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PROPERTIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. Let A be the class of functions $f(z)$ which are analytic in the open unit disk U with $f(0)=0$ and $f'(0)=1$. Two subclasses $H(\lambda,\mu)$ and $H_0(\lambda,\mu)$ of A with some inequalities for functions $f(z)$ are introduced. The object of the present paper is to consider some interesting properties for functions $f(z)$ belonging to these subclasses.

1. Introduction. Let A denote the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ which are analytic in the unit disk $E=\{z:|z|<1\}$. We denote by S the subclass of A consisting of functions which are univalent in E. A function $f(z)\in A$ is called starlike in $|z|<r$ $(0<r\leq 1)$ if it satisfies $\mathrm{Re}\{zf'(z)/f(z)\}>0$ for $|z|<r$.

For a function $f(z)\in A$, we say that $f(z)$ is in the class $H(\lambda,\mu)$ if and only if it satisfies the conditions $f(z)/z\neq 0$ for $z\in E$ and

$$\left|\frac{z^2 f'(z)}{f^2(z)}-\lambda z^2\left(\frac{z}{f(z)}\right)''-1\right|<\mu \quad (z\in E), \tag{1}$$

where λ is a complex number with $\mathrm{Re}\lambda\geq 0$ and μ is a positive real number. Also we define the class $H_0(\lambda,\mu)$ by

$$H_0(\lambda,\mu)=\{f(z)\in H(\lambda,\mu): f''(0)=0\}.$$

In [2], Nunokawa, Obradovic and Owa proved that if $f(z)\in A$ with $f(z)/z\neq 0$ for $z\in E$ and $|(z/f(z))''|\leq 1$ in E, then $f(z)\in S$. Ozaki and Nunokawa [4] showed that $H(0,1)\subset S$ and Obradovic et al. [3] considered the classes $H(0,1)$ and $H_0(0,1)$. In the present paper we investigate certain properties for the classes

$H(\lambda,\mu)$ and $H_0(\lambda,\mu)$. Our results generalize or improve the results obtained in [2], [3] and [4] and some other new results are also given.

2. Properties of the class $H(\lambda,\mu)$. Let $f(z)$ and $g(z)$ be analytic in E. Then we say that $f(z)$ is subordinate to $g(z)$ in E, written $f(z)\prec g(z)$, if there exists an analytic function $w(z)$ in E such that $|w(z)|\leq|z|$ and $f(z)=g(w(z))$ for $z\in E$. If $g(z)$ is univalent in E, then $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(E)\subset g(E)$.

We need the following lemma due to Miller and Mocanu [1].

Lemma. Let $h(z)$ be analytic and convex univalent in E, $h(0)=1$, and let $p(z)=1+p_n z^n + p_{n+1}z^{n+1}+\cdots$ $(n\in N)$ be analytic in E. If $p(z)+zp'(z)/c\prec h(z)$, where $c\neq 0$ and $\mathrm{Re}\, c\geq 0$, then

$$p(z)\prec \frac{c}{n}z^{-c/n}\int_0^z t^{c/n-1}h(t)\,dt.$$

For $\mathrm{Re}\,\lambda\geq 0$ and $\mu>0$, it is easy to verify that the function

$$f(z)=\frac{z}{(1+\sqrt{\mu/|1+2\lambda|}\,z)^2} \tag{2}$$

belongs to $H(\lambda,\mu)$ if and only if $\mu\leq|1+2\lambda|$. Applying the lemma, we derive

Theorem 1. Let $\mathrm{Re}\,\lambda\geq 0$ and $0<\mu\leq|1+2\lambda|$. Then $H(\lambda,\mu)\subset S$.

Proof. Let

$$p(z)=\frac{z^2f'(z)}{f^2(z)}=1+p_2z^2+\cdots \tag{3}$$

for $f(z)\in H(\lambda,\mu)$. Then

$$zp'(z)=-z^2\left(\frac{z}{f(z)}\right)''$$

and it follows from condition (1) that

$$p(z)+\lambda zp'(z)=\frac{z^2f'(z)}{f^2(z)}-\lambda z^2\left(\frac{z}{f(z)}\right)''\prec 1+\mu z.$$

For $\lambda\neq 0$, $\mathrm{Re}\,\lambda\geq 0$ and $\mu>0$, an application of the lemma yields

$$p(z) \prec 1+\frac{\mu}{1+2\lambda}z. \tag{4}$$

From (3), (4) and the Schwarz lemma, we have

$$\left|\frac{z^2 f'(z)}{f^2(z)}-1\right| \leq \frac{\mu}{|1+2\lambda|}|z|^2 \quad (z \in E) \tag{5}$$

for $\mathrm{Re}\,\lambda \geq 0$ and $\mu > 0$. Hence

$$\left|\frac{z^2 f'(z)}{f^2(z)}-1\right| < \frac{\mu}{|1+2\lambda|} \leq 1 \quad (z \in E) \tag{6}$$

for $\mathrm{Re}\,\lambda \geq 0$ and $0<\mu \leq |1+2\lambda|$.

Now, using Theorem 2 in [4], from (6) we conclude that $f(z) \in S$.

If we let $\mu = |1+2\lambda|$ and $\lambda \to \infty$, then condition (1) can be written as

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq 2 \quad (z \in E) \tag{7}$$

by the Schwarz lemma. Thus we obtain an improvement of the main theorem in [2]. Namely, we have

Corollary 1. Let $f(z) \in A$ with $f(z)/z \neq 0$ for $z \in E$ and let $f(z)$ satisfy (7). Then $f(z) \in S$.

Remark 1. Recently, Yang and Liu [5] showed Corollary 1 by a different method.

Corollary 2. Let $\mathrm{Re}\,\lambda \geq 0$, $0<\mu \leq |1+2\lambda|$ and

$$f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n} \in A. \tag{8}$$

If

$$\sum_{n=2}^{\infty} (n-1)|1+n\lambda||b_n| \leq \mu, \tag{9}$$

then $f(z) \in S$.

Proof. From (8) and (9) we have

$$\left|\frac{z^2f'(z)}{f^2(z)}-\lambda z^2\left(\frac{z}{f(z)}\right)''-1\right| = \left|-\sum_{n=2}^{\infty}(n-1)(1+n\lambda)b_n z^n\right|$$

$$\leqslant \sum_{n=2}^{\infty}(n-1)|1+n\lambda||b_n| \leqslant \mu$$

for $z \in E$. Therefore $f(z) \in H(\lambda,\mu) \subset S$ by using Theorem 1.

Theorem 2. Let $0 \leqslant \lambda_1 < \lambda_2$ and $\mu > 0$. Then $H(\lambda_2,\mu) \subset H(\lambda_1,\mu)$.

Proof. Let $f(z) \in H(\lambda_2,\mu)$. Then

$$\frac{z^2f'(z)}{f^2(z)}-\lambda_2 z^2\left(\frac{z}{f(z)}\right)'' \prec 1+\mu z$$

and from (4) in the proof of Theorem 1 we obtain

$$\frac{z^2f'(z)}{f^2(z)} \prec 1+\frac{\mu}{1+2\lambda_2}z \prec 1+\mu z.$$

Hence

$$\frac{z^2f'(z)}{f^2(z)}-\lambda_1 z^2\left(\frac{z}{f(z)}\right)'' = \frac{\lambda_1}{\lambda_2}\left\{\frac{z^2f'(z)}{f^2(z)}-\lambda_2 z^2\left(\frac{z}{f(z)}\right)''\right\}+\left(1-\frac{\lambda_1}{\lambda_2}\right)\frac{z^2f'(z)}{f^2(z)}$$

$$\prec 1+\mu z$$

for $0 \leqslant \lambda_1 < \lambda_2$. This implies that $f(z) \in H(\lambda_1,\mu)$.

Theorem 3. Let $\mathrm{Re}\,\lambda \geqslant 0$, $\mu > 0$ and $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in H(\lambda,\mu)$. Then

$$\left|\frac{z}{f(z)}-1+a_2 z\right| \leqslant \frac{\mu}{|1+2\lambda|}|z|^2, \tag{10}$$

$$\left|\frac{z}{f(z)}-1\right| \leqslant |z|\left(|a_2|+\frac{\mu}{|1+2\lambda|}|z|\right), \tag{11}$$

$$1-|z|\left(|a_2|+\frac{\mu}{|1+2\lambda|}|z|\right) \leqslant \mathrm{Re}\frac{z}{f(z)} \leqslant 1+|z|\left(|a_2|+\frac{\mu}{|1+2\lambda|}|z|\right) \tag{12}$$

and

$$|f(z)| \geqslant \frac{|z|}{1+|a_2||z|+(\mu/|1+2\lambda|)|z|^2}. \tag{13}$$

Equalities in (10)-(13) are attained if we take

$$f(z)=\frac{z}{1\pm bz+(\mu/|1+2\lambda|)z^2}\in H(\lambda,\mu)$$

with $0<\mu\leq|1+2\lambda|$ and $0\leq b\leq 2\sqrt{\mu/|1+2\lambda|}$.

Proof. For $f(z)=z+a_2z^2+\cdots\in H(\lambda,\mu)$, we find that

$$\int_0^z\left(\frac{f'(t)}{f^2(t)}-\frac{1}{t^2}\right)dt=\left(\frac{1}{t}-\frac{1}{f(t)}\right)\Big|_0^z=\frac{1}{z}-\frac{1}{f(z)}-a_2. \tag{14}$$

Using (5) in the proof of Theorem 1, it follows from (14) that

$$\left|\frac{1}{f(z)}-\frac{1}{z}+a_2\right|\leq\int_0^{|z|}\left|\frac{f'(t)}{f^2(t)}-\frac{1}{t^2}\right|dt\leq\frac{\mu}{|1+2\lambda|}|z|\quad(z\in E),$$

which gives (10). In view of (10), we easily have (11), (12) and (13).

Remark 2. Taking $\lambda=0$ and $\mu=1$ in (11) and (12), we get the corresponding results in [3].

Theorem 4. Let $\mathrm{Re}\,\lambda\geq 0$, $\mu>0$ and $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in H(\lambda,\mu)$. Then

$$|a_2^2-a_3|\leq\mu/|1+2\lambda|. \tag{15}$$

The result is sharp for $0<\mu\leq|1+2\lambda|$.

Proof. Since

$$\frac{z}{f(z)}-1+a_2z=(a_2^2-a_3)z^2+\sum_{n=3}^{\infty}b_nz^n,$$

from (10) in Theorem 3 we deduce that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{re^{i\theta}}{f(re^{i\theta})}-1+a_2re^{i\theta}\right|^2d\theta=|a_2^2-a_3|^2r^4+\sum_{n=3}^{\infty}|b_n|^2r^{2n}$$
$$\leq\left(\frac{\mu}{|1+2\lambda|}\right)^2r^4\qquad(0<r<1),$$

which leads to (15).

It is easy to see that the estimate (15) is best possible for the function $f(z)$ given by (2) with $0<\mu\leq|1+2\lambda|$.

3. Properties of the class $H_0(\lambda,\mu)$

Theorem 5. Let $\mathrm{Re}\lambda \geq 0$ and $f(z)\in H_0(\lambda,\mu)$.

(a) If $|1+2\lambda|/\sqrt{2} \leq \mu \leq |1+2\lambda|$, then $f(z)$ is starlike in $|z|<\sqrt{|1+2\lambda|/(\sqrt{2}\mu)}$;

(b) If $|1+2\lambda|/2 \leq \mu \leq |1+2\lambda|$, then $\mathrm{Re}f'(z)>0$ for $|z|<\sqrt{|1+2\lambda|/(2\mu)}$.

Proof. We use a technique in [5]. Let $\mathrm{Re}\lambda\geq 0$ and $0<\mu\leq|1+2\lambda|$. Then from (5) in the proof of Theorem 1 we have

$$\left|\arg\frac{z^2f'(z)}{f^2(z)}\right| \leq \arcsin\left(\frac{\mu}{|1+2\lambda|}|z|^2\right) \quad (z\in E). \tag{16}$$

Also it follows from (10) in Theorem 3 with $a_2=0$ that

$$\left|\arg\frac{z}{f(z)}\right| \leq \arcsin\left(\frac{\mu}{|1+2\lambda|}|z|^2\right) \quad (z\in E). \tag{17}$$

(a) If $|1+2\lambda|/\sqrt{2}\leq\mu\leq|1+2\lambda|$, then from (16) and (17) we obtain

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &\leq \left|\arg\frac{z^2f'(z)}{f^2(z)}\right| + \left|\arg\frac{z}{f(z)}\right| \\ &\leq 2\arcsin\left(\frac{\mu}{|1+2\lambda|}|z|^2\right) < \frac{\pi}{2}\end{aligned}$$

for $|z|<r_1=\sqrt{|1+2\lambda|/(\sqrt{2}\mu)}\leq 1$. This shows that $f(z)$ is starlike in $|z|<r_1$.

(b) If $|1+2\lambda|/2\leq\mu\leq|1+2\lambda|$, then it follows from (16) and (17) that

$$\begin{aligned}|\arg f'(z)| &\leq \left|\arg\frac{z^2f'(z)}{f^2(z)}\right| + 2\left|\arg\frac{z}{f(z)}\right| \\ &\leq 3\arcsin\left(\frac{\mu}{|1+2\lambda|}|z|^2\right) < \frac{\pi}{2}\end{aligned}$$

for $|z|<r_2=\sqrt{|1+2\lambda|/(2\mu)}\leq 1$. This implies that $\mathrm{Re}f'(z)>0$ for $|z|<r_2$.

Remark 3. Letting $\lambda=0$ and $\mu=1$ in Theorem 5, (b) improves a result of [3] and (a) is the same as in [3].

Corollary 3. Let $\mathrm{Re}\,\lambda \geq 0$ and $f(z) \in H_0(\lambda, \mu)$.

(a) If $0 < \mu \leq |1+2\lambda|/\sqrt{2}$, then $f(z)$ is starlike in E;

(b) If $0 < \mu \leq |1+2\lambda|/2$, then $\mathrm{Re}\, f'(z) > 0$ for $z \in E$.

Theorem 6. Let

$$f(z) = \frac{z}{1+\sum_{n=2}^{\infty} b_n z^n}, \tag{18}$$

$f_1(z) = z$ and $f_m(z) = z/(1 + b_2 z^2 + \cdots + b_m z^m)$ $(m \geq 2)$. If $\mathrm{Re}\,\lambda \geq 0$, $0 < \mu \leq |1+2\lambda|$ and

$$\sum_{n=2}^{\infty} (n-1)|1+n\lambda||b_n| \leq \mu, \tag{19}$$

then we have

(a) $f(z) \in H_0(\lambda, \mu) \subset S$;

(b) For $z \in E$,

$$\mathrm{Re}\, \frac{f_m(z)}{f(z)} > 1 - \frac{\mu}{m|1+(m+1)\lambda|} \tag{20}$$

and

$$\mathrm{Re}\, \frac{f(z)}{f_m(z)} > \frac{m|1+(m+1)\lambda|}{m|1+(m+1)\lambda| + \mu}. \tag{21}$$

The results are sharp for each $m \in N$.

Proof. Let $c_n = (n-1)|1+n\lambda|/\mu$ $(n \geq 2)$. Then

$$c_{n+1} > c_n \geq 1 \quad (n \geq 2) \tag{22}$$

for $\mathrm{Re}\,\lambda \geq 0$ and $0 < \mu \leq |1+2\lambda|$. From (19) and (22) we deduce that the function $f(z)$ given by (18) is analytic in E and

$$\sum_{n=2}^{m} |b_n| + c_{m+1} \sum_{n=m+1}^{\infty} |b_n| \leq \sum_{n=2}^{\infty} c_n |b_n| \leq 1 \quad (m \geq 2). \tag{23}$$

(a) Noting that $f(z) \in A$ and $f''(0) = 0$, from the proof of Corollary 2 we see that

$f(z) \in H_0(\lambda, \mu) \subset S$.

(b) Let

$$p_1(z) = c_{m+1}\left\{\frac{f_m(z)}{f(z)} - \left(1 - \frac{1}{c_{m+1}}\right)\right\}.$$

Then

$$p_1(z) = 1 + \frac{c_{m+1}\sum_{n=m+1}^{\infty} b_n z^n}{1 + \sum_{n=2}^{m} b_n z^n}$$

and from (23) we deduce that

$$\left|\frac{p_1(z)-1}{p_1(z)+1}\right| = \left|\frac{c_{m+1}\sum_{n=m+1}^{\infty} b_n z^n}{2\left(1+\sum_{n=2}^{m} b_n z^n\right) + c_{m+1}\sum_{n=m+1}^{\infty} b_n z^n}\right|$$

$$\leqslant \frac{c_{m+1}\sum_{n=m+1}^{\infty}|b_n|}{2 - 2\sum_{n=2}^{m}|b_n| - c_{m+1}\sum_{n=m+1}^{\infty}|b_n|}$$

$$\leqslant 1 \quad (z \in E).$$

Hence we conclude that $\mathrm{Re}\{f_m(z)/f(z)\} > 1 - 1/c_{m+1}$ for $z \in E$. This proves (20) for $m \geqslant 2$.

If we take

$$f(z) = \frac{z}{1 + (\mu/(m|1+(m+1)\lambda|))\, z^{m+1}}, \tag{24}$$

then $f_m(z) = z$ and $f_m(z)/f(z) \to 1 - \mu/(m|1+(m+1)\lambda|)$ as $z \to e^{i\pi/(m+1)}$. Hence the bound in (20) is best possible for each $m \geqslant 2$.

Similarly, if we put

$$p_2(z) = (1 + c_{m+1})\left\{\frac{f(z)}{f_m(z)} - \frac{c_{m+1}}{1 + c_{m+1}}\right\},$$

then it follows from (23) that

$$\left|\frac{p_2(z)-1}{p_2(z)+1}\right| = \left|\frac{-(1+c_{m+1})\sum_{n=m+1}^{\infty} b_n z^n}{2\left(1+\sum_{n=2}^{m} b_n z^n\right)-(c_{m+1}-1)\sum_{n=m+1}^{\infty} b_n z^n}\right|$$

$$\leqslant \frac{(1+c_{m+1})\sum_{n=m+1}^{\infty} |b_n|}{2-2\sum_{n=2}^{m}|b_n|-(c_{m+1}-1)\sum_{n=m+1}^{\infty}|b_n|}$$

$$\leqslant 1 \quad (z \in E).$$

Now we easily have the inequality (21) for $m \geqslant 2$ and the bound in (21) is sharp for the function $f(z)$ given by (24).

Finally, (23) becomes

$$c_2 \sum_{n=2}^{\infty} |b_n| \leqslant \sum_{n=2}^{\infty} c_n |b_n| \leqslant 1$$

when $m=1$. By using the same way as in the above, the proof of the theorem is completed.

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