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Coefficient Inequalities for Ceratin Analytic Functions

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Abstract. For real $\alpha(\alpha>1)$, subclasses $M(\alpha)$ and $N(\alpha)$ of analytic fuctions $f(z)$ with $f(0)=0$ and $f'(0)=1$ in U are introduced. The object of the present paper is to consider the coefficient inequalities for functions $f(z)$ to be in the classes $M(\alpha)$ and $N(\alpha)$. Further, the bounds of α for functions $f(z)$ to be starlike in U are considered.

1 Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. Let $M(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<\alpha \qquad (z\in U)$$

for some $\alpha(\alpha>1)$. And let $N(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}<\alpha \qquad (z\in U)$$

for some $\alpha(\alpha>1)$. Then, we see that $f(z)\in N(\alpha)$ if and only if $zf'(z)\in M(\alpha)$. Let us give examples of functions $f(z)$ in the classes $M(\alpha)$ and $N(\alpha)$.

Remark *For* $1<\alpha\leq\frac{4}{3}$, *the classes* $M(\alpha)$ *and* $N(\alpha)$ *were introduced by Uralegaddi, Ganigi and Sarangi* [2].

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Example (i)$f(z) = z(1-z)^{2(\alpha-1)} \in M(\alpha)$

(ii)$g(z) = \frac{1}{2\alpha-1}\{1-(1-z)^{2\alpha-1}\} \in N(\alpha)$

Proof. Since $f(z) \in M(\alpha)$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \alpha,$$

we can write

$$\frac{\alpha - \frac{zf'(z)}{f(z)}}{\alpha-1} = \frac{1+z}{1-z},$$

which is equivalent to

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{2(\alpha-1)}{1-z}.$$

Integrating both sides of the above equality, we have

$$f(z) = z(1-z)^{2(\alpha-1)} \in M(\alpha).$$

Next, since $g(z) \in N(\alpha)$ if and only if $zg'(z) \in M(\alpha)$,

$$zg'(z) = z(1-z)^{2(\alpha-1)}.$$

for function $g(z) \in N(\alpha)$.It follows that

$$g(z) = \frac{-1}{2\alpha-1}(1-z)^{2\alpha-1} + \frac{1}{2\alpha-1}$$

$$= \frac{1}{2\alpha-1}\left\{1-(1-z)^{2\alpha-1}\right\} \in N(\alpha).$$

□

2 Coefficient inequalities for the classes $M(\alpha)$ and $N(\alpha)$

We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

Theorem 1. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)$$

for some $\alpha(\alpha > 1)$, then $f(z) \in M(\alpha)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)$$

for $f(z) \in A$.
It sufficies to show that

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} - (2\alpha - 1)} \right| < 1 \qquad (z \in U).$$

We have

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} - (2\alpha - 1)} \right| \leq \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{2(\alpha - 1) - \sum_{n=2}^{\infty} |n - 2\alpha + 1| \, |a_n||z|^{n-1}}$$

$$< \frac{\sum_{n=2}^{\infty}(n-1)|a_n|}{2(\alpha - 1) - \sum_{n=2}^{\infty} |n - 2\alpha + 1| \, |a_n|}.$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}(n-1)|a_n| \leq 2(\alpha - 1) - \sum_{n=2}^{\infty} |n - 2\alpha + 1| \, |a_n|$$

which is equivalent to our condition

$$\sum_{n=2}^{\infty}\{(n-1) + |n - 2\alpha + 1|\}|a_n| \leq 2(\alpha - 1)$$

of the theorem. This completes the proof of the theorem. □

By using Theorem1, we have

Corollary 1. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n\{(n-1)+|n-2\alpha+1|\}|a_n| \leq 2(\alpha-1)$$

for some $\alpha(\alpha > 1)$, then $f(z) \in N(\alpha)$.

Proof. From $f(z) \in N(\alpha)$ if and only if $zf'(z) \in M(\alpha)$, replacing a_n by na_n in Theorem1 we have the corollary. □

In view of Theorem1 and Corollary1, if $1 < \alpha \leq 2$, then $n - 2\alpha + 1 \geq 0$ for all $n \geq 2$. Thus we have

Corollary 2. (i) *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq \alpha - 1$$

for some $\alpha(1 < \alpha \leq 2)$, then $f(z) \in M(\alpha)$.

(ii) *If $f(z) \in A$, satisfies*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha - 1$$

for some $\alpha(1 < \alpha \leq 2)$, then $f(z) \in N(\alpha)$.

3 Starlikeness for functions in $M(\alpha)$ and $N(\alpha)$

By Silverman [1], we know that if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n|a_n| \leq 1,$$

then $f(z) \in S^*$, where S^* denotes the subclsses of A consisting of all univalent and starlike functions $f(z)$ in U. Thus we have

Theorem 2. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq \alpha - 1$$

for some $\alpha(1 < \alpha \leq \frac{4}{3})$*, then* $f(z) \in S^* \cap M(\alpha)$*, therefore,* $f(z)$*is starlike in* U*. Further, if* $f(z) \in A$ *satisfies*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha - 1$$

for some $\alpha(1 < \alpha \leq \frac{3}{2})$*, then* $f(z) \in S^* \cap N(\alpha)$*, therefore,* $f(z)$ *is starlike in* U*.*

Proof. Let us consider α such that

$$\sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} \frac{n-\alpha}{\alpha-1}|a_n| \leq 1.$$

Then we have $f(z) \in S^* \cap M(\alpha)$ by means of Theorem1. This inequality holds true if

$$n \leq \frac{n-\alpha}{\alpha-1} \qquad (n = 2, 3, 4, \cdots).$$

Therefore, we have

$$1 < \alpha \leq 2 - \frac{2}{n+1} \qquad (n = 2, 3, 4, \cdots),$$

which shows $1 < \alpha \leq \frac{4}{3}$. Next, considering α such that

$$\sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{\alpha-1}|a_n| \leq 1,$$

we have

$$n \leq \frac{n(n-\alpha)}{\alpha-1} \qquad (n = 2, 3, 4, \cdots).$$

which is equivalent to

$$1 < \alpha \leq \frac{n+1}{2} \qquad (n = 2, 3, 4, \cdots).$$

This implies that $1 < \alpha \leq \frac{3}{2}$. □

Finally, by virtue of the result for convex functions by Silverman [1], we have if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n^2|a_n| \leq 1,$$

then $f(z) \in K$, where K denotes the subclass of A consisting of all univalent and convex functions $f(z)$ in U. Using the same manner as in the proof of Theorem2, we derive

Theorem 3. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha - 1$$

for some $\alpha(1 < \alpha \leq \frac{4}{3})$, then $f(z) \in K \cap N(\alpha)$, therefore, $f(z)$ is convex in U.

4 Bounds of α for starlikeness

Note that the sufficient for $f(z)$ to be in the class $M(\alpha)$ is given by

$$\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1).$$

Since, if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n|a_n| \leq 1,$$

then $f(z) \in S^*$ (cf. [1]), it is interesting to find the bounds of α for starlikeness of $f(z) \in M(\alpha)$.

To do this, we have to consider the following inequality

$$\begin{aligned} \sum_{n=2}^{\infty} n|a_n| &\leq \frac{1}{2(\alpha-1)} \sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\}|a_n| \\ &\leq 1 \end{aligned}$$

which is equivalent to

$$\sum_{n=2}^{\infty} \{|n - 2\alpha + 1| + (3 - 2\alpha)n\} |a_n| \geq 0.$$

Let us define

$$F(n) = |n - 2\alpha + 1| + (3 - 2\alpha)n \qquad (n \geq 2).$$

Then, if $F(n)$ satisfies

$$\sum_{n=2}^{\infty} F(n)|a_n| \geq 0,$$

then $f(z)$ belongs to S^*.

Theorem 4. *Let $f(z) \in A$ satisfy*

$$\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)$$

for some $\alpha > 1$. Further, let δ_k be defined by

$$\delta_k = \sum_{n=k}^{\infty} F(n)|a_n|.$$

Then,

(i) *if* $1 < \alpha \leq \frac{3}{2}$, *then* $f(z) \in S^*$,

(ii) *if* $\frac{3}{2} \leq \alpha \leq \min\left(\frac{13}{8}, \frac{3+\delta_3}{2}\right)$, *then* $f(z) \in S^*$,

(iii) *if* $\frac{8}{3} \leq \alpha \leq \min\left(\frac{17}{10}, \frac{12-\delta_4+\sqrt{\delta_4^2+48\delta_4+48}}{12}\right)$, *then* $f(z) \in S^*$.

Proof. For $1 < \alpha \leq \frac{3}{2}$, we know that

$$n - 2\alpha + 1 \geq 3 - 2\alpha \geq 0 \qquad (n \geq 2),$$

that is that $F(n) \geq 0$ $(n \geq 2)$. Therefore, we have

$$\sum_{n=2}^{\infty} F(n)|a_n| \geq 0.$$

If $\frac{3}{2} \leq \alpha \leq \frac{13}{8}$, then $F(2) = 3 - 2\alpha \leq 0$ and

$$\begin{aligned} F(n) &= 2n(2-\alpha) + 1 - 2\alpha \\ &\geq 13 - 8\alpha \\ &\geq 0 \end{aligned}$$

for $n \geq 3$. Further, we know that

$$|a_n| \leq \frac{2(\alpha-1)}{(n-1)+|n-2\alpha+1|} \qquad (n \geq 2),$$

which given us that $|a_2| \leq 1$. therefore, we obtain that

$$\begin{aligned} \sum_{n=2}^{\infty} F(n)|a_n| &= F(2)|a_2| + \sum_{n=3}^{\infty} F(n)|a_n| \\ &\geq 3 - 2\alpha + \delta_3 \qquad \geq 0 \end{aligned}$$

for

$$\frac{3}{2} \leq \alpha \leq \min\left(\frac{13}{8}, \frac{3+\delta_3}{2}\right),$$

Furthermore, if $\frac{13}{8} \leq \alpha \leq \frac{17}{10}$, then

$$F(2) = 3 - 2\alpha \leq 0$$

$$\begin{aligned} F(3) &= |4 - 2\alpha| + 3(3 - 2\alpha) \\ &= 13 - 8\alpha \\ &\leq 0, \end{aligned}$$

and

$$\begin{aligned} F(n) &= |n - 2\alpha + 1| + (3 - 2\alpha)n \\ &= 4n + 1 - 2(n+1)\alpha \\ &\geq \frac{3(n-4)}{5} \\ &\geq 0 \end{aligned}$$

for $n \geq 4$. Noting that $|a_2| \leq 1$ and $|a_3| \leq \frac{\alpha - 1}{3 - \alpha}$, we conclude that

$$\begin{aligned} \sum_{n=2}^{\infty} F(n)|a_n| &= F(2)|a_2| + F(3)|a_3| + \sum_{n=4}^{\infty} F(n)|a_n| \\ &\geq (3 - 2\alpha) + (13 - 8\alpha)\frac{\alpha - 1}{3 - \alpha} + \delta_4 \\ &\geq 0. \end{aligned}$$

for α which satisfies

$$6\alpha^2 - (12 - \delta_4)\alpha + 4 - 3\delta_4 \leq 0.$$

This shows that

$$\frac{8}{3} \leq \alpha \leq min\left(\frac{17}{10}, \frac{12 - \delta_4 + \sqrt{\delta_4^2 + 48\delta_4 + 48}}{12}\right),$$

This completes the proof of Theorem4.

Finally, by virtue of Theorem4, we may supporse Conjecture. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)$$

for some $1 < \alpha < 2$, then $f(z) \in S^*$. □

References

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