<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
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</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>Several properties on Aluthge transformation (Development of Operator Theory and Problems)</td>
</tr>
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Several properties on Aluthge transformation

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ABSTRACT

In 1990, Aluthge defined an operator transformation $\overline{T}$ of $T$ by $\overline{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}$, where $T = U|T|$ is the polar decomposition of $T$. This transformation has very interesting properties, and many authors call $\overline{T}$ Aluthge transformation and have studied properties of this transformation.

In this paper, firstly, we shall show properties of Aluthge transformation on operator norm, and a characterization of normaloid operators by giving a definition to n-th Aluthge transformation $\overline{T_n} = (\overline{T_{n-1}})$.

Secondly, we shall point out that there are parallelisms between Aluthge transformation and powers of operators. Moreover we shall show $\lim_{n \to \infty} \|\overline{T_n}\| = r(T)$ which is a parallel result to $\lim_{n \to \infty} \|T^n\|^\frac{1}{n} = r(T)$.

Lastly, we shall discuss relations between the orders $|\overline{T}|^p \geq |T|^p$ and $|T|^{p-1} \geq |T^*|^{p-1}$ for some positive number $p$.

1. INTRODUCTION

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $H$. For each $p > 0$, an operator $T$ is said to be $p$-hyponormal if $|T|^{2p} \geq |T^*|^{2p}$, where $|T| = (T^*T)^{\frac{1}{2}}$. Especially, an operator $T$ is said to be hyponormal if $T$ is 1-hyponormal. It is well known that "every $p$-hyponormal operator is $q$-hyponormal for $p \geq q > 0."$ And it is also well known that "for each $q > 0$, there..."
exists a $q$-hyponormal and non-$p$-hyponormal operator for any $p > q > 0$. Especially, there exists a $\frac{1}{2}$-hyponormal and non-hyponormal operator. Relating to these facts, many authors have studied some operator transformations from $\frac{1}{2}$-hyponormal operator to hyponormal operator. And the following two operator transformations were obtained:

Let $T = U|T|$ be the polar decomposition of $T$.

(i) $S = U|T|^\frac{1}{2}$.
(ii) $\overline{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}$ (Aluthge transformation [1]).

If $T$ is $\frac{1}{2}$-hyponormal, then both $S$ and $\overline{T}$ are hyponormal. Moreover, it was shown that $\sigma(T) = \sigma(\overline{T})$ in [3, 4, 11], where $\sigma(T)$ is the spectrum of $T$. So we understand that Aluthge transformation is a better transformation than (i).

In this paper, we shall show several properties of Aluthge transformation as follows: Firstly, it is well known that $\|T\| \geq \|\overline{T}\|$ holds for all operator $T$. Relating to this fact, we shall show a characterization of the condition $\|T\| = \|\overline{T}\|$, and generalize this result by giving a definition to "$n$-th Aluthge transformation". An operator $T$ is said to be \textit{normaloid} if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of $T$. It is well known that "for each $p > 0$, every $p$-hyponormal operator is normaloid." Moreover we shall show a characterization of normaloid operators via Aluthge transformation.

Secondly, we shall show a parallel result to powers of $p$-hyponormal operators for $p \in [0, 1]$ via $n$-th Aluthge transformation. And we shall show a new expression of spectral radius via Aluthge transformation.

Lastly, we shall discuss relations between the orders $|T|^p \geq |T^*|^p$ and $|\overline{T}|^{p-1} \geq |T|^{p-1}$ for some positive number $p$.

2. A CHARACTERIZATION OF NORMALOID OPERATORS

Fujii, Izumino and Nakamoto [6] showed the following characterization of normaloid operators via Aluthge transformation as follows:

\textbf{Theorem A} ([6]). Let $T \in B(H)$. Then the following assertions are mutually equivalent:

(1) $T$ is normaloid.
(2) $\|T\| = \|\overline{T}\|$ and $\overline{T}$ is normaloid (i.e., $\|\overline{T}\| = r(\overline{T})$).

In this section, we shall discuss the condition under which $\|T\| = \|\overline{T}\|$ and normaloidness of $T$ via Aluthge transformation. First, we obtain the following result:
**Theorem 1** ([Y1]). Let $T \in B(H)$. Then for each natural number $n$, the following assertions are equivalent:

1. $\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}}$.
2. $\|T\| = \|\tilde{T}^{n}\|^{\frac{1}{n}}$.

**Remark 1.** Put $n = 1$ in Theorem 1, we obtain the following equivalence relation:

(2.1) $\|T\| = \|T^{2}\|^{\frac{1}{2}} \iff \|T\| = \|\overline{T}\|$.

To prove Theorem 1, we cite the following norm inequality:

**Theorem B** ([10]). Let $A$ and $B$ be positive operators, and $X \in B(H)$. Then the following inequalities hold:

(i) $\|A^r X B^r\| \leq \|AXB\|^r \|X\|^{1-r}$ for $r \in [0, 1]$.
(ii) $\|A^r X B^r\| \geq \|AXB\|^r \|X\|^{1-r}$ for $r > 1$.

**Proof of Theorem 1.** Let $T = U|T|$ be the polar decomposition of $T$.

Proof of (1) $\Rightarrow$ (2). Assume that $\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}}$. Then we have

\[
\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}} = \|T^{\frac{1}{2}} (|T^{\frac{1}{2}} U|T^{\frac{1}{2}})^n |T^{\frac{1}{2}}\|^{\frac{1}{n+1}} \leq \|T\|^\frac{1}{n+1} \|\tilde{T}^{n}\|^{\frac{1}{n+1}}.
\]

Hence $\|T\| \leq \|\tilde{T}^{n}\|^{\frac{1}{n}} = \|\tilde{T}\| \leq \|T\|$ hold.

Proof of (2) $\Rightarrow$ (1). Assume that $\|T\| = \|\tilde{T}^{n}\|^{\frac{1}{n}}$. Then by (i) of Theorem B for $\frac{1}{2} \in [0, 1]$, we have

\[
\|T\| = \|\tilde{T}^{n}\|^{\frac{1}{n}} = \|((|T^{\frac{1}{2}} U|T^{\frac{1}{2}})^n \|^{\frac{1}{n}} = \|T^{\frac{1}{2}} (U|T^{\frac{1}{2}})^{n-1} U|T^{\frac{1}{2}}\|^{\frac{1}{n}} \leq \{\|T\| (U|T^{\frac{1}{2}})^{n-1} U|T^{\frac{1}{2}}\|^{\frac{1}{n}} \cdot \|(U|T^{\frac{1}{2}})^{n-1} U\|^{\frac{1}{2}}\}^{\frac{1}{n}} \leq \|T^{n+1}\|^{\frac{1}{n+1}} \cdot \|T^{n-1}\|^{\frac{1}{n}} \leq \|T^{n+1}\|^{\frac{1}{n+1}} \cdot \|T\|^{\frac{n-1}{n}}.
\]

Hence we obtain

\[
\|T\| \leq \|T^{n+1}\|^{\frac{1}{n+1}} \leq \|T\|.
\]

Therefore the proof of Theorem 1 is complete. \qed
By considering the following “n-th Aluthge transformation”, we obtain another generalization of (2.1).

**Definition 1** (n-th Aluthge transformation [Y1]). Let $T \in B(H)$ and $T = U|T|$ be the polar decomposition of $T$. Then for each natural number $n$, n-th Aluthge transformation $\widetilde{T_n}$ of $T$ is defined by $\widetilde{T_n} = (\widetilde{T_{n-1}})$ and $\widetilde{T_1} = \widetilde{T}$.

**Theorem 2** ([Y1]). Let $T \in B(H)$. Then for each natural number $n$, the following assertions are equivalent:

1. $\|T\| = \|T^{k+1}\|^{\frac{1}{k+1}}$ for all $k = 1, 2, \cdots, n$.
2. $\|T\| = \|\widetilde{T_k}\|$ for all $k = 1, 2, \cdots, n$.

**Proof.** We shall prove Theorem 2 by induction on $n$.

In case $n = 1$. Theorem 2 holds by (2.1).
Assume that Theorem 2 holds in case $n = m$.
In case $n = m + 1$.

Proof of (1) $\Rightarrow$ (2). Suppose that

$$\|T\| = \|T^2\|^{\frac{1}{2}} = \cdots = \|T^{m+2}\|^{\frac{1}{m+2}}.$$

Then we have

(2.2) \[\|T\| = \|\widetilde{T}\| = \cdots = \|\widetilde{T^{m+1}}\|^{\frac{1}{m+1}}\]
by Theorem 1. Put $S = \widetilde{T}$ in (2.2). Then (2.2) asserts

$$\|S\| = \|S^2\|^{\frac{1}{2}} = \cdots = \|S^{m+1}\|^{\frac{1}{m+1}}.$$

By the induction hypothesis for the case $n = m$, we have

(2.3) \[\|\widetilde{T}\| = \|S\| = \|\widetilde{S}\| = \cdots = \|\widetilde{S_m}\| = \|\widetilde{(T_m)}\| = \|\widetilde{T_{m+1}}\|\]
Hence we obtain

$$\|T\| = \|\widetilde{T}\| = \cdots = \|\widetilde{T_{m+1}}\|$$
by (2.2) and (2.3).

Proof of (2) $\Rightarrow$ (1). Suppose that

(2.4) \[\|T\| = \|\widetilde{T}\| = \cdots = \|\widetilde{T_{m+1}}\| = \|\widetilde{(T_m)}\|\]
Put $S = \widetilde{T}$ in (2.4). Then we have

$$\|S\| = \|\widetilde{S}\| = \cdots = \|\widetilde{S_m}\|$$
By the induction hypothesis for the case $n = m$, we have

$$\|T\| = \|\widetilde{T}\| = \|S\| = \|S^2\|^{\frac{1}{2}} = \cdots = \|S^{m+1}\|^{\frac{1}{m+1}} = \|\widetilde{T^{m+1}}\|^{\frac{1}{m+1}}.$$
Hence we have
\[ \|T\| = \|T^k\|^{\frac{1}{k}} \] for all \( k = 1, 2, \ldots, m+2 \)
by Theorem 1.

Therefore the proof of Theorem 2 is complete. \( \square \)

By Theorem 2, we obtain the following Corollary 3 which is a characterization of normaloid operators, immediately.

**Corollary 3 ([Y1]).** Let \( T \in B(H) \). Then the following assertions are equivalent:

1. \( T \) is normaloid.
2. \( \|T\| = \|\overline{T_n}\| \) for all natural number \( n \).

By Corollary 3, we can obtain Theorem A, easily as follows:

\( T \) is normaloid

\[ \iff \|T\| = \|\overline{T}\| = \|\overline{T_n}\| = \|\overline{(T)}_{n-1}\| \] for all natural number \( n \) by Corollary 3

\[ \iff \|T\| = \|\overline{T}\| \text{ and } \overline{T} \text{ is normaloid by Corollary 3.} \]

**Proof of Corollary 3.** We recall the following well-known result:

\( T \) is normaloid \( \iff \|T\| = \|T^n\|^{\frac{1}{n}} \) for all positive integer \( n \).

Hence we obtain Corollary 3 by Theorem 2. \( \square \)

3. **Parallel results between Aluthge transformation and powers of operators**

It was shown that "there exists a hyponormal operator \( T \) such that \( T^2 \) is not hyponormal" in [9]. Relating to this fact, Aluthge and Wang [2] showed that "if \( T \) is a \( p \)-hyponormal operator for \( p \in (0,1] \), then \( T^n \) is \( \frac{p}{n} \)-hyponormal for all natural number \( n \)." As an extension of this result, the following result was shown in [8]:

**Theorem C ([8]).** Let \( T \) be a \( p \)-hyponormal operator for \( p \in (0,1] \). Then for each natural number \( n \), the following inequalities hold:

1. \( |T|^{2(p+1)} \leq |T^2|^{p+1} \leq \cdots \leq |T^n|^{\frac{2(p+1)}{n}} \).
2. \( |T^*|^{2(p+1)} \geq |T^*2|^{p+1} \geq \cdots \geq |T^n|^{\frac{2(p+1)}{n}} \).

We remark that as a generalization of the result by Aluthge and Wang, Ito [12] showed that "if \( T \) is a \( p \)-hyponormal operator for \( p > 0 \), then \( T^n \) is \( \min\{1, \frac{p}{n}\} \)-hyponormal for all natural number \( n \)." And he showed an extension of Theorem C. As a parallel result to Theorem C, we obtain the following result:
**Theorem 4** ([Y2]). Let $T$ be a $\frac{p}{2}$-hyponormal operator for $p \in (0,1]$. Then for each natural number $n$, the following inequalities hold:

(i) $|T|^p+1 \leq |\overline{T}|^p \leq \ldots \leq |\overline{T}_n|^p$.

(ii) $|T^*|^p+1 \geq |\overline{T}^*|^p \geq \ldots \geq |\overline{T}_n^*|^p$.

To prove Theorem 4, we prepare the following results and definition:

**Theorem D** ([1, 7, 11, 16]). Let $T$ be a $\frac{p}{2}$-hyponormal operator for $p > 0$ (i.e., $|T|^p \geq |T^*|^p$). Then the following inequalities hold:

(i) In case $p \in (0,1]$. $|\overline{T}|^p+1 \geq |T|^p+1 \geq |(\overline{T}^*)^+|^p$ (i.e., $\overline{T}$ is $\overline{2}$-hyponormal).

(ii) In case $p \in [1,2]$. $|\overline{T}|^2 \geq |T|^2 \geq |(\overline{T}^*)^+|^2$ (i.e., $\overline{T}$ is hyponormal).

Theorem D was shown in [1] when $U$ is unitary, where $T = U|T|$ is the polar decomposition of $T$. And Theorem D was shown in [11, 16]. Moreover, a generalization of Theorem D was shown in [7, 11, 16].

**Definition 2** (*-Aluthge transformation [Y2]). Let $T = U|T|$ be the polar decomposition of an operator $T$. Then *-Aluthge transformation of $T$ is defined as follows:

(i) $\overline{\overline{T}} \equiv (\overline{T}^*)^* = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$ (*-Aluthge transformation).

(ii) For each natural number $n$,

$\overline{T}_n^{(*)} = (T_{n-1}^{(*)})^* = (\overline{T}_n^{(*)})^*$ and $\overline{T}_1^{(*)} = \overline{T}^{(*)}$ (n-th *-Aluthge transformation).

As relations between $\overline{T}$ and $\overline{T}^{(*)}$, we obtain the following results, immediately.

**Theorem 5** ([Y2]). Let $T \in B(H)$. Then the following assertions hold:

(i) $\sigma(\overline{T}) = \sigma(\overline{T}^{(*)}) = \sigma(T)$.

(ii) $w(\overline{T}) = w(\overline{T}^{(*)})$, where $w(T)$ is the numerical radius of $T$.

(iii) $\|\overline{T}\| = \|\overline{T}^{(*)}\|$.

We remark that (i) of Theorem 5 asserts more generalization form of $\sigma(\overline{T}) - \{0\} = \sigma(\overline{T}^{(*)}) - \{0\} = \sigma(T) - \{0\}$. And $\sigma(\overline{T}) = \sigma(T)$ has been already shown in [3, 4, 11].

**Proposition 6** ([Y2]). Let $T \in B(H)$. Then for each $p > 0$,

$\overline{T}$ is $p$-hyponormal $\iff \overline{T}^{(*)}$ is $p$-hyponormal.

**Remark 2.** If $T$ is $\frac{p}{2}$-hyponormal for $p \in (0,2]$, then we obtain the following assertions, easily. $\overline{T}_n$ is $\frac{p}{2}$-hyponormal for all natural number $n$ by using Theorem D.
several times, and also $\tilde{T}_n^{(*)}$ is $\frac{p}{2}$-hyponormal for all natural number $n$ by Theorem D and Proposition 6 several times.

Proof of Theorem 4. We shall prove Theorem 4 by induction on $n$.

Proof of (i). (a) By (i) of Theorem D, we have $|\tilde{T}|^{p+1} \geq |T|^{p+1}$.

(b) Assume that $|\tilde{T}_{n-1}|^{p+1} \geq \cdots \geq |\tilde{T}|^{p+1} \geq |T|^{p+1}$.

(c) Proof of $|\tilde{T}_n|^{p+1} \geq |T_{n-1}|^{p+1}$.

Put $S = \tilde{T}_{n-1}$. Then $S$ is also $\frac{p}{2}$-hyponormal by Remark 2. Hence we have

$$|\tilde{T}_n|^{p+1} = |S|^{p+1} \geq |S|^{p+1} = |\tilde{T}_{n-1}|^{p+1} \text{ by (i) of Theorem D.}$$

Proof of (ii). Let $T = U|T|$ be the polar decomposition of $T$.

(a) Proof of $|T^*|^{p+1} \geq |\tilde{T}^*|^{p+1}$.

$$|T^*|^{p+1} = U|T|^{p+1}U^* \geq U|\tilde{T}^*|^{p+1}U^* \text{ by (i) of Theorem D}$$

$$= U(|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}})^{p+1}U^*$$

$$= (|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}})^{p+1}$$

$$= |\tilde{T}^*|^{p+1}.$$ 

(b) Assume that $|T^*|^{p+1} \geq |\tilde{T}^*|^{p+1} \geq \cdots \geq |\tilde{T}_{n-1}^*|^{p+1}$.

(c) Proof of $|\tilde{T}_{n-1}^*|^{p+1} \geq |\tilde{T}_n^*|^{p+1}$.

Put $S = (T_{n-1}^*)^* = \tilde{T}_{n-1}^{(*)}$. Then $S$ is also $\frac{p}{2}$-hyponormal by Remark 2. By (a), we obtain

$$|\tilde{T}_{n-1}^*|^{p+1} = |S^*|^{p+1} \geq |\tilde{T}_n^*|^{p+1}.$$ 

Therefore the proof of Theorem 4 is complete. $\square$

By considering Theorem 4, we can understand that $n$-th Aluthge transformation and powers of operators have similar properties. On the other hand, $\lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = r(T)$ is a very famous and useful result. So we shall show the parallel result to this one as follows:

Theorem 7 ([Y3]). Let $T \in B(H)$. Then $\lim_{n \to \infty} \|\tilde{T}_n\| = r(T)$.

To prove Theorem 7, we prepare the following lemmas:
Lemma 8 ([Y3]). For a natural number $n$ and $k = 0, 1, \ldots, n + 1$, let

\[ nD_k = \frac{n!(n-2k+1)}{k!(n-k+1)!}. \]

Then the following assertions hold:

(i) $nD_0 = 1$ for all natural number $n$.

(ii) $nD_k + nD_{k+1} = n+1D_{k+1}$ for all natural number $n$ and $k = 0, 1, \ldots, n$.

(iii) $2n+1D_n = 2n+\ldots\cdot 2.D_{n+1}$ for all natural number $n$.

(iv) $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (n-2k+1)nD_k = 2^n$,

where $\lfloor \frac{n}{2} \rfloor$ is the largest integer satisfying $\frac{n}{2} \leq \frac{n}{2}$.

(v) $\lim_{n\to\infty} \frac{(n-2k+1)nD_k}{2^n} = 0$ for all positive integer $k$.

Lemma 9 ([Y3]). Let $T \in B(H)$. Then

\[ \| \overline{T^n} \| \leq \| T^{n+1} \|^\frac{1}{2} \| T^{n-1} \|^\frac{1}{2} \]

holds for all natural number $n$.

Proof. Let $T = U |T|$ be the polar decomposition of $T$. Then we have

\[
\| \overline{T^n} \|
= \| (|T| \frac{1}{2} U |T|^{\frac{1}{2}})^n \|
= \| |T| \frac{1}{2} (U |T|)^{n-1} U |T| \|^\frac{1}{2} \|
\leq \| T^{n+1} \|^\frac{1}{2} \| T^{n-1} \|^\frac{1}{2}
\]

by (i) of Theorem B.

Lemma 10 ([Y3]). Let $T \in B(H)$ and $m = \lfloor \frac{n}{2} \rfloor$. Then

\[ \| \overline{T_n} \| \leq \| T^{n+1} \|^\frac{nD_0}{2^n} \| T^{n-1} \|^\frac{nD_1}{2^n} \ldots \| T^{n-2k+1} \|^\frac{nD_k}{2^n} \ldots \| T^{n-2m+1} \|^\frac{nD_m}{2^n} \]

Proof. We shall prove Lemma 10 by induction on $n$.

(a) $\| \overline{T} \| \leq \| T^2 \|^\frac{1}{2}$ holds by Lemma 9.

(b) Assume that

\[ \| \overline{T_{n-1}} \| \leq \| T^n \|^\frac{nD_0}{2^n} \| T^{n-2} \|^\frac{nD_1}{2^n} \ldots \| T^{n-2k+1} \|^\frac{nD_k}{2^n} \ldots \| T^{n-2m+1} \|^\frac{nD_m}{2^n}, \]

where $m = \lfloor \frac{n-1}{2} \rfloor$. 

(c-1) In case $n = 2m + 1$ for $m = 1, 2, \cdots$. Then $[\frac{n}{2}] = [\frac{n-1}{2}] = m$. Hence by (3.2), we have

\[
\|\tilde{T}_n\| = \|\tilde{T}_{n-1}\| = \|\tilde{T}_{n-2}\| \frac{n-1}{2^{n-1}} \cdots \]

\[
\leq \|\tilde{T}_n\| \frac{n-1}{2^{n-1}} \|\tilde{T}_{n-2}\| \frac{n-1}{2^{n-1}} \cdots \]

\[
\leq \left(\|\tilde{T}^n\| \frac{1}{2} \|\tilde{T}_{n-1}\| \frac{1}{2}\right)^{n-1} \frac{n-1}{2^{n-1}} \left(\|\tilde{T}^n\| \frac{1}{2} \|\tilde{T}_{n-3}\| \frac{1}{2}\right)^{n-1} \frac{n-1}{2^{n-1}} \cdots \]

by (i) and (ii) of Lemma 8, and the last inequality holds by Lemma 9.

(c-2) In case $n = 2m + 2$ for $m = 0, 1, 2, \cdots$. Then $[\frac{n}{2}] = m + 1$ and $[\frac{n-1}{2}] = m$. Hence by (3.2), we have

\[
\|\tilde{T}_n\| = \|\tilde{T}_{n-1}\| = \|\tilde{T}_{n-2}\| \frac{n-1}{2^{n-1}} \cdots \]

\[
\leq \|\tilde{T}_n\| \frac{n-1}{2^{n-1}} \|\tilde{T}_{n-2}\| \frac{n-1}{2^{n-1}} \cdots \]

\[
\leq \left(\|\tilde{T}^n\| \frac{1}{2} \|\tilde{T}_{n-1}\| \frac{1}{2}\right)^{n} \frac{n-1}{2^{n-1}} \left(\|\tilde{T}^n\| \frac{1}{2} \|\tilde{T}_{n-3}\| \frac{1}{2}\right)^{n} \frac{n-1}{2^{n-1}} \cdots \]

by (i), (ii) and (iii) of Lemma 8, and the last inequality holds by Lemma 9. \qed
Lemma 11 ([Y3]). Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence satisfying \( \lim_{n \to \infty} a_n = a \), and for each natural number \( n \), \( \{c_{n,k}\}_{k=1}^{n} \) be a positive sequence satisfying

\[
(3.3) \quad c_{n,1} + \cdots + c_{n,k} + \cdots + c_{n,n} = 1 \quad \text{for all natural number } n
\]

and \( \lim_{n \to \infty} c_{n,k} = 0 \) for fixed \( k = 1, 2, \ldots \). Then

\[
\lim_{n \to \infty} (c_{n,1}a_1 + \cdots + c_{n,k}a_k + \cdots + c_{n,n}a_n) = a.
\]

Proof of Theorem 7. Let \( m = \left[ \frac{n}{2} \right] \). Then by Lemma 10, (iv) of Lemma 8 and Arithmetic mean-Geometric mean inequality, we have

\[
r(T) = r(T_n) \leq \|T_n\| \leq \|T^{n+1}\|^{\frac{nD_0}{2^n}} \|T^{n-1}\|^{\frac{nD_1}{2^n}} \cdots \|T^{n-2k+1}\|^{\frac{nD_k}{2^n}} \cdots \|T^{n-2m+1}\|^{\frac{nD_m}{2^n}}
\]

\[
\leq \left( \frac{n+1}{2^n} \right)^{nD_0} \|T^{n+1}\|^{\frac{1}{n+1}} + \left( \frac{n-1}{2^n} \right)^{nD_1} \|T^{n-1}\|^{\frac{1}{n-1}} + \cdots + \left( \frac{n-2k+1}{2^n} \right)^{nD_k} \|T^{n-2k+1}\|^{\frac{1}{n-2k+1}} + \cdots + \left( \frac{n-2m+1}{2^n} \right)^{nD_m} \|T^{n-2m+1}\|^{\frac{1}{n-2m+1}}
\]

\[
\to r(T) \quad \text{as } n \to \infty
\]

by \( \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = r(T) \), (iv) and (v) of Lemma 8 and Lemma 11. \( \square \)

4. Relations between the orders \(|\overline{T}|^p - 1 \geq |T|^p - 1 \) and \(|T|^p \geq |T^*|^p \)

In this section, we shall discuss properties of the order \(|\overline{T}|^p \geq |T|^p \) for some \( p > 0 \). Relating to this order, Theorem D is very famous. As a converse of Theorem D, we obtain the following result:

Theorem 12 ([Y2]). Let \( T \) be an invertible operator. Then the following assertions hold:

(i) For each \( p \in [2, 4] \), \( |\overline{T}|^p \geq |T|^p \) ensures \( |T|^{p-1} \geq |T^*|^{p-1} \).

(ii) For each \( p \geq 4 \), \( |\overline{T}|^p \geq |T|^p \) ensures \( |T|^3 \geq |T^*|^3 \).

To prove Theorem 12, we need the following result:

Theorem E ([5, 13, 14, 15]). Let \( A \) and \( B \) be positive invertible operators. Then the following assertions hold:

(i) \( A \geq B > 0 \) ensures \( (B^{\frac{t}{2}} A^p B^{\frac{1}{2}})^{\frac{1-t}{p-t}} \geq B^{1-t} \) for \( 1 \geq p \geq \frac{1}{2} \) with \( p > t \geq 0 \).

(ii) \( A \geq B > 0 \) ensures \( (B^{\frac{t}{2}} A^p B^{\frac{1}{2}})^{\frac{2p}{2p-t}} \geq B^{2p-t} \) for \( \frac{1}{2} \geq p > t \geq 0 \).
Proof of Theorem 12. Let $T = U|T|$ be the polar decomposition of $T$. Then $U$ is unitary since $T$ is invertible.

Proof of (i). By applying (i) of Theorem E to $|\tilde{T}|^p \geq |T|^p$, we have
\begin{equation}
(|T|^\frac{-1}{2} |\tilde{T}|^{pp_1} |T|^\frac{-1}{2})^{\frac{1-t_1}{p_1-1}} \geq |T|^{p(1-t_1)}
\end{equation}
for $1 \geq p_1 \geq \frac{1}{2}$ with $p_1 > t_1 \geq 0$.

Put $p_1 = \frac{2}{p}$ and $t_1 = \frac{1}{p}$ in (4.1). Then we have
\[ (|T|^\frac{1}{2} |\tilde{T}|^2 |T|^\frac{1}{2})^{p-1} \geq |T|^{p-1}. \]
It is equivalent to
\[ (|T|^\frac{1}{2} |T|^\frac{1}{2} U^* |T| U |T|^\frac{1}{2} |T|^\frac{1}{2})^{p-1} \geq |T|^{p-1}, \]
that is, $U^*|T|^{p-1} U \geq |T|^{p-1}$ since $U$ is unitary. Hence we have $|T|^{p-1} \geq U|T|^{p-1} U^* = |T^*|^{p-1}$.

Proof of (ii). By applying (ii) of Theorem E to $|\tilde{T}|^p \geq |T|^p$, we have
\begin{equation}
(|T|^\frac{-1}{2} |\tilde{T}|^{2p_1} |T|^\frac{-1}{2})^{\frac{2}{p_1-1}} \geq |T|^{p(2p_1-t_1)}
\end{equation}
for $\frac{1}{2} \geq p_1 > t_1 \geq 0$.

Put $p_1 = \frac{2}{p}$ and $t_1 = \frac{1}{p}$ in (4.2). Then we have
\[ (|T|^\frac{1}{2} |\tilde{T}|^2 |T|^\frac{1}{2})^{3} \geq |T|^3. \]
It is equivalent to
\[ (|T|^\frac{1}{2} |T|^\frac{1}{2} U^* |T| U |T|^\frac{1}{2} |T|^\frac{1}{2})^{3} \geq |T|^3, \]
that is, $U^*|T|^3 U \geq |T|^3$ since $U$ is unitary. Hence we have $|T|^3 \geq U|T|^3 U^* = |T^*|^3$. \hfill \Box

REFERENCES


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