| \left( \frac{n-1}{2^n-1} \right)
\leq \left\{ \left( \|\tilde{T}^n\| \frac{1}{2^n-1} \right) \left( \|\tilde{T}^{n-1}\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\} \left\{ \left( \|\tilde{T}^n\| \frac{1}{2^n-1} \right) \left( \|\tilde{T}^{n-1}\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\} \cdots \left\{ \left( \|\tilde{T}^3\| \frac{1}{2^n-1} \right) \left( \|\tilde{T}\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\} \left\{ \left( \|\tilde{T}\| \frac{1}{2^n-1} \right) \left( \|\tilde{T}\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\}
\leq \left\{ \left( \|\tau^{n+1}\| \frac{1}{2^n-1} \right) \left( \|T^{n-1}\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\} \left\{ \left( \|\tau^{n-1}\| \frac{1}{2^n-1} \right) \left( \|T^{n-3}\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\} \cdots \left\{ \left( \|\tau^3\| \frac{1}{2^n-1} \right) \left( \|\tau\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\} \leq \left\{ \left( \|\tau^{n+1}\| \frac{1}{2^n} \right) \left( \|\tau^{n-1}\| \frac{1}{2^n} \right) \frac{n-1}{2^n} \right\} \left\{ \left( \|\tau^{n-1}\| \frac{1}{2^n} \right) \left( \|\tau\| \frac{1}{2^n} \right) \frac{n-1}{2^n} \right\} \cdots \left\{ \left( \|\tau^3\| \frac{1}{2^n} \right) \left( \|\tau\| \frac{1}{2^n} \right) \frac{n-1}{2^n} \right\} \leq \|\tilde{T}^n\| \frac{1}{2^n} \|\tilde{T}^{n-1}\| \frac{1}{2^n} \|\tilde{T}^3\| \frac{1}{2^n} \cdots \|\tilde{T}\| \frac{1}{2^n} \|\tau\| \frac{1}{2^n} \frac{n-1}{2^n} \text{ by (i) and (ii) of Lemma 8, and the last inequality holds by Lemma 9.}
\]

(c-2) In case $n = 2m + 2$ for $m = 0, 1, 2, \cdots$. Then $\left[ \frac{n}{2} \right] = m + 1$ and $\left[ \frac{n-1}{2} \right] = m$. Hence by (3.2), we have
\[
\|\tilde{T}_n\| = \|(\tilde{T})_{n-1}\|
\leq \|\tilde{T}^n\| \left( \frac{n-1}{2^n-1} \right) \|\tilde{T}^{n-2}\| \left( \frac{n-1}{2^n-1} \right) \cdots \|\tilde{T}^3\| \left( \frac{n-1}{2^n-1} \right) \|\tilde{T}\| \left( \frac{n-1}{2^n-1} \right)
\leq \left\{ \left( \|\tilde{T}^n\| \frac{1}{2^n-1} \right) \left( \|\tilde{T}^{n-1}\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\} \left\{ \left( \|\tilde{T}^n\| \frac{1}{2^n-1} \right) \left( \|\tilde{T}^{n-1}\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\} \cdots \left\{ \left( \|\tilde{T}^3\| \frac{1}{2^n-1} \right) \left( \|\tilde{T}\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\} \left\{ \left( \|\tilde{T}\| \frac{1}{2^n-1} \right) \left( \|\tilde{T}\| \frac{1}{2^n-1} \right) \frac{n-1}{2^n-1} \right\}
\leq \left\{ \left( \|\tau^{n+1}\| \frac{1}{2^n} \right) \left( \|\tau^{n-1}\| \frac{1}{2^n} \right) \frac{n-1}{2^n} \right\} \left\{ \left( \|\tau^{n-1}\| \frac{1}{2^n} \right) \left( \|\tau\| \frac{1}{2^n} \right) \frac{n-1}{2^n} \right\} \cdots \left\{ \left( \|\tau^3\| \frac{1}{2^n} \right) \left( \|\tau\| \frac{1}{2^n} \right) \frac{n-1}{2^n} \right\} \leq \|\tilde{T}^n\| \frac{1}{2^n} \|\tilde{T}^{n-1}\| \frac{1}{2^n} \|\tilde{T}^3\| \frac{1}{2^n} \cdots \|\tilde{T}\| \frac{1}{2^n} \|\tau\| \frac{1}{2^n} \frac{n-1}{2^n} \text{ by (i) and (ii) of Lemma 8, and the last inequality holds by Lemma 9.}
\]

$\square$
Lemma 11 ([Y3]). Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence satisfying \( \lim_{n \to \infty} a_n = a \), and for each natural number \( n \), \( \{c_{n,k}\}_{k=1}^{n} \) be a positive sequence satisfying
\[
(3.3) \quad c_{n,1} + \cdots + c_{n,k} + \cdots + c_{n,n} = 1 \quad \text{for all natural number } n
\]
and \( \lim_{n \to \infty} c_{n,k} = 0 \) for fixed \( k = 1, 2, \ldots \). Then
\[
\lim_{n \to \infty} (c_{n,1}a_1 + \cdots + c_{n,k}a_k + \cdots + c_{n,n}a_n) = a.
\]

**Proof of Theorem 7.** Let \( m = [\frac{n}{2}] \). Then by Lemma 10, (iv) of Lemma 8 and Arithmetic mean-Geometric mean inequality, we have
\[
r(T) = r(\overline{T}_n) \leq \|\overline{T}_{n+1}\| \leq \|T^{n+1}\|^{\frac{nD_0}{2^n}} \frac{nD_k}{2^n} \ldots \|T^{n-2m+1}\|^{\frac{nD_m}{2^n}}
\]
\[
\leq \frac{(n+1)nD_0}{2^n} \|T^{n+1}\|^{\frac{1}{n+1}} + \frac{(n-1)nD_1}{2^n} \|T^{n-1}\|^{\frac{1}{n-1}} + \cdots + \frac{(n-2k+1)nD_k}{2^n} \|T^{n-2k+1}\|^{\frac{1}{n-2k+1}} + \cdots + \frac{(n-2m+1)nD_m}{2^n} \|T^{n-2m+1}\|^{\frac{1}{n-2m+1}}
\]
\[
\to r(T) \quad \text{as } n \to \infty
\]
by \( \lim_{n \to \infty} \|T^n\|^\frac{1}{n} = r(T) \), (iv) and (v) of Lemma 8 and Lemma 11. \( \square \)

4. Relations between the orders \( |\overline{T}|^p \geq |T|^p \) and \( |T|^p \geq |T^*|^p \)

In this section, we shall discuss properties of the order \( |\overline{T}|^p \geq |T|^p \) for some \( p > 0 \). Relating to this order, Theorem D is very famous. As a converse of Theorem D, we obtain the following result:

**Theorem 12 ([Y2]).** Let \( T \) be an invertible operator. Then the following assertions hold:

(i) For each \( p \in [2, 4] \), \( |\overline{T}|^p \geq |T|^p \) ensures \( |T|^{p-1} \geq |T^*|^{p-1} \).

(ii) For each \( p \geq 4 \), \( |\overline{T}|^p \geq |T|^p \) ensures \( |T|^3 \geq |T^*|^3 \).

To prove Theorem 12, we need the following result:

**Theorem E ([5, 13, 14, 15]).** Let \( A \) and \( B \) be positive invertible operators. Then the following assertions hold:

(i) \( A \geq B > 0 \) ensures \( (B^{\frac{-t}{2}} A^p B^{\frac{t}{2}})^{\frac{1-t}{p-1}} \geq B^{1-t} \) for \( 1 \geq p \geq \frac{1}{2} \) with \( p > t \geq 0 \).

(ii) \( A \geq B > 0 \) ensures \( (B^{\frac{-t}{2}} A^p B^{\frac{t}{2}})^{\frac{2p-1}{p-t}} \geq B^{3p-t} \) for \( \frac{1}{2} \geq p > t \geq 0 \).
Proof of Theorem 12. Let $T = U|T|$ be the polar decomposition of $T$. Then $U$ is unitary since $T$ is invertible.

Proof of (i). By applying (i) of Theorem E to $|\tilde{T}|^p \geq |T|^p$, we have

$$(|T|^{-\frac{1}{2}}|\tilde{T}|^{\frac{p}{2}} |T|^{-\frac{1}{2}})^{\frac{1-t_1}{p_1-t_1}} \geq |T|^{p(1-t_1)}$$

for $1 \geq p_1 \geq \frac{1}{2}$ with $p_1 > t_1 \geq 0$.

Put $p_1 = \frac{2}{p}$ and $t_1 = \frac{1}{p}$ in (4.1). Then we have

$$(|T|^{-\frac{1}{2}}|\tilde{T}|^2 |T|^{-\frac{1}{2}})^{p-1} \geq |T|^{p-1}.$$

It is equivalent to

$$(|T|^{-\frac{1}{2}}|\tilde{T}|^{\frac{1}{2}}U^*|T|U|T|^\frac{1}{2} |T|^{-\frac{1}{2}})^{p-1} \geq |T|^{p-1},$$

that is, $U^*|T|^{p-1} U \geq |T|^{p-1}$ since $U$ is unitary. Hence we have $|T|^{p-1} \geq U|T|^{p-1} U^* = |T^*|^{p-1}$.

Proof of (ii). By applying (ii) of Theorem E to $|\tilde{T}|^p \geq |T|^p$, we have

$$(|T|^{-\frac{1}{2}}|\tilde{T}|^{\frac{p}{2}} |T|^{-\frac{1}{2}})^{\frac{2p_1-1}{p_1-t_1}} \geq |T|^{p(2p_1-t_1)}$$

for $\frac{1}{2} \geq p_1 > t_1 \geq 0$.

Put $p_1 = \frac{2}{p}$ and $t_1 = \frac{1}{p}$ in (4.2). Then we have

$$(|T|^{-\frac{1}{2}}|\tilde{T}|^2 |T|^{-\frac{1}{2}})^{3} \geq |T|^3.$$

It is equivalent to

$$(|T|^{-\frac{1}{2}}|\tilde{T}|^{\frac{1}{2}}U^*|T|U|T|^\frac{1}{2} |T|^{-\frac{1}{2}})^{3} \geq |T|^3,$$

that is, $U^*|T|^3 U \geq |T|^3$ since $U$ is unitary. Hence we have $|T|^3 \geq U|T|^3 U^* = |T^*|^3$.

\[\square\]

REFERENCES


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