数理解析研究所講究録
1189 巻 2001 年 82-92

INEQUALITIES FOR SEMIBOUNDED OPERATORS AND LOG-HYPONORMAL OPERATORS

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1. INTRODUCTION

Let X be a linear operator on a Hilbert space $\mathfrak{H}$. Then $\mathcal{R}(X)$ and $\mathcal{N}(X)$ stand respectively for the range and the null space of X. Throughout this paper, both A and B represent bounded selfadjoint operators and also H and K do semi-bounded selfadjoint operators.

$A \geq B$ means $(Ax, x) \geq (Bx, x)$ for every x, by definition. It is well-known that $A \geq B \geq 0$ implies

$$A^{\alpha} \geq B^{\alpha} \quad (0 < \alpha < 1), \quad \log(A+\beta) \geq \log(B+\beta) \quad (0 < \beta);$$

the first inequality is called *Löwner-Heinz inequality.*

The following inequality was found by Hansen [8]:
if P is a projection, and if $A \geq 0$, then

$$(PAP)^{\alpha} \geq PA^{\alpha}P \quad (0 < \alpha < 1). \tag{1}$$

Let H and K be both bounded below with spectral families $\{\Lambda_t\}$ and $\{\Gamma_t\}$ respectively. Then we write $H \geq K$ if

$$\int_{-\infty}^{\infty} t\, d(\Lambda_t x, x) \geq \int_{-\infty}^{\infty} t\, d(\Gamma_t x, x) \quad \text{for every } x \in \mathfrak{H}.$$

It is known that for any real number λ, $H + \lambda \geq K + \lambda$ follows from $H \geq K$ and that if $H \geq K \geq 0$ and if $\mathcal{N}(K) = \{0\}$ then $K^{-1} \geq H^{-1} \geq 0$. If H and K are both bounded above, we denote $H \geq K$ if $-K \geq -H$.

Let A be a bounded self-adjoint operator with the spectral family $\{E_t\}$. If $A \geq 0$ and if $\mathcal{N}(A) = \{0\}$, then $\{0\}$ is a null set with respect to $d(E_t x, x)$ for every $x \neq 0$. Hence $\log A$ is well-defined by the functional calculus and bounded above. The following fact is obvious but important in this paper, so we give a proof for the completeness:

$$A \geq B \geq 0, \quad \mathcal{N}(B) = \{0\} \Rightarrow \log A \geq \log B. \tag{2}$$

To see this, by multiplying both A and B by a constant, we may assume that $1/2 \geq A \geq B \geq 0$. Hence we have $1 \geq A + \epsilon \geq B + \epsilon \geq \epsilon$ for $1/2 \geq \epsilon > 0$.

Let $\{E_t\}$ and $\{F_t\}$ be the spectral families of A and B respectively. Since $\log(A+\epsilon) \geq \log(B+\epsilon)$, $-\log(B+\epsilon) \geq -\log(A+\epsilon)$. This implies

$$\int_0^{1/2} -\log(t+\epsilon)\, d(F_t x, x) \geq \int_0^{1/2} -\log(t+\epsilon)\, d(E_t x, x) \quad (x \in \mathfrak{H}).$$

ϵ tending to 0, by Lebesgue's theorem, we get

$$\int_0^{1/2} -\log t\, d(F_t x, x) \geq \int_0^{1/2} -\log t\, d(E_t x, x) \quad (x \in \mathfrak{H}).$$

This implies $-\log B \geq -\log A$, and hence $\log A \geq \log B$.

Furuta ([5] and [6]) proved that if $A \geq B \geq 0$, then for $0 < r, 1 \leq t$

$$(B^{\frac{r}{2}} A^t B^{\frac{r}{2}})^{\frac{1+r}{t+r}} \geq B^{1+r}, \quad A^{1+r} \geq (A^{\frac{r}{2}} B^t A^{\frac{r}{2}})^{\frac{1+r}{t+r}},$$

and moreover, for $0 < r$, $0 \leq s \leq t$

$$(B^{\frac{r}{2}} A^t B^{\frac{r}{2}})^{\frac{s+r}{t+r}} \geq B^{\frac{r}{2}} A^s B^{\frac{r}{2}}, \quad A^{\frac{r}{2}} B^s A^{\frac{r}{2}} \geq (A^{\frac{r}{2}} B^t A^{\frac{r}{2}})^{\frac{s+r}{t+r}}. \tag{3}$$

As inequalities related to these inequalities the following were shown (see [3],[4] and [12]) :
if $A \geq B$, then for $r, t > 0$

$$(e^{rB/2} e^{tA} e^{rB/2})^{r/(t+r)} \geq e^{rB}, \quad e^{rA} \geq (e^{rA/2} e^{tB} e^{rA/2})^{r/(t+r)}. \tag{4}$$

We will establish these inequalities for semibounded operators H and K in Section 2. In Section 3, we will modify the definition of log-hyponormality and show that if T is log-hyponormal then

$$(T^{*n} T^n)^{1/n} \leq (T^{*n+1} T^{n+1})^{1/(n+1)} \quad (n = 1, 2, \cdots).$$

In Section 4, we will prove that if T is log-hyponormal and if $|T|^n = |T^n|$ for some $n \geq 3$, then the polar decomposition of T is commutative, that is, T is a quasi-normal operator.

2. SEMI-BOUNDED OPERATOR INEQUALITY

In this section we establish exponential inequalities for semi-bounded operators in the same way that we have shown (4) in [12]. For convenience, we first consider selfadjoint operators bounded below.

THEOREM 2.1. *Let H and K be bounded below and suppose $H \geq K$. Then for $0 < r$, $0 \leq s \leq t$*

$$(e^{-\frac{r}{2}H} e^{-tK} e^{-\frac{r}{2}H})^{\frac{s+r}{t+r}} \geq e^{-\frac{r}{2}H} e^{-sK} e^{-\frac{r}{2}H},$$
$$e^{-\frac{r}{2}K} e^{-sH} e^{-\frac{r}{2}K} \geq (e^{-\frac{r}{2}K} e^{-tH} e^{-\frac{r}{2}K})^{\frac{s+r}{t+r}}. \tag{5}$$

Proof. It is clear that $(1+K/n)^{-1}$ and $(1+H/n)^{-1}$ are both bounded for sufficiently large n and that $(1+K/n)^{-1} \geq (1+H/n)^{-1} \geq 0$. Therefore, from (3) it follows that

$$\{(1+\frac{H}{n})^{-\frac{nr}{2}}(1+\frac{K}{n})^{-nt}(1+\frac{H}{n})^{-\frac{n}{2}r}\}^{\frac{ns+nr}{nt+nr}}$$
$$\geq (1+\frac{H}{n})^{-\frac{nr}{2}}(1+\frac{K}{n})^{-ns}(1+\frac{H}{n})^{-\frac{n}{2}r}.$$

Since the sequence of functions $(1+\lambda/n)^{-n}$ of λ converges uniformly to $e^{-\lambda}$ on $\gamma \leq \lambda < \infty$ as $n \to \infty$, where γ is a lower bound of K, $(1+H/n)^{-nr/2}$ and $(1+K/n)^{-nt}$ converges to $e^{-rH/2}$ and e^{-tK} in the norm sense, respectively. Thus the above inequality yields

$$(e^{-\frac{r}{2}H}e^{-tK}e^{-\frac{r}{2}H})^{\frac{s+r}{t+r}} \geq e^{-\frac{r}{2}H}e^{-sK}e^{-\frac{r}{2}H}.$$

One can see the second inequality of the theorem as well. □

THEOREM 2.2. *Let H and K be bounded above and suppose $H \geq K$. Then for* $0 < r,\ 0 \leq s \leq t$

$$(e^{\frac{r}{2}K}e^{tH}e^{\frac{r}{2}K})^{\frac{s+r}{t+r}} \geq e^{\frac{r}{2}K}e^{sH}e^{\frac{r}{2}K}, \qquad e^{\frac{r}{2}H}e^{sK}e^{\frac{r}{2}H} \geq (e^{\frac{r}{2}H}e^{tK}e^{\frac{r}{2}H})^{\frac{s+r}{t+r}}. \tag{6}$$

In particular,

$$(e^{\frac{r}{2}K}e^{tH}e^{\frac{r}{2}K})^{\frac{r}{t+r}} \geq e^{rK}, \qquad e^{rH} \geq (e^{\frac{r}{2}H}e^{tK}e^{\frac{r}{2}H})^{\frac{r}{t+r}}. \tag{7}$$

Proof. Since both $-H$ and $-K$ are bounded below, and since $-K \geq -H$, (5) yields (6). Put $s=0$ in (6) to get (7). □

THEOREM 2.3. *Let A and B be bounded non-negative operators such that $\mathcal{N}(A) = \mathcal{N}(B) = \{0\}$, and suppose* $\log A \geq \log B$. *Then for* $0 < r,\ 0 \leq s \leq t$

$$(B^{\frac{r}{2}}A^tB^{\frac{r}{2}})^{\frac{s+r}{t+r}} \geq B^{\frac{r}{2}}A^sB^{\frac{r}{2}}, \qquad A^{\frac{r}{2}}B^sA^{\frac{r}{2}} \geq (A^{\frac{r}{2}}B^tA^{\frac{r}{2}})^{\frac{s+r}{t+r}}. \tag{8}$$

In particular,

$$(B^{\frac{r}{2}}A^tB^{\frac{r}{2}})^{\frac{r}{t+r}} \geq B^r, \qquad A^r \geq (A^{\frac{r}{2}}B^tA^{\frac{r}{2}})^{\frac{r}{t+r}}. \tag{9}$$

Proof. $\log A$ is a selfadjoint operator bounded above, and

$$e^{t\log A} = A^t$$

(see Section 128 of [9]). Thus (8) follows clearly from (6), and (9) is obvious. □

3. LOG-HYPONORMAL OPERATOR

From now on, T represents a bounded operator. T is said to be *subnormal* if T has a normal extension, *quasi-normal* if $T(T^*T) = (T^*T)T$, *hyponormal* if $T^*T \geq TT^*$ and *paranormal* if $\|T^2x\|\,\|x\| \geq \|Tx\|^2$ $(x \in \mathfrak{H})$ (see [7],[2]). The relations among these classes of operators are follows:
normal $\Rightarrow$ quasi-normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal $\Rightarrow$ paranormal.

For a subspace $\mathfrak{L} \subseteq \mathfrak{H}$ and the projection P onto $\mathfrak{L}$, we call $PT|_{\mathfrak{L}}$ the *compression* of T to $\mathfrak{L}$. Ando [3] showed that *if*

$$\mathcal{N}(T) \subseteq \mathcal{N}(T^*) \quad \textit{and} \quad \log A \geq \log B, \tag{10}$$

where A and B are compressions of T^*T and TT^* to $\overline{\mathcal{R}(T)}$, then T is a paranormal.

Recently in many papers ([1],[4], [10] and so on) T is called a log-hyponormal if T is *invertible* and $\log T^*T \geq \log TT^*$; the invertibility of T is necessary for $\log T^*T$ to be bounded. According to this definition, the class of log-hyponormal operators does not contain all hyponormal operators; in fact, there are many hyponormal operators which are not invertible. Therefore, we remove the condition of the invertibility from the definition of the log-hyponormality.

DEFINITION. T is said to be *log-hyponormal* if it satisfies (10).

According to this definition, we have the following relation:
hyponormal $\Rightarrow$ log-hyponormal $\Rightarrow$ paranormal.
In[11] it was shown that a subnormal operator T satisfies

$$|T| \leq |T^2|^{1/2} \leq \cdots \leq |T^n|^{1/n} \leq \cdots .$$

Yamazaki [13] showed that if $(T^*T)^s \geq (TT^*)^s$ for some $s > 0$ or if T is an *invertible* log-hyponormal operator, then

$$T^{*n}T^n \leq (T^{*n+1}T^{n+1})^{\frac{n}{n+1}}, \quad T^nT^{*n} \geq (T^{n+1}T^{*n+1})^{\frac{n}{n+1}} \quad (n = 1, 2, \cdots),$$
$$T^*T \leq (T^{*2}T^2)^{1/2} \leq \cdots \leq (T^{*n}T^n)^{1/n} \leq \cdots ,$$
$$TT^* \geq (T^2T^{*2})^{1/2} \geq \cdots \geq (T^nT^{*n})^{1/n} \geq \cdots .$$

An operator T satisfying $T^*T \leq (T^{*2}T^2)^{1/2}$ is paranormal: indeed, in this case, by Jensen's inequality, for $x \neq 0$

$$\|Tx\|^2 = (T^*Tx, x) \leq ((T^{*2}T^2)^{1/2}x, x) \leq (T^{*2}T^2x, x)^{1/2}\|x\| = \|T^2x\|\|x\|.$$

This says that the result of Yamazaki is a partial extension of that of Ando. We will show the inequalities of Yamazaki without assuming the invertibility of T, which induce a complete extension of the result of Ando. We first state a simple lemma needed later.

LEMMA. *Let $T = U|T|$ be the polar decomposition of a bounded operator T and $f(t)$ a continuous function on $[0,\infty)$ with $f(0) = 0$. Then for any $A \geq 0$*

$$f(U|T|A|T|U^*) = f(TAT^*) = Uf(|T|A|T|)U^*,$$
$$f(U^*|T^*|A|T^*|U) = f(T^*AT) = U^*f(|T^*|A|T^*|)U.$$

Proof. Since $T = U|T| = |T^*|U$ and $T^* = |T|U^* = U^*|T^*|$, calculation shows that the equalities hold in case f is a polynomial which vanishes at $t = 0$. Then for general f, we need only to take a sequence $\{p_n\}$ of polynomials with $p_n(0) = 0$ which converges uniformly to f on $[0, \|T\|^2\|A\|\,]$. □

THEORM 3.1. *If T is log-hyponormal, then*

$$T^{*n}T^n \leq (T^{*n+1}T^{n+1})^{\frac{n}{n+1}} \tag{11}$$

Proof. Let $T = U|T|$ be the polar decomposition of T and P the orthogonal projection onto $\overline{\mathcal{R}(T)}$. We denote the compression of an operator X to $\overline{\mathcal{R}(T)}$ by $[X]$. Then the log-hyponormality of T means $\log[T^*T] \geq \log[TT^*]$. Thus, by the first inequality of (9) we get

$$([TT^*]^{1/2}[T^*T][TT^*]^{1/2})^{1/2} \geq [TT^*],$$

which is equivalent to

$$\{(PTT^*P)^{1/2}(PT^*TP)(PTT^*P)^{1/2}\}^{1/2} \geq PTT^*P.$$

Since $TT^*(1-P) = 0$, this gives

$$\{(TT^*)^{1/2}PT^*TP(TT^*)^{1/2}\}^{1/2} \geq TT^*,$$

and hence

$$U^*(|T^*|PT^*TP|T^*|)^{1/2}U \geq U^*TT^*U.$$

In view of the lemma, the left hand side equals $(T^*PT^*TPT)^{1/2}$. Since $PT = T$ and $U^*TT^*U = T^*T$, from the above inequality we get

$$(T^{*2}T^2)^{1/2} \geq T^*T.$$

This means (11) holds for $n = 1$.

Assume that (11) holds for $n \leq m-1$. Therefore,

$$T^{*m-1}T^{m-1} \leq (T^{*m}T^m)^{\frac{m-1}{m}}, \tag{12}$$
$$T^*T \leq (T^{*m}T^m)^{\frac{1}{m}}. \tag{13}$$

(13) implies

$$PT^*TP \leq P(T^{*m}T^m)^{\frac{1}{m}}P \leq (PT^{*m}T^mP)^{\frac{1}{m}},$$

here the last inequality is due to (1). Thus $[T^*T] \leq [T^{*m}T^m]^{\frac{1}{m}}$. Since $\mathcal{N}([T^*T]) = 0$ because $\mathcal{R}(T)$ is orthogonal to $\mathcal{N}(T)$ by definition of log-hyponormality of T, by (2) we have

$$\log[T^*T] \leq \log[T^{*m}T^m]^{\frac{1}{m}}.$$

Since T is log-hyponormal, this gives

$$\log[TT^*] \leq \log[T^{*m}T^m]^{\frac{1}{m}}.$$

By the first inequality of (8), we obtain

$$[TT^*]^{1/2}[T^{*m}T^m]^{\frac{m-1}{m}}[TT^*]^{1/2} \leq ([TT^*]^{1/2}[T^{*m}T^m]^{\frac{m}{m}}[TT^*]^{1/2})^{\frac{m-1+1}{m+1}},$$

which is equivalent to

$$\begin{aligned}&(PTT^*P)^{1/2}(PT^{*m}T^mP)^{\frac{m-1}{m}}(PTT^*P)^{1/2}\\ &\leq \{(PTT^*P)^{1/2}(PT^{*m}T^mP)(PTT^*P)^{1/2}\}^{\frac{m}{m+1}}.\end{aligned}$$

Since $PT = T$ and $T^*P = T^*$, this gives

$$|T^*|(PT^{*m}T^mP)^{\frac{m-1}{m}}|T^*| \leq \{|T^*|(PT^{*m}T^mP)|T^*|\}^{\frac{m}{m+1}}. \tag{14}$$

From (12) it follows that

$$\begin{aligned}T^{*m}T^m &= T^*(T^{*m-1}T^{m-1})T \leq T^*(T^{*m}T^m)^{\frac{m-1}{m}}T\\ &= U^*|T^*|P(T^{*m}T^m)^{\frac{m-1}{m}}P|T^*|U\\ &\leq U^*|T^*|(PT^{*m}T^mP)^{\frac{m-1}{m}}|T^*|U \quad (by\ (1))\\ &\leq U^*\{|T^*|(PT^{*m}T^mP)|T^*|\}^{\frac{m}{m+1}}U \quad (by\ (14))\\ &= (U^*|T^*|PT^{*m}T^mP|T^*|U)^{\frac{m}{m+1}} \quad (by\ Lemma\, 3.1)\\ &= (T^{*m+1}T^{m+1})^{\frac{m}{m+1}}.\end{aligned}$$

This completes the proof. □

Note that (11) implies that $\mathcal{N}(T) = \mathcal{N}(T^2) = \cdots = \mathcal{N}(T^n) = \cdots$.

THEOREM 3.2. *If for a natural number m*

$$\mathcal{N}(T) \subseteq \mathcal{N}(T^*) \quad and \quad \log[T^*T]_n \geq \log[TT^*]_n \quad (1 \leq n \leq m), \tag{15}$$

where $[X]_n$ *means the compression of* X *to* $\overline{\mathcal{R}(T^n)}$*, then*

$$T^nT^{*n} \geq (T^{n+1}T^{*n+1})^{\frac{n}{n+1}} \quad (n = 1, 2, \ldots, m). \tag{16}$$

Proof. We prove this by the mathematical induction of m. Suppose (15) hold for $m = 1$. Since then T is log-hyponormal, from the second inequality of (9) it follows that

$$[T^*T]_1 \geq \{[T^*T]_1^{1/2}[TT^*]_1[T^*T]_1^{1/2}\}^{1/2}.$$

Denoting the projection to $\overline{\mathcal{R}(T)}$ by P, the above shows

$$PT^*TP \geq \{(PT^*TP)^{1/2}(PTT^*P)(PT^*TP)^{1/2}\}^{1/2}.$$

Let $TP = U|TP|$ be the polar decomposition of TP. Then the above leads to

$$|TP|^2 \geq \{|TP|(PTT^*P)|TP|\}^{1/2},$$

and hence

$$U|TP|^2U^* \geq U\{|TP|(PTT^*P)|TP|\}^{1/2}U^*.$$

By the lemma, we obtain

$$|PT^*|^2 \geq \{TP(PTT^*P)(TP)^*\}^{1/2}.$$

Since $PT = T$, the righthand side equals $(T^2T^{*2})^{1/2}$. Thus we get

$$TT^* \geq TPT^* \geq (T^2T^{*2})^{1/2}.$$

Consequently (16) holds for $m = 1$.

Assume that the theorem is valid for $m - 1$. To show that it is valid for m, suppose (15) hold for $n = 1, 2, \cdots, m$. From the inductive assumption (16) holds for $n = 1, 2, \ldots, m-1$; therefore, we have

$$T^{m-1}T^{*m-1} \geq (T^mT^{*m})^{\frac{m-1}{m}} \quad \text{and} \quad TT^* \geq (T^mT^{*m})^{1/m}. \tag{17}$$

The second inequality yields

$$[TT^*]_m \geq [T^mT^{*m}]_m^{1/m},$$

and hence

$$\log[TT^*]_m \geq \log([T^mT^{*m}]_m^{1/m}).$$

Therefore, by (15) with $n = m$ we obtain

$$\log[T^*T]_m \geq \log([T^mT^{*m}]_m^{1/m}).$$

Applying this to the second inequality of (8) we get

$$[T^*T]_m^{1/2}([T^mT^{*m}]_m^{1/m})^{m-1}[T^*T]_m^{1/2} \geq \{[T^*T]_m^{1/2}([T^mT^{*m}]_m^{1/m})^m[T^*T]_m^{1/2}\}^{\frac{m-1+1}{m+1}}.$$

Denote the projection to $\overline{\mathcal{R}(T^m)}$ by Q, and rewrite the above as

$$(QT^*TQ)^{1/2}(T^mT^{*m})^{\frac{m-1}{m}}(QT^*TQ)^{1/2} \geq \{(QT^*TQ)^{1/2}(T^mT^{*m})(QT^*TQ)^{1/2}\}^{\frac{m-1+1}{m+1}}.$$

Now let $TQ = V|TQ|$ be the polar decomposition of TQ. Then we have

$$|TQ|(T^mT^{*m})^{\frac{m-1}{m}}|TQ| \geq \{|TQ|(T^mT^{*m})|TQ|\}^{\frac{m-1+1}{m+1}}.$$

Multiplying this inequality by V and V^*, in virtue of the lemma, yields

$$TQ(T^mT^{*m})^{\frac{m-1}{m}}QT^* \geq \{TQ(T^mT^{*m})QT^*\}^{\frac{m-1+1}{m+1}}.$$

From this it follows that

$$T(T^mT^{*m})^{\frac{m-1}{m}}T^* \geq \{T(T^mT^{*m})T^*\}^{\frac{m-1+1}{m+1}} = (T^{m+1}T^{*m+1})^{\frac{m}{m+1}},$$

for $Q(T^mT^{*m}) = T^mT^{*m} = (T^mT^{*m})Q$. This in conjunction with the first inequality of (17) gives

$$T^mT^{*m} = T(T^{m-1}T^{*m-1})T^* \geq T(T^mT^{*m})^{\frac{m-1}{m}}T^* \geq (T^{m+1}T^{*m+1})^{\frac{m}{m+1}}.$$

This shows that (16) holds for $n = m$. Thus, we conclude the proof. □

As mentioned in the proof, (15) with $m = 1$ is satisfied for a log-hyponormal operator. Therefore,

$$TT^* \geq (T^2T^{*2})^{\frac{1}{2}}$$

is valid for a log-hyponormal operator T. The condition (15) is technical. We do not know if (15) follows from the log-hyponormality of T. However, one can easily see that (15) holds for every m if T is hyponormal. Moreover, $\mathcal{N}(T^*) = \mathcal{N}(T^{*m})$ for every m if $\mathcal{N}(T^*) = 0$ or $\mathcal{N}(T^*) = \mathcal{N}(T^{*2})$, so that we get:

COROLLARY 3.3. *If T is a log-hyponormal operator and if $\mathcal{N}(T^*) = 0$ or $\mathcal{N}(T^*) = \mathcal{N}(T^{*2})$, then for every n*

$$T^nT^{*n} \geq (T^{n+1}T^{*n+1})^{\frac{n}{n+1}}.$$

4. NORMALITY

Recall the definition of the quasi-normality of T stated in the previous section. Then we can see that T is quasi-normal if and only if the polar decomposition of T is commutative, so that a quasi-normal operator T is normal if $\mathcal{N}(T) = \mathcal{N}(T^*)$.

In this section, we give a few conditions which guarantee quasi-normality or normality of a log-hyponormal operator.

THEOREM 4.1. *If a log-hyponormal operator T satisfies $|T|^n = |T^n|$ for some $n \geq 3$, then T is quasi-normal.*

Proof. By Theorem 3.1, we get

$$|T| \leq |T^2|^{1/2} \leq \cdots \leq |T^n|^{1/n} \leq \cdots .$$

Thus the assumption implies that

$$(T^*T)^2 = T^{*2}T^2 \quad \text{and} \quad (T^*T)^3 = T^{*3}T^3. \tag{18}$$

From the first equality of (18) it follows that $T^*(TT^* - T^*T)T = 0$, and hence, for the projection P onto $\overline{\mathcal{R}(T)}$

$$PT^*TP = PTT^*P = TT^*. \tag{19}$$

By (18) and (19)

$$T^*(TT^*)^2T = (T^*T)^3 = T^{*3}T^3 = T^*(T^{*2}T^2)T = T^*(T^*T)^2T,$$

from which it follows that

$$P(TT^*)^2P = P(T^*T)^2P.$$

Since the left hand side equals $(TT^*)^2$, this and (19) yield

$$(PT^*TP)^2 = (TT^*)^2 = P(TT^*)^2P = P(T^*T)^2P.$$

Consequently,

$$PT^*T(1-P)T^*TP = 0 \quad \text{and hence} \quad T^*TP = PT^*TP.$$

From this and (19) we derive

$$(T^*T)T = (T^*TP)T = (PT^*TP)T = (TT^*)T = T(T^*T).$$

This means T is quasi-normal. □

The above theorem does not hold for the condition $n = 2$: see for a counter example p.199 of [11].

THEOREM 4.2. *If a log-hyponormal operator T satisfies*

$$|T^*|^n = |T^{*n}| \quad \textit{for some} \quad n \geq 2,$$

then T is normal.

Proof. The assumption implies $T^nT^{*n} = (TT^*)^n$ and hence

$$\mathcal{N}(T^{*n}) = \mathcal{N}(T^nT^{*n}) = \mathcal{N}((TT^*)^n) = \mathcal{N}(TT^*) = \mathcal{N}(T^*),$$

from which $\mathcal{N}(T^{*2}) = \mathcal{N}(T^*)$ follows. By Corollary 3.3

$$(T^m T^{*m})^{1/m} \geq (T^{m+1} T^{*(m+1)})^{1/(m+1)} \quad \text{for every } m.$$

Therefore, by assumption, we have

$$TT^* = (T^2 T^{*2})^{1/2} = \cdots = (T^n T^{*n})^{1/n},$$

which implies $T^2 T^{*2} = (TT^*)^2$. Denoting the projection onto $\overline{\mathcal{R}(T^*)}$ by Q, this implies that

$$QTT^*Q = QT^*TQ = T^*T.$$

On the other hand, since $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$ by assumption and since $1 - Q$ is the projection to $\mathcal{N}(T)$, we have $T^*(1 - Q) = 0$, which implies $TT^* = QTT^*Q$. Consequently, we obtain $TT^* = QTT^*Q = T^*T$. □

THEOREM 4.3. *A log-hyponormal operator T is normal if T^{*n} is log-hyponormal for some natural number n.*

Proof. As we mentioned after the proof of Theorem 3.1,

$$\mathcal{N}(T^*) \supseteq \mathcal{N}(T) = \mathcal{N}(T^m) \quad \text{for every } m.$$

On the other hand,

$$\mathcal{N}(T^*) \subseteq \mathcal{N}(T^{*2}) \subseteq \cdots \subseteq \mathcal{N}(T^{*n}) \subseteq \mathcal{N}(T^n)$$

since T^{*n} is log-hyponormal. Thus we have $\mathcal{N}(T) = \mathcal{N}(T^{*m})$ for $m = 1, 2, \ldots, n$. Denote the compression of X to $\overline{\mathcal{R}(T)}$ by $[X]$. By Theorems 3.1 and 3.2 we get

$$[T^*T] \leq [T^{*n}T^n]^{1/n}, \quad [T^n T^{*n}]^{1/n} \leq [TT^*].$$

Since $\log(A^{1/n}) = \frac{1}{n}\log A$ for $A \geq 0$ with $\mathcal{N}(A) = 0$, and since T^{*n} is log-hyponormal,

$$\log[T^*T] \leq \frac{1}{n}\log[T^{*n}T^n] \leq \frac{1}{n}\log[T^n T^{*n}] \leq \log[TT^*].$$

Therefore the log-hyponormality of T yields $\log[T^*T] = \log[TT^*]$. This implies $T^*T = TT^*$. □

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