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<td>Miyachi, Hyohe</td>
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<td>引用</td>
<td>数理解析研究所講究録 2001年2月号</td>
</tr>
<tr>
<td>発行年月</td>
<td>2001-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64739">http://hdl.handle.net/2433/64739</a></td>
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<td>テキストバージョン</td>
<td>Departmental Bulletin Paper</td>
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GL_n(F_q)のブロックにおける圏同値の組合せ論的
記述、qとの独立性と分解定数について

宮地 兵衛 (Hyohe Miyachi)

(千葉大・自然科

1 Notation

The result of this paper is a joint work with Akihiko Hida. Let
G be a finite group, and (K, O, k) be a splitting \ell-modular system for G.
Here char(K) = 0, char(k) = \ell > 0. For R \in \{O, k\}, let B_0(RG) be the
principal block of RG.
S_n denotes the symmetric group on n letters. \mathbb{F}_q denotes a field with q
elements with \ell \nmid q. Let natural numbers e(q) and r(q) be as follows:

\[ e(q) := \min \{ i \in \mathbb{N} \mid q^i \equiv 1 \pmod{\ell} \}, \]

\[ r(q) := \max \{ r \in \mathbb{N} \mid \ell^r \mid q^{e(q)} - 1 \} : \text{the } \ell\text{-part of } q^{e(q)} - 1. \]

Let A and B be blocks ideals. "A \sim_M B" means that A is Morita
(Puig) equivalent to B. "A \sim_d B" means that A is derived (splendid
Rickard) equivalent to B (see [34],[35]).
We use results on representation theory of finite general linear groups
in non-defining characteristic due to Fong-Srinivasan and Dipper-James
(see [14], [15], [9],[10],[11],[12],[13],[19]).

2 Motivations

We wish to prove the following conjectures:

Conjecture 2.1 (Broué). [2],[3],[4] Let B be an \ell-block ideal of G with
abelian defect group D. Then B and its Brauer correspondent in \mathcal{N}_G(D)
are derived equivalent?

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0The detailed version of this paper will be submitted for "Doctor thesis at Chiba
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Conjecture 2.2 (James). [20] Suppose that $\text{char}(k) = \ell > n$ and $e(q) = \ell$. Let $\zeta$ be a primitive $e$-th root of unity in $\mathbb{C}$. Then, the decomposition matrix of Dipper-James Schur algebra $S_\zeta(n, r_\mathbb{C})$ over $\mathbb{C}$ is equal to that of Dipper-James Schur algebra $S_q(n, r)_{\mathbb{k}}$ over $\mathbb{k}$?

Let $G$ be a connected reductive algebraic group over $\mathbb{F}_q$ with a Frobenius map $F$. We assume that the centre of $G$ is connected. Let $\ell$ be a prime number with $\ell \nmid q$.

Lusztig series

The following is so-called Lusztig series:

$$ E(G^F, \{s\}) := \bigcup_{(T, \theta)} \{ \chi \in \hat{G}^F | \langle \chi, R_T^G(\theta) \rangle \neq 0 \}. $$

Here, the above pair $(T, \theta)$ runs $s_1 \in \{s\}$ and $\theta \in \widehat{T}^F \leftrightarrow s_1 \in T^F$, and $R_T^G(\theta)$ is a generalized Deligne-Lusztig character.

Its modular version is given as follows:

For a semisimple $\ell'$-element $s \in G'^F$, let

$$ E_\ell(G^F, \{s\}) := \bigcup_{t} E(G^F, \{st\}), \quad t \in (C_{G^*}(s)^F)^*_F. $$

Theorem 2.3 (Broué-Michel). [5] Each set $E_\ell(G^F, \{s\})$ is a union of $\ell$-block of $G^F$.

Definition 2.1. An $\ell$-block $B$ as an algebra is unipotent, if there exists $\chi \in E_\ell(G^F, \{1\})$ such that $\chi$ belongs to $B$. In particular, $B_0(OG^F)$ is unipotent.

Theorem 2.4 (Bonnafé-Rouquier). [1] Suppose that the centre of $G$ is connected and $C_{G^*}(s)^F$ is a Levi subgroup of $G$. Then:

$$ E_\ell(G^F, \{s\}) \sim \mathcal{M} E_\ell(C_{G^*}(s)^F, \{1\}) $$

as $\ell$-block ideals. (i.e. If a block $B_s$ belongs to $E_\ell(G^F, \{s\})$, then there exists a unipotent block $B'_s$ of $C_{G^*}(s)^F$ such that $B_s$ and $B'_s$ are Morita equivalent.)
Remark 1. The Morita equivalence in the above theorem is not a Puig equivalence in general.

In particular, for finite general linear groups we may concentrate unipotent blocks by Bonnafé-Rouquier theorem.

We want to classify the block ideals of $kG(F_q)$ up to Morita equivalence, and recover its structure as algebras from some small subgroups. So, we wish to prove the following conjecture:

Conjecture 2.5. If $e(q) = e(q'), r(q) = r(q')$ then for any unipotent block ideal $B$ of $G(F_q)$ there exists a unipotent block ideal $B'$ of $G(F_{q'})$ such that $B \sim_M B'$ by an exact $\ell$-permutation $(B, B')$-bimodule. This equivalence preserves the natural indices of modules.

In this article we deal the special case concerning these three conjectures for finite general linear groups.

3 Abacus and $[w:k]$-pairs

Definition 3.1. For a $k$-core $\tau$ and a non-negative integer $w$, let $\Lambda_{k,w,\tau}$ be the set of partitions of $kw + |\tau|$ whose $k$-core is $\tau$.

Given partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, define $\beta = (\beta_1, \beta_2, \ldots)$ as follows:

$$\beta_i := r - i + \lambda_i (1 \leq i \leq r).$$

We call this $\beta$ an $r$-element $\beta$-set for $\lambda$.

Definition 3.2 (Scopes). For a non-negative integer $m$ and an $m$-core $\tau = (\tau_1, \ldots, \tau_r)$, let $\Gamma$ be the $r$-element $\beta$-set for $\tau$, and suppose that when $\Gamma$ is displayed on an abacus with $m$-runners there are $k$ more than beads in the $i$-th column than in the $(i - 1)$-th column. Let $m$-core $\overline{\tau}$ be displayed by an $r$-element $\beta$-set $\overline{\Gamma}$ satisfying

$$\overline{\Gamma}_j = \Gamma_j \quad \text{for } j \neq i, i - 1$$

$$\overline{\Gamma}_i = \Gamma_{i - 1}$$

$$\overline{\Gamma}_{i - 1} = \Gamma_i,$$

where $\Gamma_j$ is the number of beads on the $j$-th runner in the abacus configuration for $\Gamma$. In these situation, we shall say that $\Lambda_{m,w,\tau}$ and $\Lambda_{m,w,\overline{\tau}}$ form a Scopes $[w:k]$-pair.
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Scopes proved the following:

**Theorem 3.1 (Scopes).** [37] If $\Lambda_{p,w,\tau}$ and $\Lambda_{p,w,\nu}$ form a $[w : k]$-pair with $k \geq w$, then $p$-blocks $B^{w,\tau}$ and $B^{w,\nu}$ of symmetric groups are Morita equivalent.

By Jost we also know the following:

**Theorem 3.2 (Jost).** [23] If $\Lambda_{e,w,\tau}$ and $\Lambda_{e,w,\nu}$ form a $[w : k]$-pair with $k \geq w$, then unipotent $\ell$-blocks $B_{w,\tau}$ and $B_{w,\nu}$ are Morita equivalent.

**Example 1.** If $B$ is a unipotent block of $GL_n(q)$ with $e$-weight 2, then one of the following holds:

1. $B \cong B_0(kGL_{2e}(q))$.
2. $(B, \overline{B})$ forms $[2:1]$-pair for some unipotent block $\overline{B}$ of $kGL_{n-1}(q)$. (Actually, these blocks are derived equivalent to its Brauer correspondent of the $\ell$-local subgroup. (Hida-Miyachi(1999)) (The method we used is different from J. Chuang’s for $S_n$)
3. $B \sim_M B'$ for some unipotent block $B'$ of $kGL_m(q)$ with $m < n$.

4 A core $\rho$ and results of J. Chuang and R. Kessar

**Definition 4.1 (Chuang-Kessar-Rouquier).** [8] Let $\rho$ be the $e$-core which satisfies the following property: $\rho$ has an abacus configuration in which each runner other than the leftmost one (the 0-th runner) has at least $w - 1$ more beads than the runners to its immediate left.
Chuang and Kessar considered the following setting up:

\[ e = p > w. \]

\[ r := |\rho|. \]

\[ G := S_{pw+r}. \]

\( B^{w,\rho} \): the \( p \)-block of \( kG \) with \( p \)-weight \( w \) and \( p \)-core \( \rho \).

\[ D := \text{a defect group of } B^{w,\rho}. \]

\[ N := S_p \ltimes S_w \supset D. \]

\[ L := S_p \times \cdots \times S_p \times S_r. \]

\[ H := (S_p \ltimes S_w) \times S_r \supset N_G(D). \]

\[ \mathcal{O}Hf := \text{the Brauer correspondent of } B^{w,\rho} \text{ in } H. \]

Let \( X \) be the Green correspondent of \( B^{w,\rho} \) in \( G \times H \) with respect to \((G \times G, \Delta(D), G \times H)\). Chuang and Kessar proved the following:

**Theorem 4.1 (Chuang-Kessar).** [8] Suppose that \( p > w \). Then, we get an isomorphism

\[ \mathcal{O}Hf \cong \text{End}_G(X) \]

by checking \( \text{rank}_\mathcal{O}(\text{End}_\mathcal{O}(X)) \leq w! \cdot \text{rank}_\mathcal{O}(\mathcal{O}Lf) \). In particular, \( \mathcal{O}Hf \) is Morita equivalent to \( B^{w,\rho} \).

**Remark 2.**

1. \( X \) is exact.
2. \( \mathcal{O}Hf \to \text{End}_\mathcal{O}(X) \) is a split \((\mathcal{O}Hf, \mathcal{O}Hf)\)-monomorphism.
3. \( w! \text{rank}_\mathcal{O}(\mathcal{O}Lf) = \text{rank}_\mathcal{O}(\mathcal{O}Hf) \).
4. By Marcus [27] \( \mathcal{O}Hf \sim_d B_0(ON) \).
5. \((D^\lambda \otimes_{B^{w,\rho}} X) \downarrow_L \) is known, but \( D^\lambda \otimes_{B^{w,\rho}} X \) is not known.
5 A theorem of Chuang-Kessar type

We assume that $\text{char}(k) = \ell > w$. Choose a prime power $q$ with $e(q) = e$. Just mimicking Chuang and Kessar’s setting up, we consider the following:

$r := |\rho|.$

$G(q) := GL_{ew+r}(q).$

$B^{w,\rho}(q):$ the unipotent $\ell$-block of $kG(q)$ with $e$-weight $w$ and $e$-core $\rho$.

$D(q) :=$ a defect group of $B^{w,\rho}.$

$N(q) := GL_e(q) \wr S_w \supset D(q).$

$L(q) := GL_e(q) \times \cdots \times GL_e(q) \times GL_r(q).$

$H_w(q) := (GL_e(q) \wr S_w) \times GL_r(q) \supset \mathcal{N}_G(D(q)).$

$\mathcal{O}H_w(q)f_q :=$ the Brauer correspondent of $B_{w,\rho}(q)$ in $H_w(q)$.

Once we believe that an analogy of Chuang-Kessar theorem holds for finite general linear groups, we can easily prove the following:

**Proposition 5.1.** (An analogy of Chuang-Kessar theorem) Let $X(q)$ be the Green correspondent of $B^{w,\rho}(q)$ in $G(q) \times H_w(q)$ with respect to $(G(q) \times G(q), \Delta(D(q)), G(q) \times H_w(q))$. Then, we get an isomorphism

$$\mathcal{O}H_w(q)f_q \cong \text{End}_{G(q)}(X_{\mathcal{O}}(q))$$

by checking $\text{rank}_{\mathcal{O}}(\text{End}_{G(q)}(X_{\mathcal{O}}(q))) \leq w! \cdot \text{rank}_{\mathcal{O}}(\mathcal{O}L(q)f_q)$. In particular, $\mathcal{O}H_w(q)f_q$ is Morita equivalent to $B_{w,\rho}(q)$.

**Remark 3.** One must consider not only unipotent characters but also characters indexed by semisimple $\ell$-elements. We can know these characters by [9]. We also need some results by [15] in order to mimic Chuang and Kessar’s argument.
6 Indices of $B_0(GL_e(q) \wr S_w)$-modules

In this section we reformulate indices of the simple $B_0(GL_e(q) \wr S_w)$-modules to fit that of $B_{w,\rho}(q)$ via the equivalence in Proposition 5.1. For $i = 1, 2, \ldots, e$ let $\nu_i = (i, 1^{e-i}) \vdash e$. The principal block $B_0(kGL_e(q))$ has $e$ non-isomorphic irreducible modules

$$\{ D_{k,q}(\nu_i) \mid i = 1, 2, \ldots, w \}.$$ 

Fix $R \in \{K, k\}$. Let $n$ be an $e$-tuple non-negative integer of $w$. i.e. $\sum_{i=1}^{e} n_i = w$. $S_{R,q}(n) := \bigotimes_{i}(S_{R,q}(\nu_i)^{\otimes n_i})$ is an $R[GL_e(q)^\times w]$-module.

In particular, $S_{K,q}(n)$ is a simple $K[GL_e(q)^\times w]$-module. The parabolic subgroup $S_n$ act on $S_{R,q}(n)$. So, $S_{R,q}(n)$ is an $R[L(e^w) \rtimes S_n]$-module. Ind_{L(e^w) \rtimes S_n}^{GGL(e^w)} S_{R,q}(n)$ is decomposed into $\bigoplus_{\lambda \vdash n} (S_{R,q}(n) \otimes_R (\dim_R S_{R}^\lambda) \cdot S_{R}^\lambda)$ where $S_{R}^\mu$ means the Specht module of $R[\mathfrak{S}_{|\mu|}]$ corresponding to $\mu$. $S_{R}^\lambda = \bigotimes_{\lambda_i} S_{R}^{\lambda_i}$ and $S_{R,q}(n) \otimes_R S_{R}^\lambda$ is the inner tensor product of $R[L(e^w) \rtimes S_n]$-modules $S_{R,q}(n)$ and $S_{R}^\lambda$. Let

$$T_{R}^\lambda = \begin{cases} S_{R}^{\lambda_i} & \text{if } i + e \text{ is even,} \\ S_{R}^{\lambda_i'} & \text{if } i + e \text{ is odd.} \end{cases}$$

Here, $\lambda_i'$ is the conjugate partition of $\lambda_i$. Let $T_{R}^\lambda = \bigotimes_{\lambda} T_{R}^\lambda_i$.

For $\lambda \vdash n$ let $U_{R,q}(\lambda)$ be Ind_{L(e^w) \rtimes S_n}^{GGL(e^w)} (S_{R,q}(n) \otimes T_{R}^\lambda)$, and let $U_{k,q}(\lambda)^\rho$ be the $R[H_w(q)]$-module $U_{R,q}(\lambda) \otimes_R S_{R,q}(\rho)$.

Moreover, one can construct modules by using

$$\{ D_{k,q}(\nu_i) \mid i = 1, 2, \ldots, e \}$$

instead of $k(GL_e(q))$-modules $\{ S_{k,q}(\nu_i) \mid i = 1, 2, \ldots, e \}$. We denote it by $V_{k,q}(\lambda)^\rho$. 


7 Results

Now we can state our main results of this article as follows:

**Theorem 7.1 (Hida-Miyachi).** [18] For any simple $B_{w,ho}(q)$-module $D_{k,q}(\lambda)$, the Green correspondent $D_{k,q}(\lambda) \otimes_{KG} X(q)$ of $D_{k,q}(\lambda)$ is independent of $q$ in the following sense:

Assume that $e(q) = e(q')$ and $r(q) = r(q')$. Let $M_{q,q'}$ be the canonical $(kH_w(q)f_q,kH(q')f_{q'})$-bimodule which induces $kH_w(q)f_q \sim_{M} kH(q')f_{q'}$, due to A. Marcus. Then

$$D_{k,q}(\lambda) \otimes_{B_{w,q}(q)} X(q) \otimes_{kH_w(q)f_q} M_{q,q'} \otimes_{kH(q')f_{q'}} X(q')^\vee \cong D_{k,q'}(\lambda).$$

Actually, $D_{k,q}(\lambda) \otimes_{B_{w,q}(q)} X(q) \cong V_{k,q}(\bar{\lambda})^\rho$. Here, $\bar{\lambda}$ is the $e$-quotient of $\lambda$. Moreover, we know the decomposition numbers corresponding to the $e$-core $\rho$:

$$d_{\lambda,\mu} = d_{\bar{\lambda},\bar{\mu}} = [U_{k,q}(\bar{\lambda}) : V_{k,q}(\bar{\mu})].$$

(The other parts of $B_{w,q}(q)$ can be calculated by Dipper-James theory.)

**Remark 4.** First we can determine the Green correspondents of simple $B_{2,\rho}(q)$-modules in $H_2(q)$, finding two trivial source modules of $B_{2,\rho}(q)$, using the decomposition numbers for Hecke algebras of type $A$ by [33] and [22], chasing the image of Mullineux-Kleshchev map [29, p.120], the property of Specht modules [19] and induction on $\Lambda_{e,2,\rho}$.

Next we can determine the Green correspondents of simple $B_{w,\rho}(q)$-modules in $H_w(q)$ using induction on $w$ and some commutative diagrams among $B_{w,q}(q),B_0(GL_w(q)) \otimes B_{w-1,q}(q)$ and their Brauer correspondents.

In order to prove $B_{w,q}(q) \sim_{M} B_{w,q}(q')$ with the property in the above theorem we use [14],[27], and [36].

**Corollary 7.2.** [18] If there exist a sequence of $e$-cores

$$\rho = \tau^0,\tau^1, \ldots, \tau^s$$

such that $\Lambda_{e,w,\tau^i}$ and $\Lambda_{e,w,\tau^{i+1}}$ form a $[w : k_i]$-pair with $k_i \geq w-1$, Broué's conjecture is true for $B_{w,\tau^s}(q)$. 

Theorem 7.3 (Hida-Miyachi). [18] Assume that $e = e(q) = e(q')$ and $r(q) = r(q')$. If there exist a sequence of $e$-cores
\[ \rho = \tau^0, \tau^1, \ldots, \tau^s \]
such that $\Lambda_{e,w,\tau^i}$ and $\Lambda_{e,w,\tau^{i+1}}$ form a $[w:k_i]$-pair with $k_i \geq w - 1$, then
\[ B_{w,\tau^i}(q) \sim_{M} B_{w,\mathcal{T}\theta}(q') \]
Here, each $[w:w-1]$-pair is a derived (splendid) equivalence between two unipotent blocks. Moreover, the above Morita equivalence preserves natural indices (partitions) of modules. (i.e. The simple module $D_{k,q}^{(\mu)}$ (resp. the "Specht"like module $S_{k,q}^{(\mu)}$, the Young module $X_q^{(\mu)}$, PIM $P_q^{(\mu)}$ ) indexed by a partition $\mu$ corresponds to $D_{k,q'}^{(\mu)}$ (resp. $S_{k,q'}^{(\mu)}$, $X_{q'}^{(\mu)}$, $P_{q'}^{(\mu)}$ ).)

Remark 5. Just mimicking an argument in [7], constructing a generalization of [38] and using Theorem 7.1, we deduce the above results. (see also [32]).

8 A conjecture

8.1 The theory of Lascoux-Leclerc-Thibon

Let $v$ be an indeterminate over $\mathbb{Q}$. Let $U_v(\widehat{sl_n})$ be the quantized enveloping algebra over $\mathbb{Q}(v)$ corresponding to the Dynkin diagram $A_{e-1}^{(1)}$.

The so-called "Fock space"
\[ \mathcal{F}_v = \bigoplus_{\lambda} \mathbb{Q}(v)|\lambda\rangle \]
is the $\mathbb{Q}(v)$-vector space with basis $|\lambda\rangle$ indexed by the set of all partitions. In [24] Lascoux, Leclerc and Thibon introduced an algorithm to compute the canonical basis of the basis representation $M_v(\Lambda_0)$ and conjectured that it also compute the decomposition matrix of the Iwahori-Hecke algebras of type $A$ at a root of unity over $\mathbb{C}$. This is so called the LLT conjecture.

The LLT conjecture is now a theorem (see, for example, [29, Chap. 6] and the references of Chapter 6 ).
In [25], Leclerc and Thibon define a canonical basis of the \( v \)-deformed Fock space representation \( \mathcal{F}_v \) of the affine Lie algebra \( \widehat{\mathfrak{g}}_\ell \). They conjectured that the entries of the transition matrix between these bases and \( \{ |\lambda \rangle \}_{\lambda} \) are also crystalized decomposition numbers of the Dipper-James' Schur algebra for \( \zeta \) specialized at a primitive \( e \)-th root of unity.

This LT conjecture is now a theorem over \( \mathbb{Q}(\zeta) \) where \( \zeta \) is a primitive \( e \)-th root of unity over \( \mathbb{C} \), due to Varagnolo and Vasserot [39].

Leclerc and Thibon showed that

**Theorem 8.1 (Leclerc-Thibon).** There exist bases \( \{ G(\lambda) \} \) and \( \{ G^{-}(\lambda) \} \)

of \( \mathcal{F}_v \) characterized by:

1. \( \overline{G(\lambda)} = G(\lambda), \overline{G^{-}(\lambda)} = G^{-}(\lambda) \)
2. \( G(\lambda) \equiv |\lambda \rangle \mod vL, \quad G^{-}(\lambda) \equiv |\lambda \rangle \mod v^{-1}L^{-} \).

Here, \( L \) (resp. \( L^{-} \)) denotes the \( \mathbb{Z}[v] \) (resp. \( \mathbb{Z}[v^{-1}] \))-lattice in \( \mathcal{F}_v \) with basis \( \{ |\lambda \rangle \} \).

Let

\[
G(\mu) = \sum_{\lambda} d_{\lambda,\mu}(v) |\lambda \rangle, \quad G^{-}(\lambda) = \sum_{\lambda} e_{\lambda,\mu}(v) |\mu \rangle.
\]

and \( D_{m,e,0}(v) = [d_{\lambda,\mu}(v)]_{\lambda,\mu=m} \).

**Theorem 8.2 (Leclerc-Thibon, Varagnolo-Vasserot).** The matrix \( D_{m,e,0}(1) \) is equal to the decomposition matrix \( D_{m,\zeta,0} \).

**Remark**

If both \( \zeta \) and \( \zeta' \) are primitive \( e \)-th roots of unity in \( \mathbb{C} \), then \( D_{m,\zeta,0} = D_{m,\zeta',0} \). So, we may write \( D_{m,e,0} \) instead of \( D_{m,\zeta,0} \).

8.2 Our hope

Let \( l_{k,q}(\lambda) \) be the Loewy length of \( S_{k,q}(\lambda) \). For \( \ell > w \), we define \( rad_{\lambda,\mu}(v) \in \mathbb{N}[v] \) as follows:

\[
rad_{\lambda,\mu}(v) = \sum_{k=0}^{l_{k,q}(\lambda)-1} \frac{[\text{Rad}^k(U_{k,q}(\overline{\lambda})/ \text{Rad}^{k+1}(U_{k,q}(\overline{\lambda})): V_{k,q}(\overline{\mu})]} v^k.
\]
Remark 6. Note that $\text{rad}_{\lambda,\mu}(v)$ is given explicitly by some products of Littlewood-Richardson coefficients and $v$. Moreover, an explicit formula for $\text{rad}_{\lambda,\mu}(v)$ will be written in [30].

By the construction of the Loewy series of $S_{k,q}(\lambda)$ for $\lambda \in \Lambda_{e,w,\rho}$, we also know

$$\text{Rad}^i(S_{k,q}(\lambda))/\text{Rad}^{i+1}(S_{k,q}(\lambda)) \cong \text{Soc}^k(\lambda)-i(S_{k,q}(\lambda))/\text{Soc}^{k(\lambda)-i}(S_{k,q}(\lambda)).$$

**Theorem 8.3 (Geck).** [16] There exists a square lower unitriangular matrix $A$ such that each entry of $A$ is non-negative and $D_{m,q,\ell} = D_{m,e,0} \cdot A$.

By the above theorem and Theorem 7.1, we deduce

**Corollary 8.4.** If $\text{rad}_{\lambda,\mu}(v) = 0$, then $d_{\lambda,\mu}(v) = 0$ for any $\lambda, \mu \in \Lambda_{e,w,\rho}$.

Not only do we want to show that James' conjecture is true, but we want to know an explicit formula for $d_{\lambda,\mu}(v)$ which is now known to be a certain parabolic Kazhdan-Lusztig polynomial.

According to James conjecture and Rouquier-Leclerc-Thibon conjecture [29, 6.33 (see also 6.27)] on Jantzen filtrations over $\mathbb{C}$, we hope the following:

**Conjecture 8.5.** $d_{\lambda,\mu}(v) = \text{rad}_{\lambda,\mu}(v)$ for any $\lambda, \mu \in \Lambda_{e,w,\rho}$.

The first announcement of this was stated in the author's lecture "On the unipotent blocks of finite general linear groups" at a conference "Algébres de Hecke affines et groupes réductifs (CIRM,Luminy,16-20 octobre 2000)" organized by M. Geck and R. Rouquier.

**Acknowledgments**

The author would like to thank M. Geck, B. Leclerc, and R. Rouquier for stimulating discussions on James' conjecture and canonical bases.

参考文献


