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Kyoto University
Singular Values of Differences of Positive Semidefinite Matrices\textsuperscript{1}

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1 Introduction

Let $M_{n}$ be the space of $n \times n$ complex matrices. For simplicity we treat matrices here, but all our results hold for compact operators on a Hilbert space. Suppose $A, B \in M_{n}$ are positive semidefinite. We shall study the relations between the singular values of $A - B$ and those of $A - |z|B$, $A + zB$, and $A + |z|B$ where $z$ is a complex number.

A norm $\| \cdot \|$ on $M_{n}$ is called unitarily invariant if $\|UAV\| = \|A\|$ for all $A$ and all unitary $U, V$. Every unitarily invariant norm is a symmetric gauge function of the singular values. See [3, 7]. We always denote the singular values of $A$ by $s_{1}(A) \geq \cdots \geq s_{n}(A)$, and put $s(A) \equiv (s_{1}(A), \ldots, s_{n}(A))$. Familiar examples of unitarily invariant norms are the Ky Fan $k$-norms defined by $\|A\|_{(k)} = \sum_{1}^{k}s_{j}(A)$ and the Schatten $p$-norms: $\|A\|_{p} = (\sum_{1}^{n}s_{j}^{p}(A))^{1/p}$, $p \geq 1$. Note that $\| \cdot \|_{\infty}$ is just the operator (spectral) norm and $\| \cdot \|_{2}$ is the Frobenius norm.

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A unitarily invariant norm may be considered as defined on $M_n$ for all orders $n$ by the rule

$$|||A||| = |||egin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}|||,$$

i.e., adding or deleting zero singular values does not affect the value of the corresponding symmetric gauge function.

Given a real vector $x = (x_i) \in \mathbb{R}^n$, rearrange its components as $x[1] \geq \cdots \geq x[n]$. For $x = (x_i), y = (y_i) \in \mathbb{R}^n$, if

$$\sum_{1}^{k} x[i] \leq \sum_{1}^{k} y[i], \quad k = 1, 2, \ldots, n,$$

then we say $x$ is weakly majorized by $y$, denoted $x \prec_w y$. If the components of $x$ and $y$ are nonnegative and

$$\prod_{1}^{k} x[i] \leq \prod_{1}^{k} y[i], \quad k = 1, 2, \ldots, n$$

we say $x$ is weakly log-majorized by $y$, denoted $x \prec_{wlog} y$. See [6] for a discussion of this topic.

Denote the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ by $A \oplus B$. Bhatia and Kittaneh [4, Remark 5] observed that if $A, B \in M_n$ are positive semidefinite then

$$|||A - B||| \leq |||A \oplus B|||$$

(1.1)

for every unitarily invariant norm. By the Fan dominance principle [3, 7], (1.1) is equivalent to $s(A - B) \prec_w s(A \oplus B)$. We shall show that in fact each singular value of $A - B$ is not greater than the corresponding singular value of $A \oplus B$.

In another paper, Bhatia and Kittaneh [5, Thm 1] proved that for positive semidefinite $A, B \in M_n$ and any complex number $z$

$$|||A - |z|B||| \leq |||A + zB||| \leq |||A + |z|B|||$$

(1.2)
for all unitarily invariant norms. Again (1.2) is equivalent to
\[ s(A - |z|B) \prec_w s(A + zB) \prec_w s(A + |z|B). \]

We shall prove that the corresponding weak log-majorizations hold. Since weak log-majorization implies weak majorization [6,7], our result strengthens (1.2).

2 Main Results

Our first result sharpens (1.1).

**Theorem 1** Let \( A, B \in M_n \) be positive semidefinite. Then
\[
s_j(A - B) \leq s_j(A \oplus B), \quad j = 1, 2, \ldots, n. \tag{2.1}
\]

The following result sharpens (1.2).

**Theorem 2** Let \( A, B \in M_n \) be positive semidefinite. Then for any complex number \( z \)
\[
s(A - |z|B) \prec_{wlog} s(A + zB) \prec_{wlog} s(A + |z|B). \tag{2.2}
\]

The special case \( z = i = \sqrt{-1} \) of Theorem 2 says
\[
s(A - B) \prec_{wlog} s(A + iB) \prec_{wlog} s(A + B). \tag{2.3}
\]

It has been proved in [2] that for positive \( A, B \) and \( p > 1 \)
\[
s(A^p + B^p) \prec_w s((A + B)^p). \tag{2.4}
\]
When $p \geq 2$, the above relation is refined as follows:
\[ s(A^p + B^p) \prec_w s((A^2 + B^2)^{p/2}) \prec_w s(|A + iB|^p) \prec_{w \log} s((A + B)^p). \] (2.5)

The first relation in (2.5) follows from (2.4) and the third relation follows from (2.3). To see the second relation let $T = A + iB$. This is the Cartesian decomposition. From $A^2 + B^2 = (T^*T + TT^*)/2$ we get
\[ s(A^2 + B^2) \prec_w s(|A + iB|^2). \]

Note that $f(t) = t^{p/2}$ is convex and increasing on $[0, \infty)$. By a majorization principle [3, 7], applying this $f$ to the preceding weak majorization yields the second relation in (2.5).

From (2.3) and the results in [1] and [2] it follows that for $0 < p \leq 1$,
\[ s(A^p - B^p) \prec_w s(|A - B|^p) \prec_{w \log} s(|A + iB|^p) \prec_{w \log} s((A + B)^p) \prec_w s(A^p + B^p). \]

One might wonder whether the weak majorization (2.4) can be replaced by the stronger log-majorization. The answer is no, even for $p = 2$. Consider the example
\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \]
We have $\det(A^2 + B^2) = 2 > 1 = \det[(A + B)^2]$.

Recently we have generalized Theorem 1 and the second majorization result in Theorem 2 to the case of $\tau$-measurable operators affiliated with a semifinite von Neumann algebra.

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1. T. Ando, Comparison of norms $|||f(A) - f(B)|||$ and $|||f(|A - B|)|||$, Math. Z., 197(1988) 403-409.


