1. INTRODUCTION

Let $X$ be a linear operator on a Hilbert space $\mathfrak{H}$. Then $\mathcal{R}(X)$ and $\mathcal{N}(X)$ stand respectively for the range and the null space of $X$. Throughout this paper, both $A$ and $B$ represent bounded selfadjoint operators and also $H$ and $K$ do semi-bounded selfadjoint operators.

$A \geq B$ means $(Ax, x) \geq (Bx, x)$ for every $x$, by definition. It is well-known that $A \geq B \geq 0$ implies

$$A^\alpha \geq B^\alpha \ (0 < \alpha < 1), \quad \log(A + \beta) \geq \log(B + \beta) \ (0 < \beta);$$

the first inequality is called Löwner-Heinz inequality.

The following inequality was found by Hansen [8]:

\textit{if $P$ is a projection, and if $A \geq 0$, then}

$$(PAP)^\alpha \geq PA^\alpha P \ (0 < \alpha < 1).$$

Let $H$ and $K$ be both bounded below with spectral families $\{\Lambda_t\}$ and $\{\Gamma_t\}$ respectively. Then we write $H \geq K$ if

$$\int_{-\infty}^{\infty} t \, d(\Lambda_t x, x) \geq \int_{-\infty}^{\infty} t \, d(\Gamma_t x, x) \quad \text{for every } x \in \mathfrak{H}.$$ 

It is known that for any real number $\lambda$, $H + \lambda \geq K + \lambda$ follows from $H \geq K$ and that if $H \geq K \geq 0$ and if $\mathcal{N}(K) = \{0\}$ then $K^{-1} \geq H^{-1} \geq 0$. If $H$ and $K$ are both bounded above, we denote $H \geq K$ if $-K \geq -H$.

Let $A$ be a bounded self-adjoint operator with the spectral family $\{E_t\}$. If $A \geq 0$ and if $\mathcal{N}(A) = \{0\}$, then $\{0\}$ is a null set with respect to $d(E_t x, x)$ for every $x \neq 0$. Hence $\log A$ is well-defined by the functional calculus and bounded above. The following fact is obvious but important in this paper, so we give a proof for the completeness:

$$A \geq B \geq 0, \quad \mathcal{N}(B) = \{0\} \Rightarrow \log A \geq \log B.$$

To see this, by multiplying both $A$ and $B$ by a constant, we may assume that $1/2 \geq A \geq B \geq 0$. Hence we have $1 \geq A + \epsilon \geq B + \epsilon \geq \epsilon$ for $1/2 \geq \epsilon > 0$. 


Let \( \{E_t\} \) and \( \{F_t\} \) be the spectral families of \( A \) and \( B \) respectively. Since 
\[
\log(A + \epsilon) \geq \log(B + \epsilon), \quad -\log(B + \epsilon) \geq -\log(A + \epsilon).
\]
This implies 
\[
\int_0^{1/2} -\log(t + \epsilon) d(F_t, x) \geq \int_0^{1/2} -\log(t + \epsilon) d(E_t, x) \quad (x \in \mathfrak{H}).
\]
\( \epsilon \) tending to 0, by Lebesgue's theorem, we get 
\[
\int_0^{1/2} -\log t d(F_t, x) \geq \int_0^{1/2} -\log t d(E_t, x) \quad (x \in \mathfrak{H}).
\]
This implies \( -\log B \geq -\log A \), and hence \( \log A \geq \log B \).

Furuta ([5] and [6]) proved that if \( A \geq B \geq 0 \), then for \( 0 < r, 1 \leq t \)
\[
(B^{\frac{r}{2}} A^t B^{\frac{r}{2}})^{\frac{1+r}{t+r}} \geq B^{1+r}, \quad A^{1+r} \geq (B^{\frac{r}{2}} A^t B^{\frac{r}{2}})^{\frac{1+r}{t+r}},
\]
and moreover, for \( 0 < r, 0 \leq s \leq t \)
\[
(B^{\frac{r}{2}} A^t B^{\frac{r}{2}})^{\frac{s+r}{t+r}} \geq B^{\frac{r}{2}} A^s B^{\frac{r}{2}}, \quad A^{\frac{s+r}{t+r}} \geq (B^{\frac{r}{2}} A^t B^{\frac{r}{2}})^{\frac{s+r}{t+r}}.
\]
As inequalities related to these inequalities the following were shown (see [3],[4] and [12]) :
if \( A \geq B \), then for \( r, t > 0 \)
\[
(e^{rB/2} e^{tA} e^{rB/2})^{r/(t+t+r)} \geq e^{rB}, \quad e^rA \geq (e^{rA/2} e^{tB} e^{rA/2})^{r/(t+t+r)}.
\]
We will establish these inequalities for semibounded operators \( H \) and \( K \) in Section 2. In Section 3, we will modify the definition of log-hyponormality and show that if \( T \) is log-hyponormal then
\[
(T^{n+1} T^n)^{1/n} \leq (T^{n+1} T^n)^{1/(n+1)} \quad (n = 1, 2, \cdots).
\]
In Section 4, we will prove that if \( T \) is log-hyponormal and if \( |T|^n = |T^n| \) for some \( n \geq 3 \), then the polar decomposition of \( T \) is commutative, that is, \( T \) is a quasi-normal operator.

2. SEMI-BOUNDED OPERATOR INEQUALITY

In this section we establish exponential inequalities for semi-bounded operators in the same way that we have shown (4) in [12]. For convenience, we first consider selfadjoint operators bounded below.

THEOREM 2.1. Let \( H \) and \( K \) be bounded below and suppose \( H \geq K \). Then for \( 0 < r, 0 \leq s \leq t \)
\[
(e^{-\frac{r}{2}H} e^{-tK} e^{-\frac{r}{2}H})^{\frac{s+r}{t+r}} \geq e^{-\frac{r}{2}H} e^{-sK} e^{-\frac{r}{2}H},
\]
\[
e^{-\frac{r}{2}K} e^{-sH} e^{-\frac{r}{2}K} \geq (e^{-\frac{r}{2}K} e^{-tH} e^{-\frac{r}{2}K})^{\frac{s+r}{t+r}}.
\]
Proof. It is clear that \((1 + K/n)^{-1}\) and \((1 + H/n)^{-1}\) are both bounded for sufficiently large \(n\) and that \((1 + K/n)^{-1} \geq (1 + H/n)^{-1} \geq 0\). Therefore, from (3) it follows that

\[
\{(1 + \frac{H}{n})^{-\frac{nt}{2}}(1 + \frac{K}{n})^{-\frac{ns}{2}}\} \geq (1 + \frac{H}{n})^{-\frac{nt}{2}}(1 + \frac{K}{n})^{-\frac{ns}{2}}.
\]

Since the sequence of functions \((1 + \lambda/n)^{-n}\) of \(\lambda\) converges uniformly to \(e^{-\lambda}\) on \(\gamma \leq \lambda < \infty\) as \(n \to \infty\), where \(\gamma\) is a lower bound of \(K\), \((1 + H/n)^{-nr/2}\) and \((1 + K/n)^{-nt}\) converges to \(e^{-rH/2}\) and \(e^{-tK}\) in the norm sense, respectively. Thus the above inequality yields

\[
(e^{-\frac{rH}{2}}e^{-tK}e^{-\frac{sH}{2}})^{\frac{t+\tau}{t+\tau}} \geq e^{-\frac{rH}{2}}e^{-sK}e^{-\frac{tH}{2}}.
\]

One can see the second inequality of the theorem as well. \(\square\)

THEOREM 2.2. Let \(H\) and \(K\) be bounded above and suppose \(H \geq K\). Then for \(0 < r\), \(0 \leq s \leq t\)

\[
(e^{\frac{t}{2}K}e^{tH}e^{\frac{s}{2}K})^{\frac{t+\tau}{t+\tau}} \geq e^{\frac{t}{2}K}e^{sH}e^{\frac{s}{2}K}, \quad e^{\frac{s}{2}H}e^{sK}e^{\frac{s}{2}H} \geq (e^{\frac{t}{2}H}e^{tK}e^{\frac{t}{2}H})^{\frac{t+\tau}{t+\tau}}.
\]

In particular,

\[
(e^{\frac{t}{2}K}e^{tH}e^{\frac{s}{2}K})^{\frac{t+\tau}{t+\tau}} \geq e^{srK}, \quad e^{rH} \geq (e^{\frac{s}{2}H}e^{tK}e^{\frac{t}{2}H})^{\frac{t+\tau}{t+\tau}}.
\]

Proof. Since both \(-H\) and \(-K\) are bounded below, and since \(-K \geq -H\), (5) yields (6). Put \(s = 0\) in (6) to get (7). \(\square\)

THEOREM 2.3. Let \(A\) and \(B\) be bounded non-negative operators such that \(N(A) = N(B) = \{0\}\), and suppose \(\log A \geq \log B\). Then for \(0 < r\), \(0 \leq s \leq t\)

\[
(B^\frac{t}{2}A^tB^\frac{s}{2})^{\frac{t+\tau}{t+\tau}} \geq B^\frac{t}{2}A^sB^\frac{s}{2}, \quad A^\frac{s}{2}B^sA^\frac{s}{2} \geq (A^\frac{t}{2}B^tA^\frac{s}{2})^{\frac{t+\tau}{t+\tau}}.
\]

In particular,

\[
(B^\frac{t}{2}A^tB^\frac{s}{2})^{\frac{t+\tau}{t+\tau}} \geq B^r, \quad A^r \geq (A^\frac{t}{2}B^tA^\frac{s}{2})^{\frac{t+\tau}{t+\tau}}.
\]

Proof. \(\log A\) is a selfadjoint operator bounded above, and

\[
e^{t\log A} = A^t
\]

(see Section 128 of [9]). Thus (8) follows clearly from (6), and (9) is obvious. \(\square\)

3. LOG-HYPONORMAL OPERATOR
From now on, $T$ represents a bounded operator. $T$ is said to be subnormal if $T$ has a normal extension, quasi-normal if $T(T^*T) = (T^*T)T$, hyponormal if $T^*T \geq TT^*$ and paranormal if $\|T^2x\| \geq \|Tx\|^2$ ($x \in \mathfrak{H}$) (see [7],[2]). The relations among these classes of operators are follows: normal $\Rightarrow$ quasi-normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal $\Rightarrow$ paranormal.

For a subspace $\mathcal{L} \subseteq \mathfrak{H}$ and the projection $P$ onto $\mathcal{L}$, we call $PT|_{\mathcal{L}}$ the compression of $T$ to $\mathcal{L}$. Ando [3] showed that if $N(T) \subseteq N(T^*)$ and $\log A \geq \log B$, then $T$ is a paranormal.

Recently in many papers ([1],[4], [10] and so on) $T$ is called a log-hyponormal if $T$ is invertible and $\log T^*T \geq \log TT^*$; the invertibility of $T$ is necessary for $\log T^*T$ to be bounded. According to this definition, the class of log-hyponormal operators does not contain all hyponormal operators; in fact, there are many hyponormal operators which are not invertible. Therefore, we remove the condition of the invertibility from the definition of the log-hyponormality.

**DEFINITION.** $T$ is said to be log-hyponormal if it satisfies (10).

According to this definition, we have the following relation:

hyponormal $\Rightarrow$ log-hyponormal $\Rightarrow$ paranormal.

In [11] it was shown that a subnormal operator $T$ satisfies

$$|T| \leq |T^2|^{1/2} \leq \cdots \leq |T^n|^{1/n} \leq \cdots$$

Yamazaki [13] showed that if $(T^*T)^s \geq (TT^*)^s$ for some $s > 0$ or if $T$ is an invertible log-hyponormal operator, then

$$T^{*n}T^n \leq (T^{*n+1}T^{n+1})^{\frac{n}{n+1}}, \quad T^nT^{*n} \geq (T^{n+1}T^{n+1})^{\frac{n}{n+1}} \quad (n = 1, 2, \cdots),$$

$$T^*T \leq (T^*T^2)^{1/2} \leq \cdots \leq (T^nT^n)^{1/n} \leq \cdots,$$

$$TT^* \geq (TT^2)^{1/2} \geq \cdots \geq (T^nT^n)^{1/n} \geq \cdots.$$  

An operator $T$ satisfying $T^*T \leq (T^*T^2)^{1/2}$ is paranormal: indeed, in this case, by Jensen's inequality, for $x \neq 0$

$$\|Tx\|^2 = \langle T^*Tx, x \rangle \leq \langle (T^*T^2)^{1/2}x, x \rangle \leq \langle T^*T^2x, x \rangle^{1/2}\|x\| = \|T^2x\|\|x\|.$$  

This says that the result of Yamazaki is a partial extension of that of Ando. We will show the inequalities of Yamazaki without assuming the invertibility of $T$, which induce a complete extension of the result of Ando. We first state a simple lemma needed later.
LEMMA. Let $T = U|T|$ be the polar decomposition of a bounded operator $T$ and $f(t)$ a continuous function on $[0, \infty)$ with $f(0) = 0$. Then for any $A \geq 0$

$$f(U|T|A|T^{*}|U^{*}) = f(TAT^{*}) = Uf(|T|A|T|)U^{*}$$
$$f(U^{*}|T^{*}|A|T^{*}|U) = f(T^{*}A\tau) = U^{*}f(|T^{*}|A|T^{*}|)U$$

Proof. Since $T = U|T| = |T^{*}|U$ and $T^{*} = |T|U^{*} = U^{*}|T^{*}|$, calculation shows that the equalities hold in case $f$ is a polynomial which vanishes at $t = 0$. Then for general $f$, we need only to take a sequence $\{p_{n}\}$ of polynomials with $p_{n}(0) = 0$ which converges uniformly to $f$ on $[0, \|T\|^{2}\|A\|]$. □

THEOREM 3.1. If $T$ is log-hyponormal, then

$$T^{*n}T^{n} \leq (T^{*n+1}T^{n+1})^{\frac{n}{n+1}} \leq (T^{*m+1}T^{m+1})^{\frac{m}{m+1}} \leq (T^{*m}T^{m})^{\frac{1}{m}} \leq (T^{*m+1}T^{m+1})^{\frac{m}{m+1}}$$

(11)

Proof. Let $T = U|T|$ be the polar decomposition of $T$ and $P$ the orthogonal projection onto $\mathcal{R}(T)$. We denote the compression of an operator $X$ to $\mathcal{R}(T)$ by $[X]$. Then the log-hyponormality of $T$ means $\log[T^{*}T] \geq \log[TT^{*}]$. Thus, by the first inequality of (9) we get

$$([TT^{*}]^{1/2}[T^{*}T][TT^{*}]^{1/2})^{1/2} \geq [TT^{*}],$$

which is equivalent to

$$\{(PT^{*}P)^{1/2}(PT^{*}TP)(PT^{*}P)^{1/2}\}^{1/2} \geq PTT^{*}P.$$ 

Since $TT^{*}(1 - P) = 0$, this gives

$$\{(TT^{*})^{1/2}PT^{*}TP(TT^{*})^{1/2}\}^{1/2} \geq TT^{*},$$

and hence

$$U^{*}[T^{*}PT^{*}TP]^{1/2}U \geq U^{*}TT^{*}U.$$ 

In view of the lemma, the left hand side equals $(T^{*}PT^{*}PT)\frac{1}{2}$. Since $PT = T$ and $U^{*}TT^{*}U = T^{*}T$, from the above inequality we get

$$(T^{*2}T^{2})^{1/2} \geq T^{*}T.$$ 

This means (11) holds for $n = 1$.

Assume that (11) holds for $n \leq m - 1$. Therefore,

$$T^{*m-1}T^{m-1} \leq (T^{*m}T^{m})^{\frac{m-1}{m}},$$

(12)

$$T^{*}T \leq (T^{*m}T^{m})^{\frac{1}{m}}.$$ 

(13)

$$PT^{*}TP \leq P(T^{*m}T^{m})^{\frac{1}{m}}P \leq (PT^{*m}T^{m}P)^{\frac{1}{m}},$$

(13) implies
here the last inequality is due to (1). Thus \([T^*T] \leq [T^*mTm]^{1/m}\). Since \(N([T^*T]) = 0\) because \(\mathcal{R}(T)\) is orthogonal to \(N(T)\) by definition of log-hyponormality of \(T\), by (2) we have

\[
\log[T^*T] \leq \log[T^*mTm]^{1/m}.
\]

Since \(T\) is log-hyponormal, this gives

\[
\log[TT^*] \leq \log[T^*mTm]^{1/m}.
\]

By the first inequality of (8), we obtain

\[
[TT^*]^{1/2}[T^*mTm]^{m-1/m}[TT^*]^{1/2} \leq ([TT^*]^{1/2}[T^*mTm]^{m/2}[TT^*]^{1/2})^{\frac{m-1}{m+1}},
\]

which is equivalent to

\[
(PTT^*P)^{1/2}(PT^*mTmP)(PTT^*P)^{1/2} \leq \{(PTT^*P)^{1/2}(PT^*mTmP)(PTT^*P)^{1/2}\}^{\frac{m}{m+1}}.
\]

Since \(PT = T\) and \(T^*P = T^*\), this gives

\[
|T^*|(PT^*mTmP)^{m-1/m}|T^*| \leq \{|T^*|(PT^*mTmP)|T^*|\}^{m/m+1}. \tag{14}
\]

From (12) it follows that

\[
T^*mTm = T^*(T^*m^{-1}Tm^{-1})T \leq T^*(T^*mTm)^{m-1/m}T
\]

\[
= U^*|T^*|P(T^*mTm)^{m-1/m}P|T^*|U
\]

\[
\leq U^*|T^*|(PT^*mTmP)^{m-1/m}|T^*|U \quad \text{(by (1))}
\]

\[
\leq U^*\{(T^*|(PT^*mTmP)|T^*|\}^{m/m+1}U \quad \text{(by (14))}
\]

\[
= (U^*|T^*|PT^*mTmP|T^*|U)^{m/m+1} \quad \text{(by Lemma 3.1)}
\]

\[
= (T^*m+1Tm+1)^{m/m+1}.
\]

This completes the proof. \(\square\)

Note that (11) implies that \(N(T) = N(T^2) = \cdots = N(T^m) = \cdots\).

**THEOREM 3.2.** If for a natural number \(m\)

\[
N(T) \subseteq N(T^*) \quad \text{and} \quad \log[T^*T]_n \geq \log[TT^*]_n \quad (1 \leq n \leq m), \tag{15}
\]

where \([X]_n\) means the compression of \(X\) to \(\overline{\mathcal{R}(T^n)}\), then

\[
T^mT^n \geq (T^{m+1}T^{n+1})^{\frac{n}{n+1}} \quad (n = 1, 2, \ldots, m). \tag{16}
\]
Proof. We prove this by the mathematical induction of $m$. Suppose (15) hold for $m = 1$. Since then $T$ is log-hyponormal, from the second inequality of (9) it follows that

$$[T^*T]_1 \geq \{(T^*T)_1^{1/2}(TT^*)_1[T^*T]_1^{1/2}\}^{1/2}.$$  

Denoting the projection to $\overline{\mathcal{R}(T)}$ by $P$, the above shows

$$PT^*TP \geq \{(PT^*TP)^{1/2}(PTT^*)^2(PT^*TP)^{1/2}\}^{1/2}.$$  

Let $TP = U|TP|$ be the polar decomposition of $TP$. Then the above leads to

$$|TP|^2 \geq \{|TP|(PTT^*)^2|TP|\}^{1/2},$$  

and hence

$$U|TP|^2 U^* \geq U\{TP|(PTT^*)^2|TP|\}^{1/2}U^*.$$  

By the lemma, we obtain

$$|PT^*|^2 \geq \{TP(PTT^*)^2(TP)^*\}^{1/2}.$$  

Since $PT = T$, the righthand side equals $(T^2T^*T^*)^{1/2}$. Thus we get

$$TT^* \geq TPT^* \geq (T^2T^*T^*)^{1/2}.$$  

Consequently (16) holds for $m = 1$.  

Assume that the theorem is valid for $m - 1$. To show that it is valid for $m$, suppose (15) hold for $n = 1, 2, \ldots, m$. From the inductive assumption (16) holds for $n = 1, 2, \ldots, m - 1$; therefore, we have

$$T^{m-1}T^{*m-1} \geq (T^{m}T^{*m})_1^{m-1/m} \text{ and } TT^* \geq (T^{m}T^{*m})_1^{1/m}. \quad (17)$$  

The second inequality yields

$$[TT^*]_m \geq [T^{m}T^{*m}]_1^{1/m},$$  

and hence

$$\log[TT^*]_m \geq \log([T^{m}T^{*m}]_1^{1/m}).$$  

Therefore, by (15) with $n = m$ we obtain

$$\log[T^*T]_m \geq \log([T^{m}T^{*m}]_1^{1/m}).$$  

Applying this to the second inequality of (8) we get

$$[T^*T]_m^{1/2}([T^{m}T^{*m}]_1^{1/m})^{m-1}T^*T]_m^{1/2} \geq \{(T^*T)_{1/m}^{1/2}([T^{m}T^{*m}]_1^{1/m})^{m}T^{*T]_m^{1/2}}_{m+1}^{m+1}\}^{m-1/m}. $$
Denote the projection to $\overline{\mathcal{R}(T^m)}$ by $Q$, and rewrite the above as

$$(QT^*TQ)^{1/2}(T^mT^*m)^{m-1 \over m} (QT^*TQ)^{1/2} \geq \{(QT^*TQ)^{1/2}(T^mT^*m)(QT^*TQ)^{1/2}\}^{m-1 \over m+1}.$$ 

Now let $TQ = V|TQ|$ be the polar decomposition of $TQ$. Then we have

$$|TQ|(T^mT^*m)^{m-1 \over m} |TQ| \geq \{|TQ|(T^mT^*m)|TQ|\}^{m-1 \over m+1}.$$ 

Multiplying this inequality by $V$ and $V^*$, in virtue of the lemma, yields

$$TQ(T^mT^*m)^{m-1 \over m} QT^* \geq \{TQ(T^mT^*m)QT^*\}^{m-1 \over m+1}.$$ 

From this it follows that

$$T(T^mT^*m)^{m-1 \over m} T^* \geq \{T(T^mT^*m)T^*\}^{m-1 \over m+1} = (T^m+1T^*m+1)^{m \over m+1},$$

for $Q(T^mT^*m) = T^mT^*m = (T^mT^*m)Q$. This in conjunction with the first inequality of (17) gives

$$T^mT^*m = T(T^{m-1}T^{*m-1})T^* \geq T(T^mT^*m)^{m-1 \over m} T^* \geq (T^m+1T^*m+1)^{m \over m+1}.$$ 

This shows that (16) holds for $n = m$. Thus, we conclude the proof. □

As mentioned in the proof, (15) with $m = 1$ is satisfied for a log-hyponormal operator. Therefore,

$$TT^* \geq (T^2T^*2)^{1 \over 2}$$

is valid for a log-hyponormal operator $T$. The condition (15) is technical. We do not know if (15) follows from the log-hyponormality of $T$. However, one can easily see that (15) holds for every $m$ if $T$ is hyponormal. Moreover, $N(T^*) = N(T^*m)$ for every $m$ if $N(T^*) = 0$ or $N(T^*) = N(T^*2)$, so that we get:

**COROLLARY 3.3.** If $T$ is a log-hyponormal operator and if $N(T^*) = 0$ or $N(T^*) = N(T^*2)$, then for every $n$

$$T^mT^*n \geq (T^m+1T^*n+1)^{m \over n+1}.$$ 

**4. NORMALITY**

Recall the definition of the quasi-normality of $T$ stated in the previous section. Then we can see that $T$ is quasi-normal if and only if the polar decomposition of $T$ is commutative, so that a quasi-normal operator $T$ is normal if $N(T) = N(T^*)$.

In this section, we give a few conditions which guarantee quasi-normality or normality of a log-hyponormal operator.
THEOREM 4.1. If a log-hyponormal operator $T$ satisfies $|T|^n = |T^n|$ for some $n \geq 3$, then $T$ is quasi-normal.

Proof. By Theorem 3.1, we get

$$|T| \leq |T^2|^{1/2} \leq \cdots \leq |T^n|^{1/n} \leq \cdots.$$  

Thus the assumption implies that

$$(T^*T)^2 = T^{*2}T^2 \quad \text{and} \quad (T^*T)^3 = T^{*3}T^3.$$  

(18)

From the first equality of (18) it follows that $T^*(TT^* - T^*T)T = 0$, and hence, for the projection $P$ onto $\overline{R(T)}$

$$PT^*TP = PTT^*P = TT^*.$$  

(19)

By (18) and (19)

$$T^*(TT^*)^2T = (T^*T)^3 = T^{*3}T^3 = T^*(T^{*2}T^2)T = T^*(T^*T)^2T,$$

from which it follows that

$$P(TT^*)^2P = P(T^*T)^2P.$$  

Since the left hand side equals $(TT^*)^2$, this and (19) yield

$$(PT^*TP)^2 = (TT^*)^2 = P(TT^*)^2P = P(T^*T)^2P.$$  

Consequently,

$$PT^*T(1 - P)T^*TP = 0 \quad \text{and hence} \quad T^*TP = PT^*TP.$$  

From this and (19) we derive

$$(T^*T)^2T = (T^*TP)T = (PT^*TP)T = (TT^*)T = T(T^*T).$$

This means $T$ is quasi-normal. \[\square\]

The above theorem does not hold for the condition $n = 2$: see for a counter example p.199 of [11].

THEOREM 4.2. If a log-hyponormal operator $T$ satisfies

$$|T^*|^n = |T^n| \quad \text{for some} \quad n \geq 2,$$

then $T$ is normal.

Proof. The assumption implies $T^nT^* = (TT^*)^n$ and hence

$$\mathcal{N}(T^*) = \mathcal{N}(T^nT^*) = \mathcal{N}((TT^*)^n) = \mathcal{N}(TT^*) = \mathcal{N}(T^*),$$
from which $\mathcal{N}(T^{*2}) = \mathcal{N}(T^{*})$ follows. By Corollary 3.3

$$(T^m T^{*m})^{1/m} \geq (T^{m+1} T^{*m+1})^{1/(m+1)}$$

for every $m$.

Therefore, by assumption, we have

$$TT^{*} = (T^{2} T^{*2})^{1/2} = \ldots = (T^{n} T^{*n})^{1/n},$$

which implies $T^{2} T^{*2} = (TT^{*})^{2}$. Denoting the projection onto $\overline{\mathcal{R}(T^{*})}$ by $Q$, this implies that

$$QTT^{*}Q = QT^{*}TQ = T^{*}T.$$ 

On the other hand, since $\mathcal{N}(T) \subseteq \mathcal{N}(T^{*})$ by assumption and since $1 - Q$ is the projection to $\mathcal{N}(T)$, we have $T^{*}(1 - Q) = 0$, which implies $TT^{*} = QTT^{*}Q$. Consequently, we obtain $TT^{*} = QTT^{*}Q = T^{*}T$. \hfill $\square$

**Theorem 4.3.** A log-hyponormal operator $T$ is normal if $T^{*n}$ is log-hyponormal for some natural number $n$.

**Proof.** As we mentioned after the proof of Theorem 3.1,

$$\mathcal{N}(T^{*}) \supseteq \mathcal{N}(T) = \mathcal{N}(T^{m})$$

for every $m$. On the other hand,

$$\mathcal{N}(T^{*}) \subseteq \mathcal{N}(T^{*2}) \subseteq \ldots \subseteq \mathcal{N}(T^{*n}) \subseteq \mathcal{N}(T^{n})$$

since $T^{*n}$ is log-hyponormal. Thus we have $\mathcal{N}(T) = \mathcal{N}(T^{m})$ for $m = 1, 2, \ldots, n$.

Denote the compression of $X$ to $\overline{\mathcal{R}(T)}$ by $[X]$. By Theorems 3.1 and 3.2 we get

$$[T^{*}T] \leq [T^{*n}T^{n}]^{1/n}, \quad [T^{m}T^{n}]^{1/n} \leq [TT^{*}].$$

Since $\log(A^{1/n}) = \frac{1}{n} \log A$ for $A \geq 0$ with $\mathcal{N}(A) = 0$, and since $T^{*n}$ is log-hyponormal,

$$\log[T^{*}T] \leq \frac{1}{n} \log[T^{*n}T^{n}] \leq \frac{1}{n} \log[T^{m}T^{*n}] \leq \log[TT^{*}].$$

Therefore the log-hyponormality of $T$ yields $\log[T^{*}T] = \log[TT^{*}]$. This implies $T^{*}T = TT^{*}$. \hfill $\square$

**References**


[5] T. Furuta, $A \geq B \geq 0$ assures $(B^rA^pB^r)^{1/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1 + 2r)q \geq p + 2r$, Proc. Amer. Math. Soc., 101(1987), 85–88.


